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EIGENVALUES VARIATION OF THE P-LAPLACIAN UNDER THE RICCI FLOW ON SM

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Abstract. Let (M, F) be a compact Finsler manifold. Studying the eigenvalues and eigenfunctions for the linear and nonlinear geometric operators is a known problem. In this paper we will consider the eigenvalue problem for the p-laplace operator for Sasakian metric acting on the space of functions on SM. We find the first variation formula for the eigenvalues of p-Laplacian on SM evolving by the Ricci flow on M and give some examples.

Key words and Phrases: Ricci flow, Finsler manifold, p-Laplace operator

Abstrak. Misalkan (M, F) adalah suatu manifold Finsler kompak. Sejauh ini telah dipelajari fungsi eigen dan nilai eigen untuk operator-operator geometri linier dan non linier. Dalam paper ini kami akan memperhatikan masalah nilai eigen untuk operator p-Laplace untuk metrik Sasakian yang berlaku pada ruang fungsi di SM. Kami memperoleh rumus variasi pertama dan memberikan beberapa contoh untuk nilai eigen dari p-Laplacian pada SM yang melibatkan aliran Ricci pada M .

Kata kunci: aliran Ricci, manifold Finsler, operator p-Laplace

1. Introduction

For a compact Finsler manifold (M, F), studying the eigenvalues of geometric operators plays a powerful role in geometric analysis. In the classical theory of the Laplace or p-Laplace equation several main parts of mathematics are joined in a fruitful way: Calculus of Variation, Partial Differential Equation, Potential Theory,

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Analytic Function. Recently, there are many mathematicians who have investigated properties of the eigenvalues of p-Laplacian on Finsler manifolds and Riemannian manifolds to estimate the spectrum in terms of the other geometric quantities of the manifold. (see [3, 4, 9, 11, 18, 20]).

Also, geometric flows have been a topic of active research interest in mathematics and other sciences (see [5, 8, 10, 12, 13, 15, 16]). Hamilton's Ricci flow ([6]) is the best known example of a geometric evolution equation. The Ricci flow is related to dynamical systems in the infinite-dimensional space of all metrics on a given manifold. One of the aims of such flows is to obtain metrics with special properties. Special cases arise when the metric is invariant under a group of transformations and this property is preserved by the flow.
Let M be a manifold with a Finsler metric g_0 (or F_0), the family $g(t)$ (or F_t) of Finsler metrics on M is called an un-normalized Ricci flow when it satisfies the equations

$$\frac{\partial logF}{\partial t} = -\mathcal{R}ic, \tag{1}$$

with the initial condition

$$F(0) = F_0$$

or equivalently satisfies the equations

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{ij},\ g(0) = g_0 \tag{2}$$

where Ric is the Ricci tensor of $g(t)$, $Ric_{ij} = (\frac{1}{2}F^2\mathcal{R}ic)_{y^iy^j}$. In fact Ricci flow is a system of partial differential equations of parabolic type which was introduced by Hamilton on Riemannian manifolds for the first time in 1982 and Bao (see [2, 17]) studied Ricci flow equation in Finsler manifold. The Ricci flow has been proved to be a very useful tool to improve metrics in Finsler geometry, when M is compact. One often considers the normalized Ricci flow

$$\frac{\partial logF}{\partial t} = -\mathcal{R}ic + \frac{1}{vol(SM)}\int_{SM} \mathcal{R}ic\, dv,\ F(0) = F_0. \tag{3}$$

or

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{ij} + \frac{2}{vol(SM)}\int_{SM} \mathcal{R}ic\, dv g_{ij},\ g(0) = g_0 \tag{4}$$

Under this normalized flow, the volume of the solution metrics remains constant in time. Short time exitance and uniqueness for solution to the Ricci flow on $[0, T)$ have been shown by Hamilton in [5] and by DeTurk in [7] for Riemannian manifolds and by the authors in [1] for important Berwald manifolds.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. For a point $x \in M$, denote by T_xM the tangent space at $x \in M$,and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. Any element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$.

Definition 2.1. *A Finsler metric on a manifold M is a function $F : TM_0 \to [0, \infty)$ which has the following properties:*

(i): $F(x, \alpha y) = \alpha F(x, y),\ \forall \alpha > 0$;
(ii): $F(x, y)$ *is* C^∞ *on* TM_0 ;
(iii): *For any non-zero tangent vector $y \in T_xM$, the associated quadratic form $g_y : T_xM \times T_xM \to \mathbb{R}$ on TM is an inner product, where*

$$g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial r}\left[F^2(x, y + su + rv)\right]\Big|_{s=r=0}.$$

The pair (M, F) is called a Finsler manifold.

Let us denote by S_xM the set consisting of all rays $[y] := \{\lambda y|\lambda > 0\}$, where $y \in T_xM_0$. The Sphere bundle of M, i.e. SM, is the union of S_xM's :

$$SM = \underset{x}{\cup} S_xM$$

SM has a natural $(2n-1)$-dimensional manifold structure. We denote the elements of SM by $(x, [y])$ where $y \in T_xM_0$. If there is not any confusion we write (x, y) for $(x, [y])$. In a local coordinate system (x^i, y^i) we have $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$ and $(g^{ij}) := (g_{ij})^{-1}$. The geodesics of F are characterized locally by

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0$$

where

$$G^i = \frac{1}{4}g^{il}\left\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k. \tag{5}$$

Definition 2.2. *The coefficients of the Riemann curvature $R_y = R^i_k dx^i \otimes \frac{\partial}{\partial x^i}$ are given by*

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k} \tag{6}$$

and $R^i{}_{jk} := \frac{1}{3}\left(\frac{\partial R^i{}_j}{\partial y^k} - \frac{\partial R^i{}_k}{\partial y^j}\right)$, $R^{\ i}_{j\ kl} := \frac{1}{3}\left(\frac{\partial^2 R^i{}_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i{}_l}{\partial y^j \partial y^k}\right)$.

The Ricci scalar function of F is given by $\mathcal{R}ic := \frac{1}{F^2}R^i_i$. A companion of the Ricci scalar is the Ricci tensor

$$Ric_{ij} := \left(\frac{1}{2}F^2\mathcal{R}ic\right)_{y^iy^j}. \tag{7}$$

Definition 2.3. *A Finsler metric is said to be an Einstain metric if the Ricci scalar function is a function of x alone,equivalently $Ric_{ij} = \mathcal{R}(x)g_{ij}$ (see* [14, 19]*).*

Definition 2.4. *Let (M, F) be a Finsler manifold, the Sasakian metric $\widetilde{g}$ of g on TM_0 is defined as*

$$\widetilde{g} = g_{ij}dx^i \otimes dx^j + g_{ij}\frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F} \tag{8}$$

then $\widetilde{g}$ is a Riemannian metric on TM_0 and $\{\frac{\delta}{\delta x^i}, F\frac{\partial}{\partial y^i}\}$ is a coordinate base on TM_0, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j\frac{\partial}{\partial y^j}$ and $\{dx^i, \frac{\delta y^i}{F}\}$ is the dual of $\{\frac{\delta}{\delta x^i}, F\frac{\partial}{\partial y^i}\}$ where $\delta y^i = dy^i + G_j^i dx^j$.

REMARK. The Levi-Civita connection $\widetilde{\nabla}$ on TM_0 with respect to the Sasakian metric $\widetilde{g}$ is locally expressed as follows:

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} &= -(C_{i\,j}^k + \frac{1}{2}R_{ij}^k)\frac{\partial}{\partial y^k} + F_{i\,j}^k\frac{\delta}{\delta x^k} \\
\widetilde{\nabla}_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} &= C_{i\,j}^k\frac{\partial}{\partial y^k} - g_{ih}(F_{j\,k}^h - G_{j\,k}^h)g^{hk}\frac{\delta}{\delta x^k} \\
\widetilde{\nabla}_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i} &= F_{i\,j}^k\frac{\partial}{\partial y^k} + (C_{i\,j}^k + \frac{1}{2}g_{ih}R_{lj}^h g^{lk})\frac{\delta}{\delta x^k} \\
&= \widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} + G_{i\,j}^k\frac{\partial}{\partial y^k},
\end{aligned} \tag{9}$$

where

$$C_{i\,j}^k = \frac{1}{2}g^{kh}\frac{\partial g_{ij}}{\partial y^h},\;\; F_{i\,j}^k = \frac{1}{2}g^{kh}(\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}),\;\; G_{i\,j}^k = \frac{\partial G_j^k}{\partial y^i},$$

and

$$R_{ij}^k = \frac{\delta G_i^k}{\delta x^j} - \frac{\delta G_j^k}{\delta x^i},\;\; [\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] = R_{ij}^k\frac{\partial}{\partial y^k},\;\; [\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}] = G_{i\,j}^k\frac{\partial}{\partial y^k}.$$

Lemma 2.5. *For a Sasakian metric $\widetilde{g}$ and any $f : TM \to \mathbb{R}$, there exists a unique vector field $Y \in \mathcal{X}(TM)$ such that*

$$\widetilde{g}(Y, \widetilde{X}) = df(\widetilde{X}),\;\; \forall\widetilde{X} \in \mathcal{X}(TM) \tag{10}$$

where

$$\widetilde{X} = X_1^i\frac{\delta}{\delta x^i} + X_2^i F\frac{\partial}{\partial y^i}$$

and X_1^i, X_2^i are C^∞ function on TM. Here we take $Y = 0$ if $df = 0$.

Denote the vector field Y in (10) by $\tilde{\nabla} f$. We call $\tilde{\nabla} f$ the gradient of f and define the divergence $div\tilde{X}$ as follows:

$$div\tilde{X} = tr\tilde{\nabla}\tilde{X}$$

Definition 2.6. *According to the above definition, the gradient of a function f is*

$$\tilde{\nabla} f = g^{ij}\frac{\delta f}{\delta x^i}\frac{\delta}{\delta x^j} + F^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^j} \tag{11}$$

therefore, the norm of $\tilde{\nabla} f$ with respect to the Riemannian metric $\widetilde{g}$ is given by

$$|\tilde{\nabla} f|^2 = \widetilde{g}(\tilde{\nabla} f, \tilde{\nabla} f) = g^{ij}\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + F^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}. \tag{12}$$

Definition 2.7. *Let M be a compact Finsler manifold. The Laplace operator of f on TM is defined as follows:*

$$\begin{aligned}\Delta f &= g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial G^r_j}{\partial x^i}\frac{\partial f}{\partial y^r} - G^r_j\frac{\partial^2 f}{\partial x^i\partial y^r} - G^s_i\frac{\partial^2 f}{\partial y^s\partial x^j} + G^r_j G^s_i \frac{\partial^2 f}{\partial y^s\partial y^r}\right)\\ &+ g^{ij}F^2\frac{\partial^2 f}{\partial y^i\partial y^j} + g^{ij}(C^{\,k}_{i\,j} + \frac{1}{2}R^k_{ij})\frac{\delta f}{\delta x^k} - F g^{ij}F^{\,k}_{i\,j}\frac{\delta f}{\delta x^k} \qquad (13)\\ &- F g^{ij} C^{\,k}_{i\,j}\frac{\partial f}{\partial y^k} - F^2 g^{ij} g_{ih}(G^{\,h}_{j\,l} - F^{\,h}_{j\,l}) g^{lk}\frac{\delta f}{\delta y^k}.\end{aligned}$$

Definition 2.8. *Let M be a compact Finsler manifold. The p-Laplace operator of $f : SM \to \mathbb{R}$, $f \in W^{1,p}(SM)$ for $1 < p < \infty$ is defined as follows:*

$$\begin{aligned}\triangle_p f &= div(|\widetilde{\nabla} f|^{p-2}\widetilde{\nabla} f) \qquad (14)\\ &= |\widetilde{\nabla} f|^{p-2}\Delta f + (p-2)|\widetilde{\nabla} f|^{p-4}(Hessf)(\widetilde{\nabla} f, \widetilde{\nabla} f)\end{aligned}$$

where

$$(Hessf)(X, Y) = \widetilde{\nabla}(\widetilde{\nabla} f)(X, Y) = Y.(X.f) - (\widetilde{\nabla}_Y X).f, \quad X, Y \in \mathcal{X}(SM)$$

and in local coordinate, we have:

$$(Hessf)(\partial_i, \partial_j) = \partial_i\partial_j f - \widetilde{\Gamma}^k_{ij}\partial_k f.$$

NOTE. If f is a function of x alone, or suppose that is the lifting of $f : M \to \mathbb{R}$ then

$$\Delta f = g^{ij}\left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \widetilde{\Gamma}^k_{ij}\frac{\partial f}{\partial x^k}\right) \tag{15}$$

where $\widetilde{\Gamma}^k_{ij}$ is christoffel symbol of $\widetilde{\nabla}$.

2.1. **Eigenvalues of the p-Laplacian.**

Definition 2.9. *Let (M^n, F) be a compact Finsler manifold and $f : SM \to \mathbb{R}$. We say that λ is an eigenvalue of the p-Laplace operator whenever*

$$^{\widetilde{g}}\Delta_p f + \lambda|f|^{p-2}f = 0 \tag{16}$$

then f is said to be the eigenfunction associated to λ, or equivalently they satisfy in

$$\int_{SM}|\widetilde{\nabla} f|^{p-2} < \widetilde{\nabla} f, \widetilde{\nabla}\varphi > dv = \lambda\int_{SM}|f|^{p-2}f\varphi dv \qquad \forall\varphi \in W^{1,2}_0(SM) \tag{17}$$

where $W_0^{1,p}(SM)$ *is closure of* $C_0^\infty(SM)$ *in Sobolev* $W^{1,p}(SM)$.

By substitution $\varphi = f$ in (17) we have:

$$\lambda = \frac{\int_{SM} |\widetilde{\nabla} f|^p dv}{\int_{SM} |f|^p dv}. \tag{18}$$

Normalized eigenfunctions are defined as follows:

$$\int_{SM} f|f|^{p-2} dv = 0, \int_{SM} |f|^p dv = 1. \tag{19}$$

Suppose that (M^n, F_t) is a solution of the Ricci flow on the smooth manifold (M^n, F_0) in the interval $[0, T)$ and

$$\lambda(t) = \int_{SM} |\widetilde{\nabla} f(x,y)|^p dv_t \tag{20}$$

defines the evolution of an eigenvalue of P-Laplacian under the variation of F_t whose eigenfunction associated to $\lambda(t)$ is normalized.Suppose that for any metric $g(t)$ on M^n

$$Spec_p(\widetilde{g}) = \{0 = \lambda_0(g) \leq \lambda_1(g) \leq \lambda_2(g) \leq ... \leq \lambda_k(g) \leq ...\}$$

is the spectrum of $\Delta_p = {}^{\widetilde{g}}\Delta_p$. In what follows we assume the existence and C^1-differentiability of the elements $\lambda(t)$ and $f(t)$, under a Ricci flow deformation $g(t)$ of a given initial metric. We prove some propositions about the problem of the spectrum variation under a deformation of the metric given by a Ricci flow equation.

3. Variation of $\lambda(t)$

In this part, we will give some useful evolution formulas for $\lambda(t)$ under the Ricci flow. Let (M^n, F_t), $t \in [0, T)$,be a deformation of Finsler metric F_0. Assume that $\lambda(t)$ is the eigenvalue of Δ_p, $f = f(x, y, t)$ satisfies

$$\Delta_p f + \lambda |f|^{p-2} f = 0$$

and $\int_{SM} |f|^p dv = 1$, using (12), we have:

$$\begin{aligned} \frac{d}{dt}|\widetilde{\nabla} f|^2 &= \frac{\partial}{\partial t}(g^{ij})\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + g^{ij}\frac{\partial}{\partial t}(\frac{\delta f}{\delta x^i})\frac{\delta f}{\delta x^j} \\ &+ g^{ij}\frac{\delta f}{\delta x^i}\frac{\partial}{\partial t}(\frac{\delta f}{\delta x^j}) + \frac{\partial (F^2)}{\partial t} g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j} \\ &+ F^2\frac{\partial}{\partial t}(g^{ij})\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j} + 2F^2 g^{ij}\frac{\partial f'}{\partial y^i}\frac{\partial f}{\partial y^j}. \end{aligned} \tag{21}$$

where

$$\frac{\partial}{\partial t}(g^{ij}) = -g^{il}g^{jk}\frac{\partial}{\partial t}(g_{lk}) \tag{22}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}(\frac{\delta f}{\delta x^i}) &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x^i} - G_i^r\frac{\partial f}{\partial y^r}\right) \\ &= \frac{\partial f'}{\partial x^i} - G_i^r\frac{\partial f'}{\partial y^r} - \frac{\partial}{\partial t}(G_i^r)\frac{\partial f}{\partial y^r} \\ &= \frac{\delta f'}{\delta x^i} - \frac{\partial}{\partial t}(G_i^r)\frac{\partial f}{\partial y^r}\end{aligned} \tag{23}$$

therefore, a substitution of (22) and (23) in (21), implies that:

Proposition 3.1. Let $(M^n, F_t))$ be a deformation of Finsler manifold (M^n, F_0), then

$$\begin{aligned}\frac{d}{dt}|\widetilde{\nabla} f|^p &= \frac{p}{2}|\widetilde{\nabla} f|^{p-2}\left\{-g^{il}g^{jk}\frac{\partial}{\partial t}(g_{lk})\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + 2g^{ij}\frac{\delta f'}{\delta x^i}\frac{\delta f}{\delta x^j}\right. \\ &\left. -2g^{ij}\frac{\partial}{\partial t}(G_i^r)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}\right\} + \frac{p}{2}|\widetilde{\nabla} f|^{p-2}\left\{2F\frac{\partial F}{\partial t}g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}\right. \\ &\left. -F^2 g^{il}g^{jk}\frac{\partial}{\partial t}(g_{lk})\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j} + 2F^2 g^{ij}\frac{\partial f'}{\partial y^i}\frac{\partial f}{\partial y^j}\right\}.\Box\end{aligned}$$

On the other hand we have

$$\frac{d}{dt}(dv) = \left\{g^{ij}\frac{\partial}{\partial t}(g_{ij}) - n\frac{\partial}{\partial t}(logF)\right\}dv. \tag{24}$$

Now, we get the following two integrability conditions:

$$0 = \frac{d}{dt}\int_{SM}|f|^{p-2}f dv = (p-1)\int_{SM}|f|^{p-2}f' dv + \int_{SM}|f|^{p-2}f\frac{d}{dt}dv$$

therefore

$$(p-1)\int_{SM}|f|^{p-2}f' dv = -\int_{SM}|f|^{p-2}f\left\{g^{ij}\frac{\partial}{\partial t}(g_{ij}) - n\frac{\partial}{\partial t}(logF)\right\}dv \tag{25}$$

and

$$0 = \frac{d}{dt}\int_{SM}|f|^p dv = p\int_{SM}ff'|f|^{p-2}dv + \int_{SM}|f|^p\frac{d}{dt}dv$$

which implies

$$p\int_{SM}ff'|f|^{p-2}dv = -\int_{SM}|f|^p\left\{g^{ij}\frac{\partial}{\partial t}(g_{ij}) - n\frac{\partial}{\partial t}(logF)\right\}dv. \tag{26}$$

Now, if we suppose that $g(t)$ is a solution of the un-normalized Ricci flow (1) and (2), then we have:

$$
\begin{aligned}
\frac{d\lambda}{dt} &= \int_{SM}(\frac{d}{dt}|\widetilde{\nabla}f|^p)dv + \int_{SM}|\widetilde{\nabla}f|^p\frac{d}{dt}(dv)\\
&= \frac{p}{2}\int_{SM}\Big\{-g^{il}g^{jk}(-2Ric_{lk})\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j}+2g^{ij}\frac{\delta f'}{\delta x^i}\frac{\delta f}{\delta x^j}\\
&\quad -2g^{ij}\frac{\partial}{\partial t}(G^r_i)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}\Big\}|\widetilde{\nabla}f|^{p-2}dv+\frac{p}{2}\int_{SM}\Big\{2F^2(-\mathcal{R}ic)g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}\\
&\quad -F^2g^{il}g^{jk}(-2Ric_{lk})\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}+2F^2g^{ij}\frac{\partial f'}{\partial y^i}\frac{\partial f}{\partial y^j}\Big\}|\widetilde{\nabla}f|^{p-2}dv\\
&\quad +\int_{SM}|\widetilde{\nabla}f|^p\left\{g^{ij}(-2Ric_{ij})-n(-\mathcal{R}ic)\right\}dv\\
&= p\int_{SM}Ric(\widetilde{\nabla}f,\widetilde{\nabla}f)|\widetilde{\nabla}f|^{p-2}dv+p\int_{SM}\widetilde{g}(\widetilde{\nabla}f',\widetilde{\nabla}f)|\widetilde{\nabla}f|^{p-2}dv\\
&\quad +\int_{SM}|\widetilde{\nabla}f|^p\left\{-2g^{ij}Ric_{ij}+n\mathcal{R}ic\right\}dv\\
&\quad -p\int_{SM}g^{ij}\frac{d}{dt}(G^r_i)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\widetilde{\nabla}f|^{p-2}dv\\
&\quad -p\int_{SM}\mathcal{R}icF^2\,g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla}f|^{p-2}dv
\end{aligned}
$$

where $\frac{d}{dt}(G^r_i)$ is obtained as follows:

$$G^r=\frac{1}{4}g^{rl}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k,\quad G^r_i=\frac{\partial G^r}{\partial y^i}$$

Hence

$$
\begin{aligned}
\frac{d}{dt}(G^r_i) &= -\frac{1}{4}g^{r\alpha}g^{a\beta}\frac{\partial}{\partial t}(g_{\alpha\beta})g^{lb}\frac{\partial(g_{ab})}{\partial y^i}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k\\
&\quad -\frac{1}{4}g^{l\alpha}g^{b\beta}\frac{\partial}{\partial t}(g_{\alpha\beta})g^{ra}\frac{\partial(g_{ab})}{\partial y^i}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k\\
&\quad -\frac{1}{4}g^{ra}g^{lb}\frac{\partial(g'_{ab})}{\partial y^i}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k\\
&\quad -\frac{1}{4}g^{ra}g^{lb}\frac{\partial(g_{ab})}{\partial y^i}\left\{2\frac{\partial g'_{jl}}{\partial x^k}-\frac{\partial g'_{jk}}{\partial x^l}\right\}y^jy^k\\
&\quad +\frac{1}{4}g^{r\alpha}g^{l\beta}\frac{\partial(g_{\alpha\beta})}{\partial t}\left\{2\frac{\partial^2 g_{jl}}{\partial y^i\partial x^k}-\frac{\partial^2 g_{jk}}{\partial y^i\partial x^l}\right\}y^jy^k\\
&\quad +\frac{1}{4}g^{rl}\left\{2\frac{\partial^2 g'_{jl}}{\partial y^i\partial x^k}-\frac{\partial^2 g'_{jk}}{\partial y^i\partial x^l}\right\}y^jy^k\\
&\quad +\frac{1}{4}g^{r\alpha}g^{l\beta}\frac{\partial(g_{\alpha\beta})}{\partial t}\left\{2\frac{\partial g_{il}}{\partial x^k}-\frac{\partial g_{ik}}{\partial x^l}\right\}y^k+\frac{1}{4}g^{rl}\left\{2\frac{\partial g'_{il}}{\partial x^k}-\frac{\partial g'_{ik}}{\partial x^l}\right\}y^k\\
&\quad +\frac{1}{4}g^{r\alpha}g^{l\beta}\frac{\partial(g_{\alpha\beta})}{\partial t}\left\{2\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^l}\right\}y^j+\frac{1}{4}g^{rl}\left\{2\frac{\partial g'_{jl}}{\partial x^i}-\frac{\partial g'_{ij}}{\partial x^l}\right\}y^j.
\end{aligned}
\tag{27}
$$

From the un-normalized Ricci flow, we can then write

$$\begin{aligned}
\frac{d}{dt}(G_i^r) &= \frac{1}{2}g^{r\alpha}g^{a\beta}Ric_{\alpha\beta}\,g^{lb}\frac{\partial(g_{ab})}{\partial y^i}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k \qquad (28)\\
&+\frac{1}{2}g^{l\alpha}g^{b\beta}Ric_{\alpha\beta}\,g^{ra}\frac{\partial(g_{ab})}{\partial y^i}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k\\
&+\frac{1}{2}g^{ra}g^{lb}\frac{\partial(Ric_{ab})}{\partial y^i}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k\\
&+\frac{1}{2}g^{ra}g^{lb}\frac{\partial(g_{ab})}{\partial y^i}\left\{2\frac{\partial Ric_{jl}}{\partial x^k}-\frac{\partial Ric_{jk}}{\partial x^l}\right\}y^jy^k\\
&-\frac{1}{2}g^{r\alpha}g^{l\beta}Ric_{\alpha\beta}\left\{2\frac{\partial^2 g_{jl}}{\partial y^i\partial x^k}-\frac{\partial^2 g_{jk}}{\partial y^i\partial x^l}\right\}y^jy^k\\
&-\frac{1}{2}g^{rl}\left\{2\frac{\partial^2 Ric_{jl}}{\partial y^i\partial x^k}-\frac{\partial^2 Ric_{jk}}{\partial y^i\partial x^l}\right\}y^jy^k\\
&-\frac{1}{2}g^{r\alpha}g^{l\beta}Ric_{\alpha\beta}\left\{2\frac{\partial g_{il}}{\partial x^k}-\frac{\partial g_{ik}}{\partial x^l}\right\}y^k\\
&-\frac{1}{2}g^{rl}\left\{2\frac{\partial Ric_{il}}{\partial x^k}-\frac{\partial Ric_{ik}}{\partial x^l}\right\}y^k\\
&-\frac{1}{2}g^{r\alpha}g^{l\beta}Ric_{\alpha\beta}\left\{2\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^l}\right\}y^j\\
&-\frac{1}{2}g^{rl}\left\{2\frac{\partial Ric_{jl}}{\partial x^i}-\frac{\partial Ric_{ij}}{\partial x^l}\right\}y^j.
\end{aligned}$$

Using (26) we obtain

$$\begin{aligned}
p\int_{SM}\widetilde{g}(\widetilde{\nabla}f',\widetilde{\nabla}f)|\widetilde{\nabla}f|^{p-2}dv &= p\lambda\int_{SM}f'f|f|^{p-2}dv\\
&= -\lambda\int_{SM}|f|^p\left\{-2g^{ij}Ric_{ij}+n\mathcal{R}ic\right\}dv.
\end{aligned}$$

We have thus proved the following proposition:

Proposition 3.2. Let $(M^n,F_t))$ be a solution of the un-normalized Ricci flow on the smooth Finsler manifold (M^n,F_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under the Ricci flow, then

$$\begin{aligned}
\frac{d\lambda}{dt} &= p\int_{SM}Ric(\widetilde{\nabla}f,\widetilde{\nabla}f)|\widetilde{\nabla}f|^{p-2}dv\\
&+\int_{SM}(\lambda|f|^p-|\widetilde{\nabla}f|^p)\left\{2g^{ij}Ric_{ij}-n\mathcal{R}ic\right\}dv \qquad (29)\\
&-p\int_{SM}g^{ij}\frac{\partial}{\partial t}(G_i^r)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\widetilde{\nabla}f|^{p-2}dv\\
&-p\int_{SM}\mathcal{R}ic\,F^2\,g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla}f|^{p-2}dv \qquad (30)
\end{aligned}$$

where f is the associated normalized evolving eigenfunction.□

Note. Let $f : SM \to \mathbb{R}$ be a lifting of $f : M \to \mathbb{R}$. We have:

$$\begin{aligned}\frac{d\lambda}{dt} &= p\int_{SM} Ric(\widetilde{\nabla} f, \widetilde{\nabla} f)|\widetilde{\nabla} f|^{p-2}dv \\ &+ \int_{SM}(\lambda|f|^p - |\widetilde{\nabla} f|^p)\left\{-2g^{ij}Ric_{ij} + n\mathcal{R}ic\right\}dv\end{aligned}$$

and in this case, if $-2g^{ij}Ric_{ij} + n\mathcal{R}ic$ is a constant, then

$$\frac{d\lambda}{dt} = p\int_{SM} Ric(\widetilde{\nabla} f, \widetilde{\nabla} f)|\widetilde{\nabla} f|^{p-2}dv.$$

Corollary 3.3. *Let (M^n, F_t) be a solution of the un-normalized Ricci flow on the smooth Riemannian manifold (M^n, F_0), i.e. F_t, F_0 are Riemannian metric. If $\lambda(t)$ denotes the evolution of an eigenvalue under Ricci flow, then:*

$$\begin{aligned}\frac{d\lambda}{dt} &= p\int_{SM} Ric(\widetilde{\nabla} f, \widetilde{\nabla} f)|\widetilde{\nabla} f|^{p-2}dv + \int_{SM} R(\lambda|f|^p - |\widetilde{\nabla} f|^p)dv \\ &\quad -p\int_{SM} g^{ij}\frac{\partial}{\partial t}(G^i_r)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\widetilde{\nabla} f|^{p-2}dv \\ &\quad -\frac{p}{n}\int_{SM} RF^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv\end{aligned} \tag{31}$$

where R is the scalar curvature of M.

Proof:
If F_t is the Riemannian metric, then

$$\mathcal{R}ic = \frac{1}{n}R, \tag{32}$$

and

$$2g^{ij}Ric_{ij} - n\mathcal{R}ic = R, \tag{33}$$

therefore (31) is obtained by replacing (32) and (33) in (29).
□

Corollary 3.4. *Let (M^2, F_t) be a solution of the un-normalized Ricci flow on the smooth Riemannian surface (M^2, F_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under the Ricci flow, then:*

$$\begin{aligned}\frac{d\lambda}{dt} &= \frac{p}{2}\int_{SM} R|\widetilde{\nabla} f|^p dv + \int_{SM} R(\lambda|f|^p - |\widetilde{\nabla} f|^p)dv \\ &\quad -p\int_{SM} g^{ij}\frac{\partial}{\partial t}(G^i_r)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\widetilde{\nabla} f|^{p-2}dv \\ &\quad -\frac{p}{n}\int_{SM} RF^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv\end{aligned}$$

where R is the scalar curvature of M.

Proof: In dimension $n = 2$, for a Riemannian manifold, we have:

$$Ric = \frac{1}{2}Rg, \tag{34}$$

hence the corollary is obtained by replacing (34) in (29). □

Corollary 3.5. *Let (M^n, F_t) be a solution of the un-normalized Ricci flow on the smooth homogenous Riemannian manifold (M^n, F_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under the Ricci flow, then:*

$$\begin{aligned}\frac{d\lambda}{dt} &= p\int_{SM} Ric(\widetilde{\nabla} f, \widetilde{\nabla} f)|\widetilde{\nabla} f|^{p-2}dv \\ &\quad -p\int_{SM} g^{ij}\frac{\partial}{\partial t}(G^i_r)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\widetilde{\nabla} f|^{p-2}dv \\ &\quad -R\frac{p}{n}\int_{SM} F^2\, g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv\end{aligned}$$

where R is the scalar curvature of M.

Proof:

Since the evolving metric remains homogenous and a Riemannian homogenous manifold has constant scalar curvature, so the corollary is obtained by (29).□

Now, we give a variation of $\lambda(t)$ under the normalized Ricci flow which is similar to the pervious proposition.

Proposition 3.6. Let (M^n, F_t) be a solution of the normalized Ricci flow on the smooth Finsler manifold (M^n, F_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under Ricci flow, then:

$$\begin{aligned}\frac{d\lambda}{dt} &= (n-p)r\lambda + p\int_{SM} Ric(\widetilde{\nabla} f, \widetilde{\nabla} f)|\widetilde{\nabla} f|^{p-2}dv \\ &\quad +\int_{SM}(\lambda|f|^p - |\widetilde{\nabla} f|^p)\left\{2g^{ij}Ric_{ij} - n\mathcal{R}ic\right\}dv \\ &\quad -p\int_{SM} g^{ij}\frac{\partial}{\partial t}(G^s_i)\frac{\partial f}{\partial y^s}\frac{\delta f}{\delta x^j}|\widetilde{\nabla} f|^{p-2}dv \\ &\quad -p\int_{SM} F^2(\mathcal{R}ic - r)\, g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv\end{aligned} \tag{35}$$

where f is the associated normalized evolving eigenfunction, $r = \frac{\int_{SM}\mathcal{R}ic\, dv}{vol(SM)}$.

Proof: In the normalized case, the integrability conditions read as follows

$$p\int_{SM} f'f|f|^{p-2}dv = \int_{SM}|f|^p\left\{2g^{ij}Ric_{ij} - nr - n\mathcal{R}ic\right\}dv. \tag{36}$$

Since

$$\frac{d}{dt}(dv) = \left\{-2g^{ij}Ric_{ij} + nr + n\mathcal{R}ic\right\}dv \tag{37}$$

using (24), (27) and the above equation, we can then write

$$\begin{aligned}
\frac{d\lambda}{dt} &= \int_{SM}(\frac{d}{dt}|\widetilde{\nabla} f|^p)dv+\int_{SM}|\widetilde{\nabla} f|^p\frac{d}{dt}(dv_t) \qquad (38)\\
&= \frac{p}{2}\int_{SM}\Big\{-g^{il}g^{jk}(-2Ric_{lk}+2rg_{lk})\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j}+2g^{ij}\frac{\delta f'}{\delta x^i}\frac{\delta f}{\delta x^j}\Big\}|\widetilde{\nabla} f|^{p-2}dv\\
&\quad+\frac{p}{2}\int_{SM}\Big\{2F^2(-\mathcal{R}ic+r)g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}\\
&\quad -F^2g^{il}g^{jk}(-2Ric_{lk}+2rg_{lk})\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}\Big\}|\widetilde{\nabla} f|^{p-2}dv\\
&\quad+\frac{p}{2}\int_{SM}\Big\{-2g^{ij}\frac{\partial}{\partial t}(G^r_i)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}+2F^2g^{ij}\frac{\partial f'}{\partial y^i}\frac{\partial f}{\partial y^j}\Big\}|\widetilde{\nabla} f|^{p-2}dv\\
&\quad+\int_{SM}|\widetilde{\nabla} f|^p\Big\{-2g^{ij}Ric_{ij}+nr+n\mathcal{R}ic\Big\}dv\\
&= (n-p)r\lambda+p\int_{SM}Ric(\widetilde{\nabla} f,\widetilde{\nabla} f)|\widetilde{\nabla} f|^{p-2}dv+p\int_{SM}\widetilde{g}(\widetilde{\nabla} f',\widetilde{\nabla} f)|\widetilde{\nabla} f|^{p-2}dv\\
&\quad+\int_{SM}|\widetilde{\nabla} f|^p\Big\{-2g^{ij}Ric_{ij}+n\mathcal{R}ic\Big\}dv\\
&\quad-p\int_{SM}g^{ij}\frac{d}{dt}(G^r_i)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\widetilde{\nabla} f|^{p-2}dv\\
&\quad-p\int_{SM}F^2(\mathcal{R}ic-r)\,g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv
\end{aligned}$$

but

$$\begin{aligned}
p\int_{SM}\widetilde{g}(\widetilde{\nabla} f',\widetilde{\nabla} f)|\widetilde{\nabla} f|^{p-2}dv &= p\lambda\int_{SM}f'f|f|^{p-2}dv\\
&= \lambda\int_{SM}|f|^p\Big\{2g^{ij}Ric_{ij}-nr-n\mathcal{R}ic\Big\}dv \qquad (39)
\end{aligned}$$

and $\frac{\partial}{\partial t}(G^s_i)$ is obtained by replacing F' and g'_{ij} from (3) and (4), respectively, in (27). Thus the proposition is obtained by replacing (39) in (38).□
Similar to un-normalized case we have the following corollaries:

Corollary 3.7. *Let (M^n, F_t) be a solution of the normalized Ricci flow on the smooth Riemannian manifold (M^n, F_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under Ricci flow, then:*

$$\begin{aligned}
\frac{d\lambda}{dt} &= (n-p)r\lambda+p\int_{SM}Ric(\widetilde{\nabla} f,\widetilde{\nabla} f)|\widetilde{\nabla} f|^{p-2}dv\\
&\quad+\int_{SM}R(\lambda|f|^p-|\widetilde{\nabla} f|^p)dv-p\int_{SM}g^{ij}\frac{\partial}{\partial t}(G^s_i)\frac{\partial f}{\partial y^s}\frac{\delta f}{\delta x^j}|\widetilde{\nabla} f|^{p-2}dv \qquad (40)\\
&\quad-p\int_{SM}(\frac{1}{n}R-r)F^2\,g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv
\end{aligned}$$

where R is the scalar curvature of M.

Corollary 3.8. *Let (M^2, F_t) be a solution of the normalized Ricci flow on the smooth Riemannian surface (M^2, F_0). If $\lambda(t)$ denotes the evolution of an eigenvalue* *under Ricci flow, then:*

$$\begin{aligned} \frac{d\lambda}{dt} &= (2-p)r\lambda + \frac{p}{2}\int_{SM} R|\widetilde{\nabla} f|^p dv \\ &+ \int_{SM} R(\lambda|f|^p - |\widetilde{\nabla} f|^p)dv - p\int_{SM} g^{ij}\frac{\partial}{\partial t}(G^s_i)\frac{\partial f}{\partial y^s}\frac{\delta f}{\delta x^j}|\widetilde{\nabla} f|^{p-2}dv \\ &- p\int_{SM}(\frac{R}{2} - r)F^2\, g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv \end{aligned}$$

where R is the scalar curvature of M.

Corollary 3.9. *Let (M^n, F_t) be a solution of the normalized Ricci flow on the smooth homogenous Riemannian manifold (M^n, F_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under Ricci flow, then:*

$$\begin{aligned} \frac{d\lambda}{dt} &= (n-p)r\lambda + p\int_{SM} Ric(\widetilde{\nabla} f, \widetilde{\nabla} f)|\widetilde{\nabla} f|^{p-2}dv \\ &- p\int_{SM} g^{ij}\frac{\partial}{\partial t}(G^s_i)\frac{\partial f}{\partial y^s}\frac{\delta f}{\delta x^j}|\widetilde{\nabla} f|^{p-2}dv \\ &- p\int_{SM}(\frac{1}{n}R - r)F^2\, g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv \end{aligned}$$

where R is the scalar curvature of M.

4. Examples

In this section, we will find the variational formula for some of Finsler manifolds.

Example 4.1. *Let (M^n, F_0) be an Einstein manifold i.e. there exists a constant a such that $Ric(F_0) = aF_0^2$. Therefore $Ric_{ij}(g_0) = ag_{ij}(0)$. Assume we have a solution to the Ricci flow which is of the form*

$$g(t) = u(t)g_0, \quad u(0) = 1$$

where $u(t)$ is a positive function. Now (2) implies that

$$u(t) = -2at + 1,$$

so that we have

$$g(t) = (1-2at)g_0$$

which says that $g(t)$ is an Einstein metric. On the other hand it is easily seen that

$$\begin{aligned} Ric(g(t)) &= Ric(g_0) = ag_0 = \frac{a}{1-2at}g(t), \\ \mathcal{R}ic(g_t) &= \frac{1}{1-2at}\mathcal{R}ic(g_0) = \frac{a}{1-2at}, \\ F_t^2 &= (1-2at)F_0^2 \end{aligned}$$

therefore

$$2g^{ij}Ric_{ij} - n\mathcal{R}ic = \frac{an}{1-2at}$$

and

$$Ric(\widetilde{\nabla} f, \widetilde{\nabla} f) = \frac{a}{1-2at}\widetilde{g}(\widetilde{\nabla} f, \widetilde{\nabla} f) = \frac{a}{1-2at}|\widetilde{\nabla} f|^2.$$

Also

$$\begin{aligned} G^i(t) &= \frac{1}{4}g^{il}\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\}y^j y^k \\ &= \frac{1}{4}(g_0)^{il}\{2\frac{\partial (g_0)_{jl}}{\partial x^k} - \frac{\partial (g_0)_{jk}}{\partial x^l}\}y^j y^k = G^i(0) \end{aligned}$$

therefore

$$\frac{\partial}{\partial t}(G^i_r) = 0.$$

Using the un-normalized Ricci flow equation (2) and (29) ,we obtain the following relation:

$$\begin{aligned} \frac{d\lambda}{dt} &= p\int_M \frac{a}{1-2at}|\widetilde{\nabla} f|^p dv + \lambda\int_M |f|^p \frac{an}{1-2at}dv \\ &- \int_M \frac{an}{1-2at}|\widetilde{\nabla} f|^p dv - p\int_{SM} \frac{a}{1-2at}F^2\, g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv \\ &= \frac{pa}{1-2at}\left\{\lambda - \int_{SM} F^2\, g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv\right\} \end{aligned}$$

Now, If we suppose that $g_t = u(t)g_0$, $u(0) = 1$ *is a solution of the normalized Ricci flow and* $r = \frac{1}{Vol(SM)}\int_{SM}\mathcal{R}icdv$, *then from (3) we have*

$$u' g_0 = -2ag_0 + 2rug_0.$$

It implies that

$$u(t) = \frac{a}{r}(1-e^{2rt}) + e^{2rt}.$$

So that we have:

$$g(t) = \left\{\frac{a}{r}(1-e^{2rt}) + e^{2rt}\right\}$$

and

$$Ric(g(t)) = a\left(\frac{a}{r}(1-e^{2rt}) + e^{2rt}\right)^{-1} g_0$$

$$\mathcal{R}ic(g(t)) = a\left(\frac{a}{r}(1-e^{2rt}) + e^{2rt}\right)^{-1}$$

$$2g^{ij}Ric_{ij} - \mathcal{R}ic_{ij} = an\left(\frac{a}{r}(1-e^{2rt}) + e^{2rt}\right)^{-1},$$

also

$$G^i(t) = G^i(0),$$

therefore

$$\frac{\partial G^r_i}{\partial t} = 0.$$

Using (35), we obtain the following:

$$\begin{aligned}
\frac{d\lambda}{dt} &= (n-p)r\lambda + pa\lambda\bigg(\frac{a}{r}(1-e^{2rt})+e^{2rt}\bigg)^{-1} \\
&- p\int_{SM}\bigg(a\bigg(\frac{a}{r}(1-e^{2rt})+e^{2rt}\bigg)^{-1}-r\bigg)(y^i\frac{\partial f}{\partial y_i})^2|\widetilde{\nabla} f|^{p-2}dv \\
&= \bigg((n-p)r + pa\bigg(\frac{a}{r}(1-e^{2rt})+e^{2rt}\bigg)^{-1}\bigg)\lambda \\
&- p\bigg(a\bigg(\frac{a}{r}(1-e^{2rt})+e^{2rt}\bigg)^{-1}-r\bigg)\int_{SM}F^2\, g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv.
\end{aligned}$$

Remark. Let (M^n, F_0) be a Finsler manifold of dimension $n \geq 3$. Suppose that the flag curvature $k = k(x)$ is isotropic and a function of $x \in M$ alone then $k = constant$ and therefore (M^n, F_0) is Einstein and the variation of its eigenvalues is similar to example (4.1).

Definition 4.2. *A Finsler metric on an n-dimensional manifold is called a weak Einstein metric if*

$$Ric = (n-1)\{\frac{3\eta}{F}+\sigma\}F^2$$

where η is a 1-form and $\sigma = \sigma(x)$ is scalar function.

Example 4.3. *If we suppose that $F_t = u(t)F_0$, $u(0) = 1$ is a solution of the Ricci flow, then:*

$$\mathcal{R}ic(F_t) = \frac{Ric(F_t)}{F_t^2} = \frac{Ric(F_0)}{(u(t))^2F_0^2} = \frac{(n-1)\{\frac{3\eta_0}{F_0}+\sigma_0\}F_0^2}{(u(t))^2F_0^2} = \frac{(n-1)\{\frac{3\eta_0}{F_0}+\sigma_0\}}{(u(t))^2}$$

Now the Ricci flow (1) implies that

$$-\frac{(n-1)\{\frac{3\eta_0}{F_0}+\sigma_0\}}{(u(t))^2} = -\mathcal{R}ic = \frac{\partial logF}{\partial t} = \frac{F_t'}{F_t} = \frac{u'(t)}{u(t)}$$

or equivalently

$$uu' = (n-1)\{\frac{3\eta_0}{F_0}+\sigma_0\}$$

By integration we have:

$$u^2(t) = -2(n-1)\{\frac{3\eta_0}{F_0}+\sigma_0\}t + c$$

with condition $u(0) = 1$ we have:

$$u^2(t) = 1 - 2(n-1)\{\frac{3\eta_0}{F_0}+\sigma_0\}t$$

therefore

$$F_t^2 = \bigg\{1-2(n-1)\{\frac{3\eta_0}{F_0}+\sigma_0\}t\bigg\}F_0^2 \tag{41}$$

and

$$Ric(\widetilde{\nabla} f, \widetilde{\nabla} f) = \frac{1}{2}(n-1)g^{il}g^{js}\left\{(3\eta_0 F_0)_{y^l y^s} + 2\sigma_0 (g_0)_{ls}\right\}\widetilde{\nabla}_i f \widetilde{\nabla}_j f. \tag{42}$$

Also we have

$$2g^{ij}Ric_{ij} - n\mathcal{R}ic = (n-1)g^{ij}\left\{(3\eta_0 F_0)_{y^i y^j} + 2\sigma_0 (g_0)_{ij}\right\} - n\frac{(n-1)\{\frac{3\eta_0}{F_0} + \sigma_0\}}{1-2(n-1)\{\frac{3\eta_0}{F_0} + \sigma_0\}t}. \tag{43}$$

Note that

$$g_{ij}(t) = (1-2(n-1)\sigma_0 t)(g_0)_{ij} - 6(n-1)t\frac{\partial \eta}{\partial y^i}\frac{\partial F}{\partial y^j}. \tag{44}$$

By replacing (42),(43) and (44) in (29) we obtain the variation of an eigenvalue. Now, if we suppose that $F_t = u(t)F_0$, $u(0) = 1$ *is a solution of the normalized Ricci flow and* $r = \frac{1}{Vol(SM)}\int_{SM} \mathcal{R}ic\, dv$, *then from (3), we have:*

$$r - \frac{(n-1)\frac{3\eta_0}{F_0} + \sigma_0}{u^2(t)} = r - \mathcal{R}ic = \frac{\partial logF}{\partial t} = \frac{u'(t)}{u(t)}$$

. It implies that

$$uu' - u^2 r = -(n-1)\frac{3\eta_0}{F_0} + \sigma_0, \quad u(0) = 1$$

which is an ordinary differential equation and has a solution as follow:

$$u^2(t) = \frac{n-1}{r}\frac{3\eta_0}{F_0} + \sigma_0(1-e^{2rt}) + e^{2rt}$$

therefore

$$F_t^2 = \left\{\frac{n-1}{r}\frac{3\eta_0}{F_0} + \sigma_0(1-e^{2rt}) + e^{2rt}\right\}F_0^2$$

and $\frac{d\lambda}{dt}$ *is obtained from (35).*

Example 4.4. *In this example we determine the behavior of the evolving spectrum on the Ricci solitons.*

Let F_t *is a solution of the Ricci flow* $\frac{\partial logF}{\partial t} = -\mathcal{R}ic$. *If* φ *ia a time-independent isometry such that*

$$F(x,y) = \varphi^* F_t(x,y)$$

is a solution of the un-normalized Ricci flow, because of $\frac{\partial}{\partial t}logF_t(x,y) = -\mathcal{R}ic(x,y), F(0) = F_0$,

$$\frac{\partial}{\partial t}logF(x,y) = \frac{\partial}{\partial t}\varphi^* logF_t(x,y) = \frac{\partial}{\partial t}logF_t(\varphi(x), \varphi_*(y))$$

since φ *is a isometry we have*

$$\frac{\partial}{\partial t}logF(x,y) = \frac{\partial}{\partial t}logF_t(x,y) = -\mathcal{R}ic(x,y) = -\mathcal{R}ic(\varphi(x), \varphi_*(y)) = -\mathcal{R}ic(F)$$

Definition 4.5. *Let* (M, F_t) *is a solution of the Ricci flow and* φ_t *is a family of diffeomorphisms. We says* $F(t)$ *is Ricci soliton, when satisfies in*

$$F_t^2 = u(t)\varphi_t^* F_0^2. \tag{45}$$

Let (M,F) and $(\overline{M},\overline{F})$ be two closed Finsler manifolds and

$$\varphi:(M,g)\to(\overline{M},\overline{F})$$

an isometry, then for $p=2$ we have

$$^{\widetilde{g}}\Delta_p\circ\varphi^*=\varphi^*\circ{}^{\widetilde{\overline{g}}}\Delta_p.$$

Therefore given a diffeomorphism $\varphi:M\to M$ we have that

$$\varphi:(SM,\varphi^*\widetilde{g})\to(SM,\widetilde{g})$$

is an isometry, hence we conclude that $(SM,\varphi^*\widetilde{g})$, and $(SM,\widetilde{g})$ have the same spectrum

$$Spec_p(\widetilde{g})=Spec_p(\varphi^*\widetilde{g})$$

with eigenfunction f_k and φ^*f_k respectively. If $g(t)$ is a Ricci soliton on (M^n,g_0) then

$$Spec_p(\widetilde{g}(t))=\frac{1}{u(t)}Spec_p(\widetilde{g}_0)$$

so that $\lambda(t)$ satisfies

$$\lambda(t)=\frac{1}{u(t)},\ \ \frac{d\lambda}{dt}=-\frac{u'(t)}{(u(t))^2}.$$

Example 4.6. *Suppose that*

$$\mathbb{R}^n_+=\left\{(x^1,x^2,...,x^n)\in\mathbb{R}^n|x^i>0,\ i=1,...,n\right\}$$

has the metric

$$g_{ij}(x,y)=\begin{cases}\frac{\phi_t(y)}{(x^i)^{\frac{2(p+1)}{p}}} & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}\tag{46}$$

where it is a solution for the un-normalized Ricci flow and $\phi_t(y)$ is a strictly positive C^∞ homogeneous function of degree zero and $p>0$. We use the formula

$$g^{ij}=\begin{cases}\frac{(x^i)^{\frac{2(p+1)}{p}}}{\phi_t(y)} & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}\tag{47}$$

for $i=1,...,n$, we have

$$G^i=-\frac{p+1}{2p}\frac{(y^i)^2}{x^i}$$

and

$$F^2\mathcal{R}ic(g(t))=0$$

therefore $Ric(g(t))=0$ and the un-normalized Ricci flow equation implies that

$$\frac{\partial g(t)}{\partial t}=0$$

hence, $g(t)=g_0$ and $\lambda(t)=\lambda(0)$.

Example 4.7. *Suppose that* $\mathbb{R}^2$ *has the metric*

$$g_{ij}(x,y) = \phi_t(y) \begin{pmatrix} 4(x^1)^2+1 & -2x^2 \\ -2x^1 & 1 \end{pmatrix} \tag{48}$$

which is a solution for the the un-normalized Ricci flow, where $\phi_t(y)$ *is a strictly positive* C^∞ *homogeneous function of degree zero. We obtain*

$$G^1 = 0,\ G^2 = -(y^1)^2$$

and

$$F^2 \mathcal{R}ic(g(t)) = 0$$

therefore $Ric(g(t)) = 0$ *and the un-normalized Ricci flow equation implies that*

$$\frac{\partial g(t)}{\partial t} = 0$$

hence, $g(t) = g_0$ *and* $\lambda(t) = \lambda(0)$.

References

[1] Azami, S., Razavi, A., "Existence and uniqueness for solution of Ricci flow on Finsler manifolds", *International Journal of Geometric Methods in Modern Physics*, **10**, (2013).

[2] Bao, D., " On two curvature-driven problems in Riemann-Finsler geometry", *Advanced Studies in Pure Mathematics* **48** (2007), 19-71.

[3] Bartheleme, T., "A natural Finsler-Laplace operator", *Israel J. Math.* **196** (2013), 375-412.

[4] Cheng, Q. M., Yang, H. C., " Estimates on eigenvalues of Laplacian", *Math. Ann.* **331** (2005), 445-460.

[5] Chow, B., Knopf, D., *The Ricci flow: An Introduction*, Mathematical Surveys and Monographs, AMS, 2004.

[6] Chow, B., Lu, P., Ni, L., *Hamilton's Ricci flow*, Vol. 77, AMS, 2006.

[7] DeTurck, D.M., " Deforming metrics in the direction of their Ricci tensors", *J. Differential Geometry*, **18** (1983), 157-162.

[8] Friedan, D., *Nonlinear Models in* $2+\epsilon$ *Dimensions*, PhD, University of California, Berkeley, USA, 1980.

[9] Harrell, E. M., Michel, P. L., "Commutator bounds for eigenvalues with applications to spectral geometry", *Comm. Partial Differential Equations*, **19** (1994), 2037-2055.

[10] Headrick, M., Wiseman, T., "Ricci Flow and Black Holes", *Class. Quantum Grav.* **23** (2006), 6683-6707.

[11] Leung, P. F., "On the consecutive eigenvalues of the Laplacian of a compact minimal submanifold in a sphere", *J. Aust. Math. Soc.* **50** (1991), 409-426.

[12] Miron, R., Anastasiei, M., *Vector Bundles and Lagrange Spaces with Applications to Relativity*, Geometry Balkan Press, Bukharest, 1997.

[13] Miron, R., Anastasiei, M., *The Geometry of Lagrange paces: Theory and Applications*, FTPH 59 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1994).

[14] Mo, X., *An introduction to finsler geometry*, World Scientific Publishing Co. Pte. Ltd., 2006.

[15] Nitta, M., "Conformal Sigma Models with Anomalous Dimensions and Ricci Solitons", *Mod. Phys. Lett.*, **20** (2005), 577-584.

[16] Perelman, G., "The entropy formula for the Ricci flow and its geometric application", *ArXiv Preprint Server*, 2002.

[17] Sadeghzadeh, N., Razavi, A., "Ricci flow equation on C-reducible metrics", *International Journal of Geometric Methods in Modern Physics*, **8** (2011), 773-781.

[18] Shen, Z., "The non-linear Laplacian for Finsler manifold,The theory of Finslerian Laplacians and applications", *Math. Appl.*, **459** (1998), 187-198.

[19] Shen, Z., *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.

[20] Wang, G., Xio, C. "A sharp lower bound for the first eigenvalue on Finsler manifolds", *Ann. I. H. Poincaré-AN*, **30** (2013), 983-996.