EIGENVALUES VARIATION OF THE P-LAPLACIAN UNDER THE RICCI FLOW ON SM

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Abstract. Let \((M, F)\) be a compact Finsler manifold. Studying the eigenvalues and eigenfunctions for the linear and nonlinear geometric operators is a known problem. In this paper we will consider the eigenvalue problem for the \(p\)-laplace operator for Sasakian metric acting on the space of functions on \(SM\). We find the first variation formula for the eigenvalues of \(p\)-Laplacian on \(SM\) evolving by the Ricci flow on \(M\) and give some examples.

Key words and Phrases: Ricci flow, Finsler manifold, \(p\)-Laplace operator

1. Introduction

For a compact Finsler manifold \((M, F)\), studying the eigenvalues of geometric operators plays a powerful role in geometric analysis. In the classical theory of the Laplace or \(p\)-Laplace equation several main parts of mathematics are joined in a fruitful way: Calculus of Variation, Partial Differential Equation, Potential Theory,
Analytic Function. Recently, there are many mathematicians who have investigated properties of the eigenvalues of $p$-Laplacian on Finsler manifolds and Riemannian manifolds to estimate the spectrum in terms of the other geometric quantities of the manifold. (see [3, 4, 9, 11, 18, 20]).

Also, geometric flows have been a topic of active research interest in mathematics and other sciences (see [5, 8, 10, 12, 13, 15, 16]). Hamilton’s Ricci flow ([6]) is the best known example of a geometric evolution equation. The Ricci flow is related to dynamical systems in the infinite-dimensional space of all metrics on a given manifold. One of the aims of such flows is to obtain metrics with special properties. Special cases arise when the metric is invariant under a group of transformations and this property is preserved by the flow.

Let $M$ be a manifold with a Finsler metric $g_0$ (or $F_0$), the family $g(t)$ (or $F_t$) of Finsler metrics on $M$ is called an un-normalized Ricci flow when it satisfies the equations

$$\frac{\partial \log F}{\partial t} = -Ric,$$

with the initial condition

$$F(0) = F_0$$

or equivalently satisfies the equations

$$\frac{\partial g_{ij}}{\partial t} = -2 Ric_{ij}, \ g(0) = g_0$$

where $Ric$ is the Ricci tensor of $g(t)$, $Ric_{ij} = (\frac{1}{2} F^2 Ric)_{y'y'}$. In fact Ricci flow is a system of partial differential equations of parabolic type which was introduced by Hamilton on Riemannian manifolds for the first time in 1982 and Bao (see [2, 17]) studied Ricci flow equation in Finsler manifold. The Ricci flow has been proved to be a very useful tool to improve metrics in Finsler geometry, when $M$ is compact. One often considers the normalized Ricci flow

$$\frac{\partial \log F}{\partial t} = -Ric + \frac{1}{vol(SM)} \int_{SM} Ric dv, \ F(0) = F_0,$$

or

$$\frac{\partial g_{ij}}{\partial t} = -2 Ric_{ij} + \frac{2}{vol(SM)} \int_{SM} Ric dv g_{ij}, \ g(0) = g_0$$

Under this normalized flow, the volume of the solution metrics remains constant in time. Short time exitance and uniqueness for solution to the Ricci flow on $[0,T)$ have been shown by Hamilton in [5] and by DeTurk in [7] for Riemannian manifolds and by the authors in [1] for important Berwald manifolds.
2. Preliminaries

Let $M$ be an $n$-dimensional $C^\infty$ manifold. For a point $x \in M$, denote by $T_x M$ the tangent space at $x \in M$, and by $TM = \bigcup_{x \in M} T_x M$ the tangent bundle of $M$. Any element of $TM$ has the form $(x, y)$, where $x \in M$ and $y \in T_x M$.

**Definition 2.1.** A Finsler metric on a manifold $M$ is a function $F : TM_0 \to [0, \infty)$ which has the following properties:

(i): $F(x, \alpha y) = \alpha F(x, y)$, $\forall \alpha > 0$;

(ii): $F(x, y)$ is $C^\infty$ on $TM$;

(iii): For any non-zero tangent vector $y \in T_x M$, the associated quadratic form $g_y : T_x M \times T_x M \to \mathbb{R}$ on $TM$ is an inner product, where $g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial r} \left[ F^2 (x, y + su + rv) \right] \bigg|_{s=r=0}$.

The pair $(M, F)$ is called a Finsler manifold.

Let us denote by $S_x M$ the set consisting of all rays $[y] := \{ \lambda y | \lambda > 0 \}$, where $y \in T_x M$. The Sphere bundle of $M$, i.e. $SM$, is the union of $S_x M$’s:

$SM = \bigcup_x S_x M$

$SM$ has a natural $(2n-1)$-dimensional manifold structure. We denote the elements of $SM$ by $(x, [y])$ where $y \in T_x M$. If there is not any confusion we write $(x, y)$ for $(x, [y])$.

In a local coordinate system $(x^i, y^j)$ we have $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} (F^2(x, y))$ and $(g^{ij}) := (g_{ij})^{-1}$. The geodesics of $F$ are characterized locally by

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0$$

where

$$G^i = \frac{1}{4} \delta^{ij} \left( \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} \right) y^j y^k.$$  \hfill (5)

**Definition 2.2.** The coefficients of the Riemann curvature $R_y = R^i_k dx^i \otimes \frac{\partial}{\partial x^k}$ are given by

$$R^i_k := \frac{2}{4} \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k} + \frac{\partial G^j}{\partial y^k} \frac{\partial G^i}{\partial y^j}.$$ \hfill (6)

and $R^i_{jk} := \frac{1}{3} \left( \frac{\partial R^i_j}{\partial y^k} - \frac{\partial R^i_k}{\partial y^j} \right)$, $R^i_{jkl} := \frac{1}{3} \left( \frac{\partial^2 R^i_j}{\partial y^l \partial y^k} - \frac{\partial^2 R^i_k}{\partial y^l \partial y^j} \right)$.

The Ricci scalar function of $F$ is given by $Ric := \frac{1}{F^2} R^i_i$. A companion of the Ricci scalar is the Ricci tensor

$$Ric_{ij} := \left( \frac{1}{2} F^2 Ric \right)_{y^i y^j}.$$  \hfill (7)
Definition 2.3. A Finsler metric is said to be an Einstein metric if the Ricci scalar function is a function of \( x \) alone, equivalently \( \text{Ric}_{ij} = \mathcal{R}(x)g_{ij} \) (see [14, 19]).

Definition 2.4. Let \((M, F)\) be a Finsler manifold, the Sasakian metric \( \tilde{g} \) of \( g \) on \( TM_0 \) is defined as

\[
\tilde{g} = g_{ij}dx^i \otimes dx^j + g_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}
\]

then \( \tilde{g} \) is a Riemannian metric on \( TM_0 \) and \( \{ \frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^i} \} \) is a coordinate base on \( TM_0 \), where \( \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_{jk}^i \frac{\partial}{\partial y^j} \) and \( \{ \frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^i} \} \) is the dual of \( \{ \frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^i} \} \) where \( \delta y^i = dy^i + G_{ij} \frac{dx^j}{F} \).

Remark. The Levi-Civita connection \( \tilde{\nabla} \) on \( TM_0 \) with respect to the Sasakian metric \( \tilde{g} \) is locally expressed as follows:

\[
\tilde{\nabla} \frac{\delta}{\delta x^i} = -(C_{ij}^k + \frac{1}{2} R_{ij}^k) \frac{\partial}{\partial y^k} + F_{ij}^k \frac{\delta}{\delta x^k}
\]

\[
\tilde{\nabla} \frac{\partial}{\partial y^i} = F_{ij}^k \frac{\partial}{\partial y^k} + (C_{ij}^k + \frac{1}{2} g_{ik} R_{kj}^l) \frac{\delta}{\delta x^k} = \tilde{\nabla} \frac{\delta}{\delta x^i} + G_{ij}^k \frac{\partial}{\partial y^k},
\]

where

\[
C_{ij}^k = \frac{1}{2} g^{kh} \frac{\partial g_{ij}}{\partial y^h}, \quad F_{ij}^k = \frac{1}{2} g^{kh} (\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}), \quad G_{ij}^k = \frac{\partial G_{ij}}{\partial y^k},
\]

and

\[
R_{ij}^k = \frac{\delta G_{ij}^k}{\delta x^k} - \frac{\delta G_{ij}^k}{\delta x^i} \quad \left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = R_{ij}^k \frac{\partial}{\partial y^k}, \quad \left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^j} \right] = G_{ij}^k \frac{\partial}{\partial y^k}.
\]

Lemma 2.5. For a Sasakian metric \( \tilde{g} \) and any \( f : TM \to \mathbb{R} \), there exists a unique vector field \( Y \in \mathcal{X}(TM) \) such that

\[
\tilde{g}(Y, \tilde{X}) = df(\tilde{X}), \quad \forall \tilde{X} \in \mathcal{X}(TM)
\]

where

\[
\tilde{X} = X_1^i \frac{\delta}{\delta x^i} + X_2^i F \frac{\partial}{\partial y^i}
\]

and \( X_1, X_2 \) are \( C^\infty \) function on \( TM \). Here we take \( Y = 0 \) if \( df = 0 \).

Denote the vector field \( Y \) in (10 ) by \( \hat{\nabla} f \). We call \( \hat{\nabla} f \) the gradient of \( f \) and define the divergence \( \text{div} \tilde{X} \) as follows:

\[
\text{div} \tilde{X} = \text{tr} \hat{\nabla} \tilde{X}
\]
Definition 2.6. According to the above definition, the gradient of a function $f$ is

$$\tilde{\nabla} f = g^{ij} \frac{\delta f}{\delta x^i} \frac{\delta}{\delta x^j} + F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial}{\partial y^j}$$  \hspace{1cm} (11)$$

therefore, the norm of $\tilde{\nabla} f$ with respect to the Riemannian metric $\tilde{g}$ is given by

$$|\tilde{\nabla} f|^2 = \tilde{g}(\tilde{\nabla} f, \tilde{\nabla} f) = g^{ij} \frac{\delta f}{\delta x^i} \frac{\delta f}{\delta x^j} + F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}. \hspace{1cm} (12)$$

Definition 2.7. Let $M$ be a compact Finsler manifold. The Laplace operator of $f$ on $TM$ is defined as follows:

$$\Delta f = g^{ij} \left( \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial G^i_j}{\partial y^k} - G^i_j \frac{\partial^2 f}{\partial y^i \partial x^j} - \frac{G^i_j}{G^s_k} \frac{\partial^2 f}{\partial y^s \partial y^k} - \frac{\partial G^i_j}{G^s_k} \frac{\partial^2 f}{\partial y^i \partial y^k} \right)$$

$$+ g^{ij} F^2 \frac{\partial^2 f}{\partial y^i \partial y^j} + g^{ij} (C^k_{ij} + \frac{1}{2} R^k_{ij}) \frac{\delta f}{\partial x^k} - F g^{ij} F^k_i \frac{\delta f}{\partial x^k}$$

$$- F g^{ij} C^k_{ij} \frac{\partial f}{\partial y^k} - F^2 \frac{\partial^2 f}{\partial y^i \partial y^j} (\text{Hess}f)(\tilde{\nabla} f, \tilde{\nabla} f). \hspace{1cm} (13)$$

Definition 2.8. Let $M$ be a compact Finsler manifold. The $p$-Laplace operator of $f : SM \to \mathbb{R}$, $f \in W^{1,p}(SM)$ for $1 < p < \infty$ is defined as follows:

$$\triangle_p f = \text{div}(|\tilde{\nabla} f|^{p-2} \tilde{\nabla} f) \hspace{1cm} (14)$$

where

$$(\text{Hess}f)(X, Y) = \tilde{\nabla} (\tilde{\nabla} f)(X, Y) = Y.(X, f) - (\tilde{\nabla} Y, X, f), \hspace{0.5cm} X, Y \in \mathcal{X}(SM)$$

and in local coordinate, we have:

$$(\text{Hess}f)(\partial_i, \partial_j) = \partial_i \partial_j f - \tilde{\Gamma}^k_{ij} \partial_k f.$$  \hspace{1cm} \text{Note.}$$

If $f$ is a function of $x$, or suppose that is the lifting of $f : M \to \mathbb{R}$ then

$$\Delta f = g^{ij} \left( \frac{\partial^2 f}{\partial x^i \partial x^j} - \tilde{\Gamma}^k_{ij} \frac{\partial f}{\partial x^k} \right) \hspace{1cm} (15)$$

where $\tilde{\Gamma}^k_{ij}$ is christoffel symbol of $\tilde{\nabla}$.

2.1. Eigenvalues of the $p$-Laplacian.

Definition 2.9. Let $(M^n, F)$ be a compact Finsler manifold and $f : SM \to \mathbb{R}$. We say that $\lambda$ is an eigenvalue of the $p$-Laplace operator whenever

$$\tilde{g} \Delta_p f + \lambda |f|^{p-2} f = 0 \hspace{1cm} (16)$$

then $f$ is said to be the eigenfunction associated to $\lambda$, or equivalently they satisfy in

$$\int_{SM} |\tilde{\nabla} f|^{p-2} < \tilde{\nabla} f, \tilde{\nabla} \varphi > dv = \lambda \int_{SM} |f|^{p-2} f \varphi dv \hspace{1cm} \forall \varphi \in W^{1,2}_0(SM) \hspace{1cm} (17)$$

Note.
where $W_0^{1,p}(SM)$ is closure of $C_0^\infty(SM)$ in Sobolev $W^{1,p}(SM)$.

By substitution $\varphi = f$ in (17) we have:

$$\lambda = \frac{\int_{SM} \tilde{\nabla} f |f|^p dv}{\int_{SM} |f|^p dv}. \quad (18)$$

Normalized eigenfunctions are defined as follows:

$$\int_{SM} f |f|^p dv = 0, \int_{SM} |f|^p dv = 1. \quad (19)$$

Suppose that $(M^n, F_t)$ is a solution of the Ricci flow on the smooth manifold $(M^n, F_0)$ in the interval $[0, T)$ and

$$\lambda(t) = \int_{SM} |\tilde{\nabla} f(x,y)|^p dv, \quad (20)$$

defines the evolution of an eigenvalue of $P$-Laplacian under the variation of $F_t$ whose eigenfunction associated to $\lambda(t)$ is normalized. Suppose that for any metric $g(t)$ on $M^n$

$$Spec_p(g) = \{0 = \lambda_0(g) \leq \lambda_1(g) \leq \lambda_2(g) \leq ... \leq \lambda_k(g) \leq ...\}$$

is the spectrum of $\Delta_p = \bar{g} \Delta_p$. In what follows we assume the existence and $C^1$-differentiability of the elements $\lambda(t)$ and $f(t)$, under a Ricci flow deformation $g(t)$ of a given initial metric. We prove some propositions about the problem of the spectrum variation under a deformation of the metric given by a Ricci flow equation.

### 3. Variation of $\lambda(t)$

In this part, we will give some useful evolution formulas for $\lambda(t)$ under the Ricci flow. Let $(M^n, F_t)$, $t \in [0, T)$, be a deformation of Finsler metric $F_0$. Assume that $\lambda(t)$ is the eigenvalue of $\Delta_p$, $f = f(x,y,t)$ satisfies

$$\Delta_p f + \lambda |f|^{p-2} f = 0$$

and $\int_{SM} |f|^p dv = 1$, using (12), we have:

$$\frac{d}{dt} |\tilde{\nabla} f|^2 = \frac{\partial}{\partial t} (g^{ij}) \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} + g^{ij} \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial x^i} \right) \frac{\partial f}{\partial x^j}$$

$$+ \ g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial x^j} \right) + \frac{\partial (F^2)}{\partial t} g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}
$$

$$+ \ F^2 \frac{\partial}{\partial t} (g^{ij}) \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} + 2F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}. \quad (21)$$

where

$$\frac{\partial}{\partial t} (g^{ij}) = -g^{il} g^{jk} \frac{\partial}{\partial t} (g_{lk}) \quad (22)$$
and
\[ \frac{\partial}{\partial t} \left( \frac{\delta f}{\delta x^i} \right) = \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial x^i} - G'_i \frac{\partial f}{\partial y^r} \right) \]
\[ = \frac{\partial f'}{\partial x^i} - G'_i \frac{\partial f'}{\partial y^r} - \frac{\partial}{\partial t} \left( G'_i \right) \frac{\partial f}{\partial y^r} \]
\[ = \frac{\delta f'}{\delta x^i} - \frac{\partial}{\partial t} \left( G'_i \right) \frac{\partial f}{\partial y^r} \]  

therefore, a substitution of (22) and (23) in (21), implies that:

**Proposition 3.1.** Let \((M^n, F_t)\) be a deformation of Finsler manifold \((M^n, F_0)\), then

\[ \frac{d}{dt} \left| \tilde{\nabla} f \right|^p = \frac{p}{2} \tilde{\nabla} f \left| f \right|^{p-2} \left\{ -g^{ij} g^{jk} \frac{\partial}{\partial t} (g_{ik}) \frac{\delta f}{\delta x^i} \frac{\delta f}{\delta x^j} + 2g^{ij} \frac{\delta f'}{\delta x^i} \frac{\delta f}{\delta x^j} \right\} \]
\[ -2g^{ij} \frac{\partial}{\partial t} \left( G'_i \right) \frac{\partial f}{\partial y^r} \left( g_{rk} \right) \frac{\delta f}{\delta y^i} \frac{\delta f}{\delta y^j} \]
\[ -F^2 g^{ij} g^{jk} \frac{\partial}{\partial t} \left( g_{ik} \right) \frac{\delta f}{\delta y^i} \frac{\delta f}{\delta y^j} + 2F^2 g^{ij} \frac{\partial f'}{\partial y^r} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \} \]

On the other hand we have

\[ \frac{d}{dt} (dv) = \left\{ g^{ij} \frac{\partial}{\partial t} (g_{ij}) - n \frac{\partial}{\partial t} (\log F) \right\} dv. \]  

(24)

Now, we get the following two integrability conditions:

\[ 0 = \frac{d}{dt} \int_{S_M} |f|^{p-2} f dv = (p - 1) \int_{S_M} |f|^{p-2} f' dv + \int_{S_M} |f|^{p-2} f \frac{d}{dt} dv \]

therefore

\[ (p - 1) \int_{S_M} |f|^{p-2} f' dv = - \int_{S_M} |f|^{p-2} f \left\{ g^{ij} \frac{\partial}{\partial t} (g_{ij}) - n \frac{\partial}{\partial t} (\log F) \right\} dv \]  

(25)

and

\[ 0 = \frac{d}{dt} \int_{S_M} |f|^p dv = p \int_{S_M} f f' |f|^{p-2} dv + \int_{S_M} |f|^p \frac{d}{dt} dv \]

which implies

\[ p \int_{S_M} f f' |f|^{p-2} dv = - \int_{S_M} |f|^p \left\{ g^{ij} \frac{\partial}{\partial t} (g_{ij}) - n \frac{\partial}{\partial t} (\log F) \right\} dv. \]  

(26)
Now, if we suppose that $g(t)$ is a solution of the un-normalized Ricci flow (1) and (2), then we have:
\[
\frac{d\lambda}{dt} = \int_{SM} \left( \frac{d}{dt} (\nabla f)^p \right) dv + \int_{SM} \nabla f d\lambda dt (dv)
\]
\[
= \frac{p}{2} \int_{SM} \left\{ g^{ij} \left( \frac{\partial}{\partial x^i} (G_t^r) \frac{\partial f}{\partial y^j} + 2g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} ight) - 2g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \right\} (\nabla f)^p dv
\]
\[
+ \int_{SM} (2F^2 (\tilde{\nabla} f)^p - 2g^{ij} \tilde{\nabla} f^p - n (\tilde{\nabla} f)^p) dv
\]
\[
= \frac{p}{2} \int_{SM} \left\{ \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} - 2g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \right\} (\nabla f)^p dv
\]
\[
+ \int_{SM} (g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} - n (\tilde{\nabla} f)^p) dv
\]
where $\frac{d}{dt} (G_t^r)$ is obtained as follows:
\[
G_t^r = \frac{1}{4} g^{il} \left\{ 2 \frac{\partial g_{it}}{\partial x^k} - \frac{\partial g_{it}}{\partial x^k} \right\} y^i y^k, \quad G_t^r = \frac{\partial G_t^r}{\partial y^i}
\]
\[
\frac{d}{dt} (G_t^r) = -\frac{1}{4} g^{il} g_{s}^{\alpha} \frac{\partial}{\partial x^i} (g_{s}^{\alpha} g_{t}^{\beta} \frac{\partial (g_{s}^{\beta})}{\partial y^t} \left\{ 2 \frac{\partial g_{it}}{\partial x^k} - \frac{\partial g_{it}}{\partial x^k} \right\} y^i y^k
\]
\[
- \frac{1}{4} g^{il} g_{s}^{\alpha} \frac{\partial}{\partial x^i} (g_{s}^{\alpha} g_{t}^{\beta} \frac{\partial (g_{s}^{\beta})}{\partial y^t} \left\{ 2 \frac{\partial g_{it}}{\partial x^k} - \frac{\partial g_{it}}{\partial x^k} \right\} y^i y^k
\]
\[
- \frac{1}{4} g^{il} \frac{\partial}{\partial x^i} (g_{s}^{\alpha} g_{t}^{\beta} \frac{\partial (g_{s}^{\beta})}{\partial y^t} \left\{ 2 \frac{\partial g_{it}}{\partial x^k} - \frac{\partial g_{it}}{\partial x^k} \right\} y^i y^k
\]
\[
+ \frac{1}{4} g^{il} \left\{ 2 \frac{\partial g_{it}}{\partial y^i} \frac{\partial g_{it}}{\partial y^k} - \frac{\partial g_{it}}{\partial y^i} \frac{\partial g_{it}}{\partial y^k} \right\} y^i y^k
\]
\[
+ \frac{1}{4} g^{il} \frac{\partial}{\partial x^i} (g_{s}^{\alpha} g_{t}^{\beta} \frac{\partial (g_{s}^{\beta})}{\partial y^t} \left\{ 2 \frac{\partial g_{it}}{\partial x^k} - \frac{\partial g_{it}}{\partial x^k} \right\} y^i y^k
\]
\[
+ \frac{1}{4} g^{il} \frac{\partial}{\partial x^i} (g_{s}^{\alpha} g_{t}^{\beta} \frac{\partial (g_{s}^{\beta})}{\partial y^t} \left\{ 2 \frac{\partial g_{it}}{\partial x^k} - \frac{\partial g_{it}}{\partial x^k} \right\} y^i y^k
\]
(27)
From the un-normalized Ricci flow, we can then write

\[
\frac{d}{dt}(G_i^r) = \frac{1}{2} g^{ra} \frac{a_\alpha}{a_\beta} g_{ra} g_{ab} \left( \frac{\partial (g_{ab})}{\partial x^b} \left( 2 \frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^j} \right) y^j y^k \right)
\]

(28)

Using (26) we obtain

\[
p \int_{SM} \tilde{g}(\tilde{\nabla} f, \tilde{\nabla} f)|\tilde{\nabla} f|^{p-2} dv = p \lambda \int_{SM} f |f|^{p-2} dv
\]

\[
= -\lambda \int_{SM} |f|^p \{-2g^{ij}Ric_{ij} + nRic\} dv.
\]

We have thus proved the following proposition:

**Proposition 3.2.** Let \((M^n, F_t)\) be a solution of the un-normalized Ricci flow on the smooth Finsler manifold \((M^n, F_0)\). If \(\lambda(t)\) denotes the evolution of an eigenvalue under the Ricci flow, then

\[
\frac{d\lambda}{dt} = p \int_{SM} Ric(\tilde{\nabla} f, \tilde{\nabla} f)|\tilde{\nabla} f|^{p-2} dv
\]

\[
+ \int_{SM} (\lambda |f|^p - |\tilde{\nabla} f|^p) \left\{ 2g^{ij}Ric_{ij} - nRic \right\} dv
\]

(29)

\[
- \int_{SM} g^{ij} \frac{\partial}{\partial t}(G_i^r) \frac{\partial f}{\partial y^i} \frac{\delta f}{\partial x^j} \tilde{\nabla} f|^{p-2} dv
\]

\[
- \int_{SM} Ric F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\delta f}{\partial y^j} \tilde{\nabla} f|^{p-2} dv
\]

(30)

where \(f\) is the associated normalized evolving eigenfunction. \(\Box\)
Note. Let $f : SM \to \mathbb{R}$ be a lifting of $f : M \to \mathbb{R}$. We have:

\[
\frac{d\lambda}{dt} = p \int_{SM} \text{Ric}(\tilde{\nabla} f, \tilde{\nabla} f)|\tilde{\nabla} f|^p - 2 \mathcal{R} \text{dv}
\]

and in this case, if $-2g^{ij}\text{Ric}_{ij} + n\text{Ric}$ is a constant, then

\[
\frac{d\lambda}{dt} = p \int_{SM} \text{Ric}(\tilde{\nabla} f, \tilde{\nabla} f)|\tilde{\nabla} f|^p - 2 \mathcal{R} \text{dv}.
\]

Corollary 3.3. Let $(M^n, F_t)$ be a solution of the un-normalized Ricci flow on the smooth Riemannian manifold $(M^n, F_0)$, i.e. $F_t, F_0$ are Riemannian metric. If $\lambda(t)$ denotes the evolution of an eigenvalue under Ricci flow, then:

\[
\frac{d\lambda}{dt} = p \int_{SM} \text{Ric}(\tilde{\nabla} f, \tilde{\nabla} f)|\tilde{\nabla} f|^p - 2 \mathcal{R} \text{dv}
\]

where $\mathcal{R}$ is the scalar curvature of $M$.

Proof: If $F_t$ is the Riemannian metric, then

\[
\text{Ric} = \frac{1}{n} R,
\]

and

\[
2g^{ij}\text{Ric}_{ij} - n\text{Ric} = R,
\]

therefore (31) is obtained by replacing (32) and (33) in (29).

Corollary 3.4. Let $(M^2, F_t)$ be a solution of the un-normalized Ricci flow on the smooth Riemannian surface $(M^2, F_0)$. If $\lambda(t)$ denotes the evolution of an eigenvalue under the Ricci flow, then:

\[
\frac{d\lambda}{dt} = \frac{p}{2} \int_{SM} R|\tilde{\nabla} f|^p \text{dv} + \int_{SM} R(\lambda|f|^p - |\tilde{\nabla} f|^p) \text{dv}
\]

where $\mathcal{R}$ is the scalar curvature of $M$.

Proof: In dimension $n = 2$, for a Riemannian manifold, we have:

\[
\text{Ric} = \frac{1}{2} \mathcal{R} g,
\]
hence the corollary is obtained by replacing (34) in (29). □

**Corollary 3.5.** Let \((M^n, F_t)\) be a solution of the un-normalized Ricci flow on the smooth homogenous Riemannian manifold \((M^n, F_0)\). If \(\lambda(t)\) denotes the evolution of an eigenvalue under the Ricci flow, then:

\[
\frac{d\lambda}{dt} = p \int_{SM} \text{Ric}(\nabla f, \nabla f) |\nabla f|^{p-2} dv
- p \int_{SM} g^{ij} \partial_t (G^i_r) \frac{\partial f}{\partial y^r} \frac{\partial f}{\partial \delta x^j} |\nabla f|^{p-2} dv
- R \frac{p}{n} \int_{SM} F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\nabla f|^{p-2} dv
\]

where \(R\) is the scalar curvature of \(M\).

**Proof:**

Since the evolving metric remains homogenous and a Riemannian homogenous manifold has constant scalar curvature, so the corollary is obtained by (29). □

Now, we give a variation of \(\lambda(t)\) under the normalized Ricci flow which is similar to the previous proposition.

**Proposition 3.6.** Let \((M^n, F_t)\) be a solution of the normalized Ricci flow on the smooth Finsler manifold \((M^n, F_0)\). If \(\lambda(t)\) denotes the evolution of an eigenvalue under Ricci flow, then:

\[
\frac{d\lambda}{dt} = (n-p) r \lambda + p \int_{SM} \text{Ric}(\nabla f, \nabla f) |\nabla f|^{p-2} dv
+ \int_{SM} (\lambda |f|^p - |\nabla f|^p) \{2g^{ij} \text{Ric}_{ij} - nRic\} dv \tag{35}
- p \int_{SM} g^{ij} \partial_t (G^i_r) \frac{\partial f}{\partial y^r} \frac{\partial f}{\partial \delta x^j} |\nabla f|^{p-2} dv
- p \int_{SM} F^2 (\text{Ric} - r) g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\nabla f|^{p-2} dv
\]

where \(f\) is the associated normalized evolving eigenfunction, \(r = \frac{\int_{SM} \text{Ric} dv}{v_{vol}(SM)}\).

**Proof:** In the normalized case, the integrability conditions read as follows

\[
p \int_{SM} f'|f|^{p-2} dv = \int_{SM} |f|^{p} \{2g^{ij} \text{Ric}_{ij} - nr - nRic\} dv. \tag{36}
\]

Since

\[
\frac{d}{dt} (dv) = \{-2g^{ij} \text{Ric}_{ij} + nr + nRic\} dv \tag{37}
\]
using (24), (27) and the above equation, we can then write

\[
\frac{d\lambda}{dt} = \int_{SM} \left( \frac{\partial}{\partial t} (\nabla^f p) \right) dv + \int_{SM} (\nabla^f p^2) dv \tag{38}
\]

\[
= \frac{p}{2} \int_{SM} \left\{ -g^{ij} g^{jk} (-2Ric_{ik} + 2r g_{ik}) \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} + 2g_{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right\} \nabla f |^2 dv
\]

\[
+ \frac{p}{2} \int_{SM} \left\{ 2g^{ij} \left( -Ric_{ij} + r g_{ij} \right) \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \right\} (\nabla f) |^2 dv
\]

\[
+ \lambda \int_{SM} \left\{ -2g^{ij} Ric_{ij} + nr + nRic \right\} dv
\]

but

\[
p \int_{SM} \tilde{g} (\tilde{\nabla} f', \tilde{\nabla} f) |^2 dv = p \lambda \int_{SM} f' f |^2 dv
\]

\[
= \lambda \int_{SM} \left\{ 2g^{ij} Ric_{ij} - nr - nRic \right\} dv \tag{39}
\]

and \( \frac{\partial}{\partial t} (G^t) \) is obtained by replacing \( F' \) and \( g_{ij}' \) from (3) and (4), respectively, in (27). Thus the proposition is obtained by replacing (39) in (38). □

Similar to un-normalized case we have the following corollaries:

**Corollary 3.7.** Let \((M^n, F_0)\) be a solution of the normalized Ricci flow on the smooth Riemannian manifold \((M^n, F_0)\). If \(\lambda(t)\) denotes the evolution of an eigenvalue under Ricci flow, then:

\[
\frac{d\lambda}{dt} = (n - p)r \lambda + p \int_{SM} \left( Ric (\tilde{\nabla} f, \tilde{\nabla} f) \right) (\tilde{\nabla} f) |^2 dv
\]

\[
+ \int_{SM} R(\lambda |^2 f - (\tilde{\nabla} f)^2) dv - p \int_{SM} g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \tilde{\nabla} f |^2 dv \tag{40}
\]

\[
- p \int_{SM} \left( \frac{1}{n} R - r \right) F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \tilde{\nabla} f |^2 dv
\]

where \( R \) is the scalar curvature of \( M \).

**Corollary 3.8.** Let \((M^2, F_0)\) be a solution of the normalized Ricci flow on the smooth Riemannian surface \((M^2, F_0)\). If \(\lambda(t)\) denotes the evolution of an eigenvalue
under Ricci flow, then:

\[
\frac{d\lambda}{dt} = (2 - p)r \lambda + \frac{p}{2} \int_{SM} R|\nabla f|^p dv \\
+ \int_{SM} R(\lambda|f|^p - |\tilde{\nabla} f|^p) dv - p \int_{SM} g^{ij} \frac{\partial}{\partial y^i} (G^s_i) \frac{\partial f}{\partial y^s} \frac{\delta f}{\delta x^j} |\tilde{\nabla} f|^p dv \\
- p \int_{SM} \left(\frac{R}{2} - r\right) F^2 g^{ij} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^p dv
\]

where \( R \) is the scalar curvature of \( M \).

**Corollary 3.9.** Let \((M^n, F_0)\) be a solution of the normalized Ricci flow on the smooth homogenous Riemannian manifold \((M^n, F_0)\). If \(\lambda(t)\) denotes the evolution of an eigenvalue under Ricci flow, then:

\[
\frac{d\lambda}{dt} = (n - p)r \lambda + \frac{p}{2} \int_{SM} \text{Ric}(\tilde{\nabla} f, \tilde{\nabla} f)|\tilde{\nabla} f|^p dv \\
- p \int_{SM} g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^p dv \\
- p \int_{SM} \left(\frac{1}{n}R - r\right) F^2 g^{ij} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^p dv
\]

where \( R \) is the scalar curvature of \( M \).

### 4. Examples

In this section, we will find the variational formula for some of Finsler manifolds.

**Example 4.1.** Let \((M^n, F_0)\) be an Einstein manifold i.e. there exists a constant \( a \) such that \( \text{Ric}(F_0) = aF_0^2 \). Therefore \( \text{Ric}_{ij}(g_0) = ag_{ij}(0) \). Assume we have a solution to the Ricci flow which is of the form

\[
g(t) = u(t)g_0, \quad u(0) = 1
\]

where \( u(t) \) is a positive function. Now (2) implies that

\[
u(t) = -2at + 1,
\]

so that we have

\[
g(t) = (1 - 2at)g_0
\]

which says that \( g(t) \) is an Einstein metric. On the other hand it is easily seen that

\[
\text{Ric}(g(t)) = \text{Ric}(g_0) = ag_0 = \frac{a}{1 - 2at} g(t),
\]

\[
\mathcal{R}(g_t) = \frac{1}{1 - 2at} \mathcal{R}(g_0) = \frac{a}{1 - 2at},
\]

\[
F^2_t = (1 - 2at)F^2_0
\]

therefore

\[
2g^{ij} \text{Ric}_{ij} - n\text{Ric} = \frac{an}{1 - 2at}
\]
and
\[ \text{Ric}(\tilde{\nabla} f, \tilde{\nabla} f) = \frac{a}{1 - 2at} \tilde{g}(\tilde{\nabla} f, \tilde{\nabla} f) = \frac{a}{1 - 2at} |\tilde{\nabla} f|^2. \]

Also
\[ G^i(t) = \frac{1}{4} g^{ij} \left\{ 2 \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^l} \right\} y^j y^k \]
\[ = \frac{1}{4} (g_0)^{ij} \left\{ 2 \frac{\partial (g_0)_{il}}{\partial x^k} - \frac{\partial (g_0)_{ik}}{\partial x^l} \right\} y^j y^k = G^i(0) \]

therefore
\[ \frac{\partial}{\partial t} (G^i_r) = 0. \]

Using the un-normalized Ricci flow equation (2) and (29), we obtain the following relation:
\[ \frac{d\lambda}{dt} = p \int_M \frac{a}{1 - 2at} |\tilde{\nabla} f|^p dv + \lambda \int_M |f|^p a \frac{1}{1 - 2at} dv \]
\[ - \int_M \frac{a}{1 - 2at} |\tilde{\nabla} f|^p dv - p \int_M \frac{a}{1 - 2at} F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^{p-2} dv \]
\[ = \frac{pa}{1 - 2at} \left\{ \lambda - \int_{SM} F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^{p-2} dv \right\} \]

Now, if we suppose that \( g_t = u(t) g_0 \), \( u(0) = 1 \) is a solution of the normalized Ricci flow and \( r = \frac{1}{\sqrt{\text{Vol} (SM)}} \int_{SM} \text{Ric} dv \), then from (3) we have
\[ u' g_0 = -2a g_0 + 2ru g_0. \]

It implies that
\[ u(t) = \frac{a}{r} \left( 1 - e^{2rt} \right) + e^{2rt}. \]

So that we have:
\[ g(t) = \left\{ \frac{a}{r} \left( 1 - e^{2rt} \right) + e^{2rt} \right\} \]

and
\[ \text{Ric}(g(t)) = a \left( \frac{a}{r} \left( 1 - e^{2rt} \right) + e^{2rt} \right)^{-1} g_0 \]
\[ \text{Ric}(g(t)) = a \left( \frac{a}{r} \left( 1 - e^{2rt} \right) + e^{2rt} \right)^{-1} \]
\[ 2g^{ij} \text{Ric}_{ij} - \text{Ric}_{ij} = an \left( \frac{a}{r} \left( 1 - e^{2rt} \right) + e^{2rt} \right)^{-1}, \]

also
\[ G^i(t) = G^i(0), \]

therefore
\[ \frac{\partial G^i_r}{\partial t} = 0. \]
Using (35), we obtain the following:
\[
\frac{d\lambda}{dt} = (n - p)\lambda + pa\lambda \left( \frac{a}{r} - 1 \right) + \bar{\lambda}_j \partial f / \partial y_j \left( y^i \frac{\partial f}{\partial y^i} \right) |\nabla f|^{p-2} dv
\]

\[
\begin{align*}
&= \left( n - p \right) \lambda + pa\left( \frac{a}{r} - 1 \right) \lambda + \int_{\mathcal{M}} \left( y^i \frac{\partial f}{\partial y^i} \right) \left( y^i \frac{\partial f}{\partial y^i} \right) |\nabla f|^{p-2} dv \\
&= \left( n - p \right) \lambda + pa\left( \frac{a}{r} - 1 \right) \lambda + \int_{\mathcal{M}} \left( y^i \frac{\partial f}{\partial y^i} \right) \left( y^i \frac{\partial f}{\partial y^i} \right) |\nabla f|^{p-2} dv.
\end{align*}
\]

**Remark.** Let \((M^n, F_0)\) be a Finsler manifold of dimension \(n \geq 3\). Suppose that the flag curvature \(k = k(x)\) is isotropic and a function of \(x \in M\) alone then \(k = \text{constant}\) and therefore \((M^n, F_0)\) is Einstein and the variation of its eigenvalues is similar to example (4.1).

**Definition 4.2.** A Finsler metric on an \(n\)-dimensional manifold is called a weak Einstein metric if
\[\text{Ric} = (n - 1) \left( \frac{3\eta}{F} + \sigma \right) F^2\]
where \(\eta\) is a 1-form and \(\sigma = \sigma(x)\) is scalar function.

**Example 4.3.** If we suppose that \(F_t = u(t)F_0, u(0) = 1\) is a solution of the Ricci flow, then:
\[\text{Ric}(F_t) = \frac{\text{Ric}(F_t)}{F_t^2} = \frac{\text{Ric}(F_0)}{(u(t))^2 F_0^2} = \frac{(n - 1) \left( \frac{3\eta}{F_0} + \sigma_0 \right)}{(u(t))^2 F_0^2} = \frac{(n - 1) \left( \frac{3\eta}{F_0} + \sigma_0 \right)}{(u(t))^2} \]

Now the Ricci flow (1) implies that
\[\frac{(n - 1) \left( \frac{3\eta}{F_0} + \sigma_0 \right)}{(u(t))^2} = -\text{Ric} = \frac{\partial \log F}{\partial t} = \frac{F_t'}{F_t} = \frac{u'(t)}{u(t)}\]
or equivalently
\[uu' = (n - 1) \left( \frac{3\eta_0}{F_0} + \sigma_0 \right)\]

By integration we have:
\[u^2(t) = -2(n - 1) \left( \frac{3\eta_0}{F_0} + \sigma_0 \right) t + c\]
with condition \(u(0) = 1\) we have:
\[u^2(t) = 1 - 2(n - 1) \left( \frac{3\eta_0}{F_0} + \sigma_0 \right) t\]
therefore
\[F_t^2 = \left\{ 1 - 2(n - 1) \left( \frac{3\eta_0}{F_0} + \sigma_0 \right) t \right\} F_0^2 \] (41)
and
\[ \text{Ric}(\tilde{\nabla} f, \tilde{\nabla} f) = \frac{1}{2} (n-1) g^{ij} \{ (3 \eta_0 F_0)_{y^i y^j} + 2 \sigma_0 (g_0)_{ij} \} \tilde{\nabla}_i f \tilde{\nabla}_j f. \] (42)

Also we have
\[ 2g^{ij} \text{Ric}_{ij} - n \text{Ric} = (n-1) g^{ij} \{ (3 \eta_0 F_0)_{y^i y^j} + 2 \sigma_0 (g_0)_{ij} \} - n \frac{(n-1) \left( \frac{3 \eta_0}{F_0} + \sigma_0 \right)}{1 - 2(n-1) \left( \frac{3 \eta_0}{F_0} + \sigma_0 \right) t}. \] (43)

Note that
\[ g_{ij}(t) = (1 - 2(n-1) \sigma_0 t) (g_0)_{ij} - 6(n-1) t \frac{\partial \eta}{\partial y^i} \partial F \partial y^j. \] (44)

By replacing (42), (43) and (44) in (29) we obtain the variation of an eigenvalue.

Now, if we suppose that \( F_t = u(t) F_0, \ u(0) = 1 \) is a solution of the normalized Ricci flow and \( r = \frac{1}{\text{Vol}(SM)} \int_{SM} \text{Ric} dv \), then from (3), we have:
\[ r - \frac{(n-1) \frac{3 \eta_0}{F_0} + \sigma_0}{u^2(t)} = r - \text{Ric} = \frac{\partial \log F}{\partial t} = \frac{u'(t)}{u(t)}. \]
It implies that
\[ uu' - u^2 r = -(n-1) \frac{3 \eta_0}{F_0} + \sigma_0, \ u(0) = 1 \]
which is an ordinary differential equation and has a solution as follow:
\[ u^2(t) = \frac{n-1}{r} \frac{3 \eta_0}{F_0} + \sigma_0 (1 - e^{2rt}) + e^{2rt} \]
therefore
\[ F_t^2 = \left\{ \frac{n-1}{r} \frac{3 \eta_0}{F_0} + \sigma_0 (1 - e^{2rt}) + e^{2rt} \right\} F_0^2 \]
and \( \frac{d \lambda}{dt} \) is obtained from (35).

**Example 4.4.** In this example we determine the behavior of the evolving spectrum on the Ricci solitons.

Let \( F_t \) is a solution of the Ricci flow \( \frac{\partial \log F}{\partial t} = -\text{Ric} \). If \( \varphi \) is a time-independent isometry such that
\[ F(x, y) = \varphi^* F_t(x, y) \]
is a solution of the un-normalized Ricci flow, because of \( \frac{\partial}{\partial t} \log F_t(x, y) = -\text{Ric}(x, y), F(0) = F_0 \),
\[ \frac{\partial}{\partial t} \log F_t(x, y) = \frac{\partial}{\partial t} \varphi^* \log F_t(x, y) = \frac{\partial}{\partial t} \log F_t(\varphi(x), \varphi^*(-y)) \]
since \( \varphi \) is a isometry we have
\[ \frac{\partial}{\partial t} \log F_t(x, y) = \frac{\partial}{\partial t} \log F_t(x, y) = -\text{Ric}(x, y) = -\text{Ric}(\varphi(x), \varphi^*(-y)) = -\text{Ric}(F) \]

**Definition 4.5.** Let \((M, F_t)\) is a solution of the Ricci flow and \( \varphi_t \) is a family of diffeomorphisms. We says \( F_t \) is Ricci soliton, when satisfies in
\[ F_t^2 = u(t) \varphi_t^* F_0^2. \] (45)
Let $(M, F)$ and $(\overline{M}, \overline{F})$ be two closed Finsler manifolds and 
\[ \varphi : (M, g) \to (\overline{M}, \overline{F}) \]
an isometry, then for $p = 2$ we have 
\[ \overline{g} \Delta_p \circ \varphi^* = \varphi^* \overline{\Delta}_p. \]
Therefore given a diffeomorphism $\varphi : M \to M$ we have that 
\[ \varphi : (SM, \varphi^* \overline{g}) \to (SM, \overline{g}) \]
is an isometry, hence we conclude that $(SM, \varphi^* \overline{g})$, and $(SM, \overline{g})$ have the same spectrum 
\[ \text{Spec}_p(\overline{g}) = \text{Spec}_p(\varphi^* \overline{g}) \]
with eigenfunction $f_k$ and $\varphi^* f_k$ respectively. If $g(t)$ is a Ricci soliton on $(M^n, g_0)$ then 
\[ \text{Spec}_p(\overline{g}(t)) = \frac{1}{u(t)} \text{Spec}_p(\overline{g}_0) \]
so that $\lambda(t)$ satisfies 
\[ \lambda(t) = \frac{1}{u(t)} \frac{d\lambda}{dt} = -\frac{u'(t)}{(u(t))^2}. \]

**Example 4.6.** Suppose that 
\[ \mathbb{R}^n_+ = \{(x^1, x^2, ..., x^n) \in \mathbb{R}^n | x^i > 0, \ i = 1, ..., n\} \]
has the metric 
\[ g_{ij}(x, y) = \begin{cases} \frac{\phi_\ast(y)}{(x^i)^{p+1}} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]
where it is a solution for the un-normalized Ricci flow and $\phi_\ast(y)$ is a strictly positive $C^\infty$ homogeneous function of degree zero and $p > 0$. We use the formula 
\[ g^{ij} = \begin{cases} \frac{(x^i)^{2(p+1)}}{\phi_\ast(y)} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]
for $i = 1, ..., n$, we have 
\[ G^i = -\frac{p + 1}{2p} \frac{(g^j)^2}{x^i} \]
and 
\[ F^2 \text{Ric}(g(t)) = 0 \]
therefore $\text{Ric}(g(t)) = 0$ and the un-normalized Ricci flow equation implies that 
\[ \frac{\partial g(t)}{\partial t} = 0 \]
hence, $g(t) = g_0$ and $\lambda(t) = \lambda(0)$. 

Example 4.7. Suppose that $\mathbb{R}^2$ has the metric
\[ g_{ij}(x, y) = \phi_t(y) \begin{pmatrix} 4(x_1)^2 + 1 & -2x_2 \n \end{pmatrix} \]
which is a solution for the the un-normalized Ricci flow, where $\phi_t(y)$ is a strictly positive $C^\infty$ homogeneous function of degree zero. We obtain
\[ G^1 = 0, \ G^2 = -(y_1)^2 \]
and
\[ F^2 \text{Ric}(g(t)) = 0 \]
therefore $\text{Ric}(g(t)) = 0$ and the un-normalized Ricci flow equation implies that
\[ \frac{\partial g(t)}{\partial t} = 0 \]
hence, $g(t) = g_0$ and $\lambda(t) = \lambda(0)$.

References
Eigenvalues variation of the $P$-Laplacian


