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AN INTRODUCTION TO DISTANCE D MAGIC GRAPHS

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Abstract. For a graph G of order $|V(G)| = n$ and a real-valued mapping $f : V(G) \to \mathbb{R}$, if $S \subset V(G)$ then $f(S) = \sum_{w \in S} f(w)$ is called the weight of S under f. When there exists a bijection $f : V(G) \to [n]$ such that the weight of all open neighborhoods is the same, the graph is said to be 1-vertex magic, or Σ labeled. In this paper we generalize the notion of 1-vertex magic by defining a graph G of diameter d to be D-vertex magic when for $D \subset \{0, 1, \ldots, d\}$, we have that $\sum_{u \in N_D(v)} f(u)$ is constant for all $v \in V(G)$. We provide several existence criteria for graphs to be D-vertex magic and use them to provide solutions to several open problems presented at the IWOGL 2010 Conference. In addition, we extend the notion of vertex magic graphs by providing measures describing how close a non-vertex magic graph is to being vertex magic. The general viewpoint is to consider how to assign a set W of weights to the vertices so as to have an equitable distribution over the D-neighborhoods.

Key words: Graph Labeling, vertex magic, Σ labeling, distance magic graphs, neighborhood sums.

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Abstrak. Untuk suatu graf G berorde $|V(G)| = n$ dan sebuah pemetaan bernilai riil $f : V(G) \to \mathbb{R}$, jika $S \subset V(G)$ maka $f(S) = \sum_{w\in S} f(w)$ disebut bobot dari S oleh f. Ketika terdapat sebuah bijeksi $f : V(G) \to [n]$ sehingga bobot dari semua himpunan buka adalah sama, graf dikatakan menjadi ajaib 1-titik, atau dilabelkan Σ. Pada paper ini kami memperumum ide dari ajaib 1-titik dengan pendefinisian sebuah graf G berdiameter d menjadi ajaib D-titik jika untuk $D \subset \{0, 1, \ldots, d\}$, kita mempunyai bahwa $\sum_{u\in N_D(v)} f(u)$ adalah konstan untuk semua $v \in V(G)$. Kami memberikan beberapa kriteria keberadaan untuk graf-graf menjadi ajaib D-titik dan menggunakan mereka untuk memberikan solusi beberapa masalah terbuka yang disajikan di konferensi IWOGL 2010. Lebih jauh, kami memperluas ide graf-graf ajaib titik dengan pemberian ukuran-ukuran yang menggambarkan seberapa dekat sebuah graf ajaib bukan titik mnejadi ajaib titik. Titik pandang umum adalah bagaimana untuk menyatakan suatu himpunan W dari bobot-bobot ke titik-titik sehingga mempunyai suatu distribusi yang seimbang sepanjang ketetanggaan-D.

Kata kunci: Pelabelan graf, ajaib titik, pelabelan Σ, graf ajaib jarak, jumlah ketetanggaan.

1. **Introduction**

One type of graph labeling problem involves labeling the vertices of a graph G and then computing a value $g(v)$ for each $v \in V(G)$, where $g(v)$ is determined by the labels on some set $S(v) \subset V(G)$. Properties of the graph can be defined based on the permissible sets of values that are produced by the set of labelings $\{g(v) : v \in V(G)\}$. For example, for a graph $G = (V, E)$ of order n, one can define a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ and then for each vertex, sum the labels in its open (or closed) neighborhood. One case that has been studied is the case where the set of resulting open neighborhood sums are all equal. Vilfred [9] called such a labeling a Σ labeling and any graph for which such a labeling exists a Σ graph. Miller et al. [2] referred to such a labeling as a 1-vertex magic labeling. More recently Sugeng et al. [8] have referred to such a labeling as a distance magic labeling. When the closed neighborhood sums are all equal, Beena [1] has referred to the labeling as a Σ' labeling and the graph as a Σ' graph. Each of these works has focused on either the open or closed neighborhood case.

In her presentation at the 2010 IWOGL Conference [5], Rinovia Simanjuntak introduced the notion of distance magic labelings for a fixed distance other than one. Her presentation suggests a generalization of the notion of distance magic to arbitrary sets of distances. In this paper we generalize the notion to an arbitrary set of distances $D \subset \{0, 1, \ldots, d\}$ where d is the diameter of the graph. A graph G will be defined to be D-vertex magic when the sums of the vertex labels at vertices whose distances from v are in D are all constant. For example, a Σ graph is a $\{1\}$-vertex magic and a Σ' graph is $\{0, 1\}$-vertex magic. We provide existence criteria for graphs to be D-vertex magic.

Each of the works mentioned has focused on the existence/non-existence of labelings for particular graphs. We extend the existence question and define measures that can be used to classify how close a particular graph is to being distance magic. For example, when a graph is not Σ labeled, we measure how close is it to being Σ labeled.

2. **Definitions**

In this section we give our definitions, including that of a graph G being D-vertex magic for an arbitrary set $D \subset \{0, 1, \ldots, diam(G)\}$, and we illustrate these definitions for the House graph H in Figure 1(a). Throughout the paper we assume graph G has order $|V(G)| = n$, diameter $diam(G) = d$, and that $D \subset \{0, 1, \ldots, d\}$.

Definition 2.1. *For $v \in V(G)$, the D-neighborhood of v, denoted by $N_D(v)$, is defined as $N_D(v) = \{u \in V(G) : d(v, u) \in D\}$.*

If $D = \{1\}$, then $N_D(v)$ is the open neighborhood of the vertex v. We will adopt the notation that $N(v) = N_{\{1\}}(v)$. We take $d(v, v) = 0$. Thus, if $D = \{0, 1\}$, then $N_D(v)$ is the closed neighborhood of the vertex v. We will adopt the notation that $N[v] = N_{\{0,1\}}(v)$.

Definition 2.2. *Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The distance D adjacency matrix, denoted by $A_D = [a_{i,j}]$, is defined to be the $n \times n$ binary matrix with $a_{i,j} = 1$ if and only if $d(v_i, v_j) \in D$.*

If $D = \{1\}$, then $A_D = A$ is the adjacency matrix of G in the usual fashion. If $D = \{0, 1\}$, then $A_D = A + I = N$ where I is the $n \times n$ identity matrix, and N is called the closed neighborhood matrix of G. Also note that for any $D \subset \{0, 1, \ldots, d\}$ we have that A_D is symmetric.

Definition 2.3. *Graph G is defined to be (D, r)-regular if for all $v \in V(G)$, $\sum_{u \in N_D(v)} 1 = r$, that is, all D-neighborhoods have the same cardinality.*

Note that for a graph G to be $(\{1\}, r)$-regular or, equivalently, $(\{0, 1\}, r + 1)$-regular corresponds to G being r-regular.

Definition 2.4. *Let $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ be a set (or, more generally, a multiset) of real numbers referred to as weights. For a bijection $f : V(G) \to W$ and a subset $S \subset V(G)$, the weight of S under f, denoted by $f(S)$, is defined as $f(S) = \sum_{v \in S} f(v)$.*

We first consider bijections f that minimize the maximum weight of a D-neighborhood.

Definition 2.5. *Let $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ be a set (or, more generally, a multiset). For a bijection $f : V(G) \to W$, we define the D-neighborhood sum of f, denoted by $NS(f; D)$, as $NS(f; D) = max\{f(N_D(v))|v \in V(G)\}$.*

When $D = \{1\}$ we will by convention shorten the notation to $NS(f) = NS(f; \{1\})$. Similarly, when $D = \{0, 1\}$ we will by convention shorten the notation to $NS[f] = NS(f; \{0, 1\})$.

Definition 2.6. *Let $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ be a set (or, more generally, a multiset). The W-valued D-neighborhood sum of G, denoted by $NS_W(G; D)$, is defined as $NS_W(G; D) = min\{NS(f; D)|f : V(G) \to W$ is a bijection $\}$.*

When $D = \{1\}$ we will adopt the previous convention and shorten the notation to $NS_W(G) = NS_W(G; \{1\})$. Likewise, when $D = \{0, 1\}$ we shorten the notation to $NS_W[G] = NS_W(G; \{0, 1\})$.

For a graph $G = (V, E)$ of order $|V(G)| = n$ we are generally interested in the set of weights $W = [n] = \{1, 2, \ldots, n\}$. When this is the case, we will by convention shorten the notation to $NS(G; D) = NS_W(G; D)$. So for the open neighborhood sum case (that is, when $D = \{1\}$) with weight set $W = [n]$, our notation will be $NS(G) = NS_W(G; \{1\})$. For the closed neighborhood sum case (that is, when $D = \{0, 1\}$) with weight set $W = [n]$, our notation will be $NS[G] = NS_W(G; \{0, 1\})$. By following these conventions, our notation matches what has been introduced by Schneider and Slater [6, 7].

Example 2.7. *Consider the graph $H = (V, E)$ in Figure 1(a) which has diameter 2. We claim that $NS(H) = 8$, $NS[H] = 11$, and $NS(H; \{2\}) = 6$.*

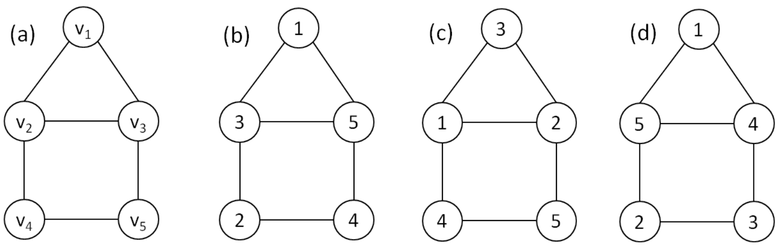

FIGURE 1. Minimax and maximin labelings of House graph H

To see that $NS(H) = 8$, consider that $N(v_2) \cup N(v_4) = V(H)$ and $N(v_2) \cap N(v_4) = \varnothing$. Hence, for any bijection $f : V(H) \to [5]$, we must have that $f(N(v_2)) + f(N(v_4)) = 15$. It follows that one of $\{f(N(v_2)), f(N(v_4))\}$ is greater than or equal to 8. From this we have that $NS(H) \geq 8$. The bijection shown in Figure 1(b) demonstrates that $NS(H) \leq 8$. Hence, we conclude that $NS(H) = 8$.

To see that $NS[H] = 11$, let $f : V(H) \to [5]$ be an arbitrary bijection. Notice that one of $\{f(v_4), f(v_5)\}$ is no more than 4. If $f(v_4) \leq 4$, then $f(N[v_3]) = 15 - f(v_4) \geq 11$. Similarly, if $f(v_5) \leq 4$, then $f(N[v_2]) = 15 - f(v_5) \geq 11$. Thus $NS[H] \geq 11$. The bijection shown in Figure 1(c) demonstrates that $NS[H] \leq 11$. Therefore, $NS[H] = 11$.

Finally, to see that $NS(H; \{2\}) = 6$, let $f : V(H) \to [5]$ be an arbitrary bijection. Notice that if $f(v_1) = 5$ or $f(v_2) = 5$, then $f(N_{\{2\}}(v_5)) = f(v_1)+f(v_2) \geq 6$. If $f(v_3) = 5$, then $f(N_{\{2\}}(v_4)) = f(v_1) + f(v_3) \geq 6$. If $f(v_4) = 5$ or $f(v_5) = 5$, then $f(N_{\{2\}}(v_1)) = f(v_4) + f(v_5) \geq 6$. Thus $NS(H; \{2\}) \geq 6$. The bijection shown in Figure 1(b) demonstrates that $NS(H; \{2\}) \leq 6$. Therefore, $NS(H; \{2\}) = 6$.

Next we consider maximizing the minimum weight of a D-neighborhood.

Definition 2.8. *Let $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ be a set (or, more generally, a multiset). For a bijection $f : V(G) \to W$, we define the lower D-neighborhood sum of f, denoted by $NS^-(f; D)$, as $NS^-(f; D) = min\{f(N_D(v))|v \in V(G)\}$.*

When $D = \{1\}$ we will by convention shorten the notation to $NS^-(f) = NS^-(f; D)$. Similarly, when $D = \{0, 1\}$ we will by convention shorten the notation to $NS^-[f] = NS^-(f; D)$.

Definition 2.9. *Let $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ be a set (or, more generally, a multiset). The W-valued lower D-neighborhood sum of G, denoted by $NS^-_W(G; D)$, is defined as $NS^-_W(G; D) = max\{NS^-(f; D)|f : V(G) \to W \text{ is a bijection }\}$.*

When $D = \{1\}$ we will adopt the previous convention and shorten the notation to $NS^-_W(G) = NS^-_W(G; D)$. Likewise, when $D = \{0, 1\}$ we shorten the notation to $NS^-_W[G] = NS^-_W(G; D)$.

When our weight set is $W = [n] = \{1, 2, \ldots, n\}$, we will again adopt the notation $NS^-(G; D) = NS^-_W(G; D)$. So, for the open neighborhood sum case, where $D = \{1\}$, with weight set $W = [n]$, our notation will be $NS^-(G) = NS^-_W(G; D)$. For the closed neighborhood sum case, where $D = \{0, 1\}$, with weight set $W = [n]$, our notation will be $NS^-[G] = NS^-_W(G; D)$. By following these conventions, our notation matches what has been introduced by O'Neal and Slater [3].

Example 2.10. *Consider the graph $H = (V, E)$ from Figure 1(a). We claim that $NS^-(H) = 7$, $NS^-[H] = 9$, and $NS^-(H; \{2\}) = 4$.*

To see that $NS^-(H) = 7$, again notice that $N(v_2) \cup N(v_4) = V(H)$ and $N(v_2) \cap N(v_4) = \varnothing$. Hence for any bijection $f : V(H) \to [5]$ we must have that $f(N(v_2)) + f(N(v_4)) = 15$. It follows that one of $\{f(N(v_2)), f(N(v_4))\}$ is less than or equal to 7; hence we have that $NS^-(H) \leq 7$. The bijection shown in Figure 1(b) demonstrates that $NS^-(H) \geq 7$. Therefore, we conclude that $NS^-(H) = 7$.

Figure 1(d) demonstrates that $NS^-[H] \geq 9$. If there exists a bijection $f : V(H) \to [5]$ such that $NS^-[f] \geq 10$, then $f(N[v_1]) \geq 10$, and hence $f(v_4)+f(v_5) \leq$

5. *However, this would imply that one of* $\{f(N[v_4]), f(N[v_5])\}$ *is no more than 9. Therefore, such a bijection does not exist, and we conclude that* $NS^-[H] = 9$.

Finally, we show that $NS^-(H;\{2\}) = 4$. *Consider that one of* $\{f(v_4), f(v_5)\}$ *is less than or equal 4. If* $f(v_4) \leq 4$, *then* $f(N_{\{2\}}(v_3)) = f(v_4) \leq 4$. *If* $f(v_5) \leq 4$, *then* $f(N_{\{2\}}(v_2)) = f(v_5) \leq 4$. *Hence* $NS^-(H;\{2\}) \leq 4$. *The bijection shown in Figure 1(c) demonstrates that* $NS^-(H;\{2\}) \geq 4$. *Therefore,* $NS^-(H;\{2\}) = 4$.

Consider the closed neighborhood sum $NS[H]$ and the lower closed neighborhood sum $NS^-[H]$. For bijection f in Figure 1(c), we have closed neighborhood sums $\{6, 10, 11, 10, 11\}$ achieving $NS[H] = 11 = f(N[v_3]) = f(N[v_5])$. Note that $f(N[v_1]) = 6$. For bijection g of Figure 1(d), we have closed neighborhood sums $\{10, 12, 13, 10, 9\}$ with $NS^-[H] = g(N[v_5]) = 9$. Is there a bijection $h : V(G) \to [5]$ that simultaneously achieves $NS[H]$ and $NS^-[H]$? Our third measure of equitability considers the range of values in $\{f(N_D(v_1)), \ldots, f(N_D(v_n))\}$.

Definition 2.11. *Let* $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ *be a set (or, more generally, a multiset). For a bijection* $f : V(G) \to W$, *we define the D-neighborhood spread of* f, *denoted by* $NS^{sp}(f;D)$, *as* $NS^{sp}(f;D) = NS(f;D) - NS^-(f;D)$.

As before, when $D = \{1\}$ we will by convention shorten the notation to $NS^{sp}(f) = NS^{sp}(f;D)$. Similarly, when $D = \{0,1\}$ we will by convention shorten the notation to $NS^{sp}[f] = NS^{sp}(f;D)$.

Definition 2.12. *Let* $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ *be a set (or, more generally, a multiset). The W-valued D-neighborhood spread of G, denoted by* $NS^{sp}_W(G;D)$, *is defined as* $NS^{sp}_W(G;D) = min\{NS(f;D) - NS^-(f;D) | \ f : V(G) \to W \text{ is a bijection }\}$.

When $D = \{1\}$ we will adopt the previous convention and shorten the notation to $NS^{sp}_W(G) = NS^{sp}_W(G;D)$. Likewise, when $D = \{0,1\}$ we shorten the notation to $NS^{sp}_W[G] = NS^{sp}_W(G;D)$.

When our weight set is $W = [n] = \{1, 2, \ldots, n\}$, we will again shorten the notation to $NS^{sp}(G;D) = NS^{sp}_W(G;D)$. So for the open neighborhood sum case, where $D = \{1\}$, with weight set $W = [n]$, our notation will be $NS^{sp}(G) = NS^{sp}_W(G;D)$. For the closed neighborhood sum case, where $D = \{0,1\}$, with weight set $W = [n]$, our notation will be $NS^{sp}[G] = NS^{sp}_W(G;D)$. By following these conventions, our notation matches what has been introduced by O'Neal and Slater [3].

Definition 2.13. *Graph G is said to be D-vertex magic, or equivalently D-distance magic, if there exists a bijection* $f : V(G) \to [n]$ *and a constant c such that for all* $v \in V(G)$, $\sum_{u \in N_D(v)} f(u) = c$, *that is, f has D-neighborhood spread zero.*

Notice that a graph is Σ labeled if and only if it is D-vertex magic where $D = \{1\}$. A graph is Σ' labeled if and only if it is D-vertex magic where $D = \{0,1\}$.

Similarly, for a graph with diameter d and for $0 < i < d$, the graph is i-vertex magic if and only if the graph is D-vertex magic with $D = \{i\}$.

For any graph $G = (V, E)$ with order n, $NS(G)$, $NS^-(G)$, and $NS^{sp}(G)$ are each measures of how close the graph is to being 1-vertex magic (Σ labeled). More generally, for $D \subset \{0, 1, \ldots, d\}$ where d is the diameter of G, $NS(G; D)$, $NS^-(G; D)$, and $NS^{sp}(G; D)$ give us measures of how close G is to being D-vertex magic.

Example 2.14. *For the graph H from Figure 1(a), we claim that $NS^{sp}(H) = 1$, $NS^{sp}[H] = 4$, and $NS^{sp}(H; \{2\}) = 4$.*

Since $NS(H) = 8$ and $NS^-(H) = 7$, clearly $NS^{sp}(H) \geq 1$. The bijection shown in Figure 1(b) demonstrates that $NS^{sp}(H) \leq 1$. Therefore, $NS^{sp}(H) = 1$.

The bijection from Figure 1(b) also demonstrates that $NS^{sp}[H] \leq 4$. Assume there exists a bijection $f : V(H) \to [5]$ such that $NS^{sp}[f] \leq 3$. Since $f(N[v_2]) - f(N[v_1]) = f(v_4)$, we must have that $f(v_4) \leq 3$. Similarly, we must have that $f(v_5) \leq 3$. Hence, one of $\{f(v_4), f(v_5)\}$ is less than or equal 2. If $f(v_4) \leq 2$, then $f(N[v_3]) \geq 13$. If $f(v_5) \leq 2$, then $f(N[v_2]) \geq 13$. But since $NS^-[H] \leq 9$, we have that $NS^{sp}[f] \geq 4$, which is a contradiction. Therefore, $NS^{sp}[H] = 4$.

Finally, notice that the bijection from Figure 1(b) also demonstrates that $NS^{sp}(H; \{2\}) \leq 4$. Assume there exists a bijection $f : V(H) \to [5]$ such that $NS^{sp}(H; \{2\}) \leq 3$. In this case $f(N_{\{2\}}(v_1)) - f(N_{\{2\}}(v_2)) = f(v_4) \leq 3$. Similarly, $f(N_{\{2\}}(v_1)) - f(N_{\{2\}}(v_3)) = f(v_5) \leq 3$. Thus, one of $\{f(v_4), f(v_5)\}$ is less than or equal 2. If $f(v_4) \leq 2$, then we have $f(N_{\{2\}}(v_3)) = f(v_4) \leq 2$. If $f(v_5) \leq 2$, then we have $f(N_{\{2\}}(v_2)) = f(v_5) \leq 2$. In either case, since $NS(H; \{2\}) \geq 6$, we have that $NS^{sp}(f; \{2\}) \geq 4$, which is a contradiction. Therefore, $NS^{sp}(H; \{2\}) = 4$.

These examples demonstrate that there are graphs where $NS^{sp}[G] > NS[G] - NS^-[G]$. Where this is the case, there cannot be a single bijection that achieves both $NS[G]$ and $NS^-[G]$. We note that there are graphs such that $NS^{sp}(G) > NS(G) - NS^{sp}(G)$.

Also notice that for the graph H in these examples, we had that $NS[H] + NS^-(H; \{2\}) = \frac{n(n+1)}{2} = 15 = NS^-[H] + NS[H; \{2\}]$ and that $NS^{sp}[H] = NS^{sp}(H; \{2\})$. This result was not a coincidence, and the more general result will be proven in the next section in Theorem 3.1.

Example 2.15. *For graph G and weight set $W = \{w_1, w_2, \ldots, w_n\}$, if $D = \{0, 1, 2, \ldots, d\}$, then it is trivially true that $NS_W[G; D] = NS^-_W[G; D] = \sum_{i=1}^{n} w_i$ and that $NS^{sp}_W[G; D] = 0$. Since the diameter of the graph H in the previous example is 2, when $D = \{0, 1, 2\}$, we have that $NS[H; D] = NS^-[H; D] = 15$ and $NS^{sp}[H; D] = 0$.*

3. Existence Theorems for D-vertex Magic Graphs

Theorem 3.1. *Let $D \subset \{0, 1, \ldots, d\}$ and let $D^\# = \{0, 1, \ldots, d\} - D$. Then G is D-vertex magic if and only if G is $D^\#$-vertex magic.*

Proof. Let $f : V(G) \to [n]$ be a bijection and c a constant such that for all $v \in V(G)$, $\sum_{u \in N_D(v)} f(u) = c$. Then for all $v \in V(G)$, we also have that $\sum_{u \in N_{D^{\#}}(v)} f(u) = \sum_{u \notin N_D(v)} f(u) = \frac{n(n+1)}{2} - \sum_{u \in N_D(v)} f(u) = \frac{n(n+1)}{2} - c$. Thus G is $D^{\#}$-vertex magic. Since the set D was arbitrary, this suffices to prove the converse as well. $\square$

Theorem 3.2. *Let $D \subset \{0, 1, \ldots, d\}$. If G is (D, r) regular and A_D^{-1} exists, then G is not D-vertex magic.*

Proof. Notice that if G is D-vertex magic, then there exists a vector $\overrightarrow{x} \in \mathbb{R}^n$ where $\overrightarrow{x}$ is a permutation of the vector $[1, 2, \ldots, n]^T$, and a constant c, such that $A_D \overrightarrow{x} = c\overrightarrow{1}$ where $\overrightarrow{1} \in \mathbb{R}^n$ is an all ones vector. Since G is (D, r) regular, the vector $\overrightarrow{y} \in \mathbb{R}^n$ where each element of $\overrightarrow{y}$ is c/r is such that $A_D \overrightarrow{y} = c\overrightarrow{1}$. If A_D^{-1} exists, then $\overrightarrow{y}$ is the unique solution to $A_D \overrightarrow{x} = c\overrightarrow{1}$, and hence there cannot exist a solution $\overrightarrow{x}$ that is a permutation of the vector $[1, 2, \ldots, n]^T$. Hence G is not D-vertex magic.

Corollary 3.3. *Let $G = (V, E)$ be any regular graph. If A^{-1} exists, then G is not $\{1\}$-vertex magic, that is, G is not Σ labeled.*

Corollary 3.4. *Let $G = (V, E)$ be any regular graph. For the closed neighborhood matrix N, if N^{-1} exists, then G is not $\{0, 1\}$-vertex magic, that is, G is not Σ' labeled.*

In Miller et al. [2] it was proved that there does not exist a $\{1\}$-vertex r-regular graph for odd r. In the next result we extend this idea to arbitrary neighborhood sets.

Theorem 3.5. *Let $G = (V, E)$ have even order n. Let $D \subset \{0, 1, \ldots, d\}$. If G is (D, r) regular with r odd, then G is not D-vertex magic.*

Proof. If G is D-vertex magic, then there exists a bijection $f : V(G) \to [n]$ such that for every $v \in V(G)$, $f(N_D(v)) = \sum_{u \in N_D(v)} f(u)$ is the same. Since every $u \in V(G)$ is a summand for $|N_D(u)| = r$ vertices, $f(N_D(v)) = \frac{r}{n} \sum_{u \in V(G)} f(u) = \frac{n(n+1)}{2} \times \frac{r}{n} = \frac{(n+1)r}{2}$. Since $n + 1$ and r are both odd, this sum is not an integer, which is a contradiction. Hence, no such bijection f exists, and therefore, G is not D-vertex magic. $\square$

Corollary 3.6. *If G is (D, r) regular with r odd, then G is not D-vertex magic.*

Proof. Since $0 \notin D$, all the elements on the diagonal of A_D are 0. Since A_D is symmetric, there are an even number of non-zero entries in A_D, thus nr is even.

Since r is odd, A_D has an odd number r of 1's in each row, and we must have n even. Thus the result follows from Theorem 3.5. □

Corollary 3.7. *There does not exist a graph of even order that is both* $\{1\}$*-vertex magic and* $\{0,1\}$*-vertex magic. That is, there does not exist a graph of even order that is both* Σ *labeled and* $\Sigma^{'}$ *labeled.*

We will now use these theoretical results to provide solutions to problems that were posed at the 2010 IWOGL Conference. For each of the following examples, the question was posed whether there existed $\{d\}$-vertex magic labelings of the graphs, where d is the diameter of the graph.

Example 3.8. *Consider the graph* $G1$ *in Figure 2(a) which has order 8 and diameter 2.* $G1$ *is regular of degree 3. Let* $f : V(G1) \to [8]$ *be any bijection. Since* $f(N(v_1)) = f(v_2) + f(v_4) + f(v_6) \neq f(v_2) + f(v_4) + f(v_7) = f(N(v_8))$*,* $G1$ *is not* $\{1\}$*-vertex magic, or equivalently,* $G1$ *is not* Σ *labeled. By Theorem 3.1,* $G1$ *is not* $\{0,2\}$*-vertex magic. Since* $f(N[v_2]) = f(v_1) + f(v_2) + f(v_3) + f(v_8) \neq f(v_1) + f(v_3) + f(v_4) + f(v_8) = f(N[v_4])$*,* $G1$ *is not* $\{0,1\}$*-vertex magic, or equivalently,* $G1$ *is not* $\Sigma^{'}-$*labeled. Since* $G1$ *is not* $\{0,1\}$*-vertex magic, by Theorem 3.1,* $G1$ *is not* $\{2\}$*-vertex magic.*

Making use of Theorem 3.2, we could also consider the closed neighborhood matrix N *and notice that* $det(N) = -16 \neq 0$*. Hence* $G1$ *is not* $\{0,1\}$*-vertex magic. By Theorem 3.1 it follows that* $G1$ *is not* $\{2\}$*-vertex magic. The adjacency matrix for* $G1$ *is singular, so we cannot apply Theorem 3.2 in that case. However since* $G1$ *has even order and is* $(\{1\},3)$ *regular, we can apply Theorem 3.5 to conclude that* $G1$ *is not* $\{1\}$*-vertex magic, and then apply Theorem 3.1 to conclude that* $G1$ *is not* $\{0,2\}$*-vertex magic.*

Example 3.9. *Consider the Petersen Graph* P *in Figure 2(b). The order of* P *is 10, the diameter of* P *is 2, and* P *is 3-regular.* $det(A) = 48 \neq 0$ *where* A *is the adjacency matrix for* P*. Hence* P *is not* $\{1\}$*-vertex magic by Theorem 3.2, and by Theorem 3.1,* P *is not* $\{0,2\}$*-vertex magic. We could have also used Theorem 3.5 to conclude that* P *was not* $\{1\}$*-vertex magic.*

We have $det(N) = 128 \neq 0$ *where* N *is the closed neighborhood matrix for* P*. So by Theorem 3.2,* P *is not* $\{0,1\}$*-vertex magic, and then by Theorem 3.1,* P *is not* $\{2\}$*-vertex magic.*

Example 3.10. *Consider the graph* $G2$ *in Figure 3(a) which has order 8 and diameter 2.* $G2$ *is regular of degree 3. Let* $f : V(G2) \to [8]$ *be any bijection. If* $G2$ *is* $\{1\}$*-vertex magic, then* $f(N(v_3)) = f(N(v_6))$*. However,* $f(N(v_3)) = f(v_2) + f(v_4) + f(v_7) \neq f(v_2) + f(v_5) + f(v_7) = f(N(v_6))$*. Hence,* $G2$ *is not* $\{1\}$*-vertex magic. Then by Theorem 3.1,* $G2$ *is not* $\{0,2\}$*-vertex magic.*

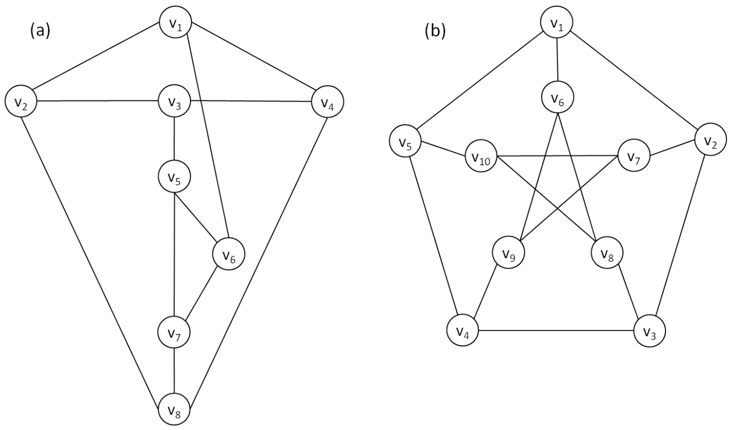

Figure 2. 3-Regular graph $G1$ and Petersen graph P

If $G2$ is $\{0,1\}$-vertex magic, then by O'Neal and Slater [3] *we know that each closed neighborhood sum must equal 18. Hence $f(N[v_8]) = f(v_1) + f(v_4) + f(v_7) + f(v_8) = 18$ and $f(N[v_2]) = f(v_1) + f(v_2) + f(v_3) + f(v_6) = 18$. It follows that $2f(v_1)+f(v_2)+f(v_3)+f(v_4)+f(v_6)+f(v_7)+f(v_8) = 36$. But we also have that $f(v_1) + \cdots + f(v_8) = 36$. Subtracting we get that for $G2$ to be $\{0,1\}$-vertex magic, we must have $f(v_1) - f(v_5) = 0$, which would be a contradiction. Hence, $G2$ is not $\{0,1\}$-vertex magic. From Theorem 3.1 it follows that $G2$ is not $\{2\}$-vertex magic.*

Making use of Theorem 3.2, we could also consider the adjacency matrix A and notice that $det(A) = -3 \neq 0$. Hence $G2$ is not $\{1\}$-vertex magic, and by Theorem 3.1, $G2$ is not $\{0,2\}$-vertex magic. Alternatively, since $G2$ has even order 8 and is $(\{1\},3)$ regular, by Theorem 3.5, $G2$ is not $\{1\}$-vertex magic. The closed neighborhood matrix N for $G2$ is singular, so we cannot apply Theorem 3.1 to conclude that $G2$ is not $\{0,1\}$-vertex magic.

Proposition 3.11. *Let M_n be a Mobius Ladder of order $2n$ with $n > 2$. M_n is neither $\{1\}$-vertex magic nor $\{0,1\}$-vertex magic.*

Proof. Notice that if $n > 2$, then there exists vertices $u, v \in V(M_n)$ such that $|N[u] \triangle N[v]| = 1$ *or* 2, where $\triangle$ denotes symmetric difference. Hence, by Theorem 9 Beena [1], M_n is not $\{0,1\}$-vertex magic. Since M_n has even order and is regular of degree 3, by Theorem 3.5, M_n is not $\{1\}$-vertex magic. Note that if $n = 2$, then $M_2 = K_4$ and hence M_2 is $\{0,1\}$-vertex magic but not $\{1\}$-vertex magic. $\square$

Example 3.12. *Consider the graph $G3$ in Figure 3(b) which has order 15 and diameter 2. $G3$ is regular of degree 4. $det(A) = 1280 \neq 0$ so by Theorem 3.2 $G3$ is not $\{1\}$-vertex magic, and thus by Theorem 3.1, $G3$ is not $\{0,2\}$-vertex magic. $det(N) = 6400 \neq 0$ where N is the closed neighborhood matrix of $G3$, and so by*

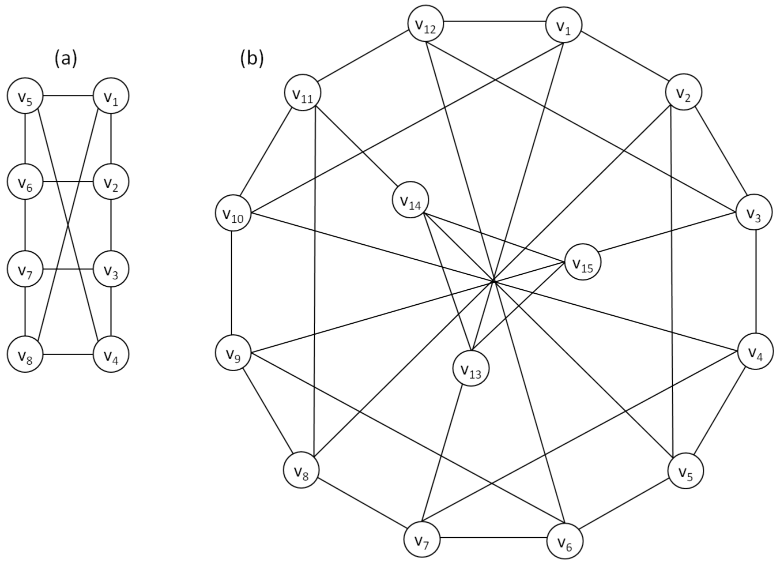

FIGURE 3. Mobius ladder $G2$ and graph $G3$

Theorem 3.2, $G3$ *is not* $\{0, 1\}$*-vertex magic. Hence by Theorem 3.1,* $G3$ *is not* $\{2\}$*-vertex magic.*

Example 3.13. *Consider the graph G*4 *in Figure 4 which has order 24 and diameter 2. G*4 *is regular of degree 5.* $det(A) = 6, 298, 560 \neq 0$ *and so by Theorem 3.2, G*4 *is not* $\{1\}$*-vertex magic, and thus by Theorem 3.1, G*4 *is not* $\{0, 2\}$*-vertex magic. We could have also used Theorem 3.5 to conclude that G*4 *was not* $\{1\}$*-vertex magic.* $det(N) = 34, 012, 224 \neq 0$ *where N is the closed neighborhood matrix of G*4*, and so by Theorem 3.2, G*4 *is not* $\{0, 1\}$*-vertex magic. Hence by Theorem 3.1, G*4 *is not* $\{2\}$*-vertex magic.*

Example 3.14. *Consider the graph G*5 *in Figure 5 which has order 20 and diameter 3. G*5 *is regular of degree 3.* $det(A) = 12 \neq 0$ *and so by Theorem 3.1, G*5 *is not* $\{1\}$*-vertex magic, and thus by Theorem 3.1, G*5 *is not* $\{0, 2, 3\}$*-vertex magic. We could also apply Theorem 3.5 to achieve the same result. Similarly* $det(A_{\{1,2\}}) = 20, 736 \neq 0$ *and we can conclude that G*5 *is neither* $\{1, 2\}$*-vertex magic nor* $\{0, 3\}$*-vertex magic. Again Theorem 3.5 could be used to achieve this same result.* $det(A_{\{0,2\}}) = -47, 068 \neq 0$ *so G*5 *is neither* $\{0, 2\}$*-vertex magic nor* $\{1, 3\}$*-vertex magic.* $A_{\{0,1\}}$ *and* $A_{\{0,1,2\}}$ *are both singular.*

We do have that $rank(A_{\{0,1\}}) = 19$ *and so* $dim(N(A_{\{0,1\}})) = 1$*. Further* $y = [-1, 3, -1, -1, -1, 3, -1, -1, -1, 3, -1, -1, -1, 3, -1, -1, -1, 3, -1, -1]^T$ *is a basis vector for* $N(A_{\{0,1\}})$*. Since* $z = (\frac{c}{4})\overrightarrow{1}$ *is a solution to the non-homogeneous*

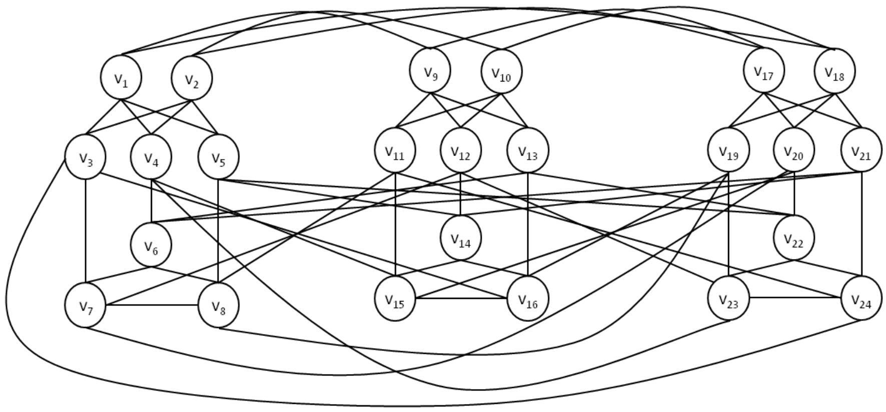

FIGURE 4. 5-Regular graph $G4$ of order 24 and diameter 2

system $A_{\{0,1\}}x = c\overrightarrow{1}$, *any solution to the non-homogeneous system must equal* $ay + z$ *for some constant* a. *Since no value of* a *produces a solution that is a permutation of* $[1, 2, \ldots, 20]^T$, *we can conclude that* $G5$ *is not* $\{0, 1\}$*-vertex magic, and thus by Theorem 3.1,* $G5$ *is not* $\{2, 3\}$*-vertex magic.*

We also have that $rank(A_{\{0,1,2\}}) = 19$. *Using the null space basis vector* $y = [0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1]^T$ *and the same logic, we can conclude that* $G5$ *is not* $\{0, 1, 2\}$*-vertex magic, and hence,* $G5$ *is not* $\{3\}$*-vertex magic.*

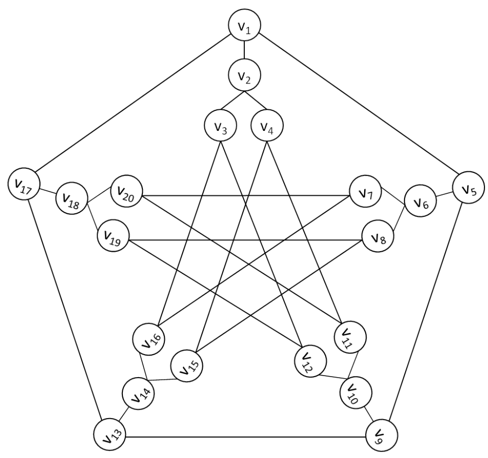

FIGURE 5. 3-Regular graph $G5$ of order 20 and diameter 3

4. **General Results**

In Section 2 we introduced terminology that allows us to make a statement about how close any graph is to being vertex magic. In this section we provide some basic results that build upon the terminology introduced.

Theorem 4.1. *Let* $D \subset \{0,1,\ldots,d\}$ *and* $D^{\#} = \{0,1,\ldots,d\} - D$. *Let* $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ *be a set (or, more generally, a multiset). Then*
(i) For a bijection $f : V(G) \to W$ *and a vertex* $v \in V(G)$, $NS(f;D) = f(N_D(v))$ *if and only if* $NS^{-}(f;D^{\#}) = f(N_{D^{\#}}(v))$;
(ii) For a bijection $f : V(G) \to W$, $NS_W(G;D) = NS(f;D)$ *if and only if* $NS^{-}_W(G;D^{\#}) = NS^{-}(f;D^{\#})$;
(iii) $NS_W(G;D) + NS^{-}_W(G;D^{\#}) = \sum\limits_{i=1}^{n} w_i$; *and*
(iv) $NS^{sp}_W(G;D) = NS^{sp}_W(G;D^{\#})$.

Proof. First note that for any $v \in V(G)$ we have $V(G) = N_D(v) \cup N_{D^{\#}}(v)$ and $N_D(v) \cap N_{D^{\#}}(v) = \varnothing$. Hence for any bijection $f : V(G) \to W$ and for any $u \in V(G)$ it follows that $f(N_D(u)) + f(N_{D^{\#}}(u)) = \sum\limits_{i=1}^{n} w_i$. So if $v \in V(G)$ is such that $NS(f;D) = f(N_D(v)) \geq f(N_D(u))$ for all $u \in V(G)$, then $f(N_{D^{\#}}(u)) = \sum\limits_{i=1}^{n} w_i - f(N_D(u)) \geq \sum\limits_{i=1}^{n} w_i - f(N_D(v)) = f(N_{D^{\#}}(v))$ for all $u \in V(G)$. Therefore, $NS^{-}(f;D^{\#}) = f(N_{D^{\#}}(v))$. The converse can be proved in a similar fashion.

Let $f : V(G) \to W$ be a bijection such that $NS_W(G;D) = NS(f;D)$. Hence $NS(f;D) \leq NS(g;D)$ for all bijections $g : V(G) \to W$, and there exists a $v \in V(G)$ such that $NS(f;D) = f(N_D(v)) \geq f(N_D(u))$ for all $u \in V(G)$. Let $g : V(G) \to W$ be an arbitrary bijection and let $u \in V(G)$ be such that $NS^{-}(g;D^{\#}) = g(N_{D^{\#}}(u))$. Then $NS^{-}(g;D^{\#}) = g(N_{D^{\#}}(u)) = \sum\limits_{i=1}^{n} w_i - g(N_D(u)) = \sum\limits_{i=1}^{n} w_i - NS(g;D) \leq \sum\limits_{i=1}^{n} w_i - NS(f;D) = \sum\limits_{i=1}^{n} w_i - f(N_D(v)) = f(N_{D^{\#}}(v))$. Therefore, $NS^{-}_W(G;D^{\#}) = NS^{-}(f;D^{\#})$. The converse can be proved in a similar fashion.

Next let $f : V(G) \to W$ be a bijection such that $NS_W(G;D) = NS(f;D)$ and let $v \in V(G)$ be such that $NS(f;D) = f(N_D(v))$. By parts 1 and 2 it follows that $NS^{-}_W(G;D^{\#}) = NS^{-}(f;D^{\#}) = f(N_{D^{\#}}(v))$. Hence $NS_W(G;D) + NS^{-}_W(G;D^{\#}) = f(N_D(v)) + f(N_{D^{\#}}(v)) = \sum\limits_{i=1}^{n} w_i$.

Finally, if we let $g : V(G) \to W$ be any bijection, and let $u, v \in V(G)$ be such that $NS(g;D) = g(N_D(u))$ and $NS^{-}(g;D) = g(N_D(v))$. Then by definition we have $NS^{sp}(g;D) = g(N_D(u)) - g(N_D(v))$. By part 1 we get that $NS(g;D^{\#}) = g(N_{D^{\#}}(v))$ and $NS^{-}(g;D^{\#}) = g(N_{D^{\#}}(u))$. Hence $NS^{sp}(g;D^{\#}) = NS(g;D^{\#}) - NS^{-}(g;D^{\#}) = g(N_{D^{\#}}(v)) - g(N_{D^{\#}}(u)) = \sum\limits_{i=1}^{n} w_i - g(N_D(v)) -$

$\sum_{i=1}^{n} w_i + g(N_D(u)) = NS^{sp}(g;D)$. Consider the bijection $f : V(G) \to W$ where $NS^{sp}_W(G;D) = NS^{sp}(f;D)$. Since the relationship above holds for the arbitrary bijection g, we mush have that $NS^{sp}_W(G;D) = NS^{sp}(f;D) = NS^{sp}(f;D^{\#}) = NS_W(G;D^{\#})$. The last equality must hold, for if there exists a bijection $h : V(G) \to W$ such that $NS^{sp}(h;D^{\#}) < NS^{sp}(f;D^{\#})$, then $NS^{sp}(h;D) < NS^{sp}(f;D)$, but this would be a contradiction. □

Corollary 4.2. *Let $D \subset \{0,1,\ldots,d\}$ and $D^{\#} = \{0,1,\ldots,d\} - D$. Then*

(i) For a bijection $f : V(G) \to [n]$ and a vertex $v \in V(G)$, $NS(f;D) = f(N_D(v))$ if and only if $NS^-(f;D^{\#}) = f(N_{D^{\#}}(v))$;

(ii) For a bijection $f : V(G) \to [n]$, $NS(G;D) = NS(f;D)$ if and only if $NS^-(G;D^{\#}) = NS^-(f;D^{\#})$;

(iii) $NS(G;D) + NS^-(G;D^{\#}) = \frac{n(n+1)}{2}$; and

(iv) $NS^{sp}(G;D) = NS^{sp}(G;D^{\#})$.

Example 4.3. *For the graph H from Figure 1(a), we showed in Example 2.7 that $NS[H] = 11$, and we showed in Example 2.10 that $NS^-(H;\{2\}) = 4$. Notice that $NS[H] + NS^-(H;\{2\}) = \frac{5(6)}{2} = 15$. In Example 2.10 we showed that $NS^-[H] = 9$, and in Example 2.7 we showed that $NS(H;\{2\}) = 6$. Notice that $NS^-[H] + NS(H;\{2\}) = 15$. In Example 2.14 we showed that $NS^{sp}[H] = 4$ and that $NS^{sp}(H;\{2\}) = 4$.*

Example 4.4. *For the graph H from Figure 1(a), we showed in Example 2.7 that $NS(H) = 8$, in Example 2.10 that $NS^-(H) = 7$, and in Example 2.14 that $NS^{sp}(H) = 1$. Using Corollary 4.2 we conclude that $NS(H;\{0,2\}) = 8$, $NS^-(H;\{0,2\}) = 7$, and that $NS^{sp}(H;\{0,2\}) = 1$.*

The following corollary was proven as Theorem 1 by O'Neal and Slater [3] and is included here for completeness.

Corollary 4.5. *Let G^C be the complement of G. Let $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ be a set (or, more generally, a multiset). Then*

(i) $NS_W(G;\{0,1\}) + NS^-_W(G^C;\{1\}) = \sum_{i=1}^{n} w_i = NS_W(G;\{1\}) + NS^-_W(G^C;\{0,1\})$, and

(ii) $NS^{sp}_W(G;\{0,1\}) = NS^{sp}_W(G^C;\{1\})$.

Proof. Notice that if we take $D = \{0,1\}$ and $D^{\#} = \{2,3,\ldots\}$, then for every $v \in V(G)$, $N_{D^{\#}}(v)$ in G contains exactly the same set of vertices as the $N_{\{1\}}(v)$ in G^C. □

Corollary 4.6. *Let G^C be the complement of G. Then*
(i) $NS(G;\{0,1\})+NS^-(G^C;\{1\}) = \frac{n(n+1)}{2} = NS(G;\{1\})+NS^-(G^C;\{0,1\})$, and
(ii) $NS^{sp}(G;\{0,1\}) = NS^{sp}(G^C;\{1\})$.

Example 4.7. *Consider again the graph H from Figure 1(a). The complement of H is the path P_5. Using Corollary 4.6 and the results from previous examples we can make the following observations:*

(i) $NS[P_5] = 15 - NS^-(H) = 8$.
(ii) $NS^-[P_5] = 15 - NS(H) = 7$.
(iii) $NS^{sp}[P_5] = NS^{sp}(H) = 1$.
(iv) $NS(P_5) = 15 - NS^-[H] = 6$.
(v) $NS^-(P_5) = 15 - NS[H] = 4$.
(vi) $NS^{sp}(P_5) = NS^{sp}(H) = 4$.

Example 4.8. *Consider the graph $G6$ in Figure 6 whose complement is C_6. From O'Neal and Slater [4] we know that $NS(C_6) = 9$, $NS^-(C_6) = 5$, and $NS^{sp}(C_6) = 4$. One can also show that $NS[C_6] = 11$, $NS^-[C_6] = 10$, and $NS^{sp}[C_6] = 1$. Using Corollary 4.6 we can conclude that $NS(G6) = 11$, $NS^-(G6) = 10$, $NS^{sp}(G6) = 1$, $NS[G6] = 16$, $NS^-[G6] = 12$, and $NS^{sp}[G6] = 4$. The labeling for the open neighborhood case is shown in Figure 6(a) and the labeling for the closed neighborhood case is shown in Figure 6(b).*

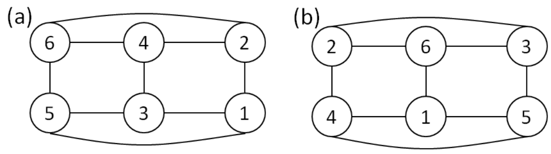

FIGURE 6. Labelings of $G6$ showing $NS(G6) = 11$, $NS^-(G6) = 10$, $NS^{sp}(G6) = 1$, $NS[G6] = 16$, $NS^-[G6] = 12$, $NS^{sp}[G6] = 4$

Corollary 4.9. *A graph G is Σ labeled if and only if its complement G^C is Σ' labeled.*

Example 4.10. *From O'Neal and Slater [4] we know that a 2-regular graph is Σ labeled if and only if it is the union of 4 cycles. Figure 7(a) shows a Σ labeling of the union of two 4 cycles. Figure 7(b) shows the Σ' labeling of the graph's complement.*

Corollary 4.11. *Define the graph $H = (V,E)$ by $V(H) = V(G)$ and for all $u, v \in V(G)$ let $uv \in E(H)$ if and only if $d(u,v) \in D$. Then:*
(i) $NS(G;\{0\} \cup D) = NS[H]$,
(ii) $NS^-(G;\{0\} \cup D) = NS^-[H]$.

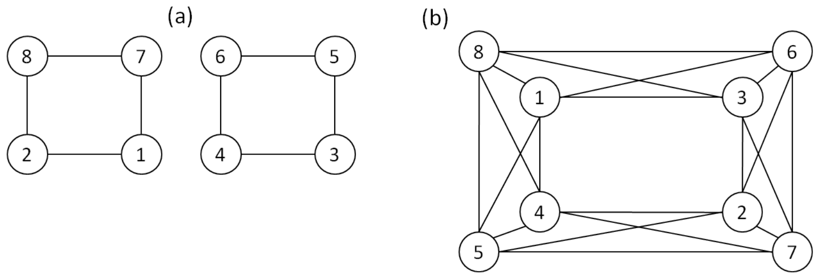

FIGURE 7. Σ labeling of the union of 2 C_4's, and $\Sigma^{'}$ labeling of its complement

(iii) $NS^{sp}(G; \{0\} \cup D) = NS^{sp}[H]$.
(iv) $NS(G; D) = NS(H)$.
(v) $NS^{-}(G; D) = NS^{-}(H)$.
(vi) $NS^{sp}(G; D) = NS^{sp}(H)$.

PROOF. Notice that for any choice of D, for all $v \in V(G)$, the $N_D(v)$ in G contains exactly the same set of vertices as the $N_{\{1\}}(v)$ in H. $\square$

Example 4.12. *Consider the 6 cycle C_6 shown in Figure 8(b). C_6 has diameter 3. Take $D = \{3\}$ and then form the graph $G7$ which is shown in Figure 8(a). The labeling shown in Figure 8(a) demonstrates that $G7$ is $\Sigma^{'}$ labeled, hence $NS[G7] = 7$, $NS^{-}[G7] = 7$, and $NS^{sp}[G7] = 0$. Making use of Corollary 4.11 we can conclude that $NS(C_6; \{0, 3\}) = 7$, $NS^{-}(C_6; \{0, 3\}) = 7$, and $NS^{sp}(C_6; \{0, 3\}) = 0$. That is, C_6 is $\{0, 3\}$-vertex magic. From Theorem 3.1 we can also conclude that C_6 is $\{1, 2\}$-vertex magic.*

Notice that any bijection $f : V(G7) \to [6]$ will be such that $NS(f) = 6$, $NS^{-}(f) = 1$, $NS^{sp}(f) = 5$. Hence, $NS(G7) = 6$, $NS^{-}(G7) = 1$ and $NS^{sp}(G7) = 5$. Making use of Corollary 4.11 we can conclude that $NS(C_6; \{3\}) = 6$, $NS^{-}(C_6; \{3\}) = 1$, and $NS(C_6; \{3\}) = 5$. From Corollary 4.2 we can also conclude that $NS(C_6; \{0, 1, 2\}) = 20$, $NS^{-}(C_6; \{0, 1, 2\}) = 15$, and $NS(C_6; \{0, 1, 2\}) = 5$.

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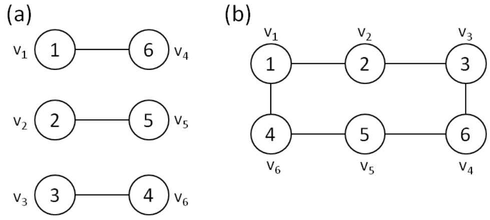

FIGURE 8. Graph $G7$ and C_6 labeled to show a $\{0, 3\}$ and $\{1, 2\}$-vertex magic labeling of C_6

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