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Extreme Vertices of the Psi-Divisible Graph of the Group $\mathbb{Z}_{p^n}$

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Abstract. The Ψ-divisible graph of a finite group G, denoted by Ψ_G is a special type of simple undirected graph, in which the set of vertices contains non-trivial subgroups of G and two distinct vertices u and v are adjacent if and only if u is a proper subgroup of v such that $\Psi(u)|\Psi(v)$ or v is a proper subgroup of u such that $\Psi(v)|\Psi(u)$. The existence of extreme vertices in the Ψ-divisible graph of the finite cyclic group $\mathbb{Z}_{p^n}$ is described in this article.

Key words and Phrases: Extreme vertices, Ψ-divisible graph of a group, finite cyclic group $\mathbb{Z}_{p^n}$.

1. INTRODUCTION

Even though Cayley's work dates back to the nineteenth century [1], the connection between group and graph theories became well known in the 1950s. Numerous graphs are defined, particularly on a finite group G. Without going too far, we recall the following:

(1) Chakrabarty et al. [2] initially investigated the power graph of a finite group G, $\mathbb{P}(G)$, as the graph whose vertex set consists of the elements of the finite group G and two different vertices $x_1 \neq x_2$ are adjacent if and only if $\langle x_2 \rangle \subseteq \langle x_1 \rangle$ or $\langle x_1 \rangle \subseteq \langle x_2 \rangle$. (See [3, 4])
(2) For a finite group G, the reduced power graph is a graph with a set of vertices G such that two vertices $x, y \in G$ are adjacent if and only if $x \neq y$ and $\langle x \rangle \subsetneq \langle y \rangle$ or $\langle y \rangle \subsetneq \langle x \rangle$. (See [5, 6, 7, 8])
(3) For a finite group G, an equal-square graph [9] is an undirected graph with group G as its vertex set, having distinct vertices x and y adjacent if and only if $x^2 = y^2$.
(4) The undirected superpower graph of a finite group G, $\mathbb{S}(G)$ is a simple graph in which the set of vertices G and two distinct vertices u, v are

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adjacent if $o(u)|o(v)$ or $o(v)|o(u)$. The concept of $\mathbb{S}(G)$ is an extension of the power graph $\mathbb{P}(G)$ of the finite group G.(See [10, 11, 12, 13])

(5) For a finite abelian group $(G, +)$, a square power graph [14] is an undirected graph with an abelian group G as its vertex set, having distinct vertices x and y adjacent if and only if $x + y = 2z$ for some $z \in G$ and $2z \neq 0$.

Numerous graphs, including the co-maximal ideal graph, co-prime graph, co-prime order graph, commuting graph, intersection graph, Gruenberg-Kegel graph, zero-divisor graphs, and many more, are associated with algebraic structures in the literature. In [15], the concept of a Ψ-divisible graph was introduced.

The Ψ-divisible graph of a finite group G, denoted by Ψ_G is a special type of simple undirected graph in which the set of vertices contains non-trivial subgroups of G and two distinct vertices u and v are adjacent if and only if u is a proper subgroup of v such that $\Psi(u)|\Psi(v)$ or v is a proper subgroup of u such that $\Psi(v)|\Psi(u)$.

The Ψ-divisible graph of a group has emerged as a significant topic of research, with vertices representing subgroups and edges linking elements that exhibit a particular algebraic relationship, generally concerning the divisible order sum properties of the group's subgroups. The connection between Ψ-divisibility and the square-free order property of finite groups was further investigated in [16]. Some results Ψ-divisible graph for finite cyclic groups were investigated in [17].

In a graph Γ, if the neighborhood of a vertex u induces a complete subgraph, then the vertex u is said to be an extreme vertex [18]. Important research on the extreme vertices of the power graph for the finite abelian, dicyclic, and dihedral groups was carried out by AbuGhneim et al. [19]. Kumari et al. [18] extended this work to the power graph for the permutation group S_n.

In this article, the extreme vertices of the Ψ-divisible graph of a finite cyclic group $\mathbb{Z}_{p^n}$ are examined. The rest of the paper is organized as follows. The fundamental findings of group theory and graph theory are presented in Section 2. Our study of the extreme vertices of the Ψ-divisible graph of a finite cyclic group $\mathbb{Z}_{p^n}$ is presented in Section 3.

2. Notations and Preliminaries

This section provides an overview of the notation and fundamental results used throughout the paper. The following notations are used:

$o(x)$	order of element x,
$\Psi(H)$	sum of elements orders of H,
Ψ_G	Ψ-divisible graph of G,
uv	vertices u, v are adjacent,
$H \leqslant K$	H is subgroup of K.

In addition, we make use of the following key results from group theory.

Lemma 2.1. [20] *"Every subgroup of a cyclic group is cyclic. Moreover, if $|\langle a\rangle| = n$, then the order of any subgroup of $\langle a\rangle$ is a divisor of n; and, for each positive divisor k of n, the group $\langle a\rangle$ has exactly one subgroup of order k".*

Lemma 2.2. [20] *"If H and K are two subgroups of a finite cyclic group G and $|H|||K|$, then $H \leqslant K$.".*

Lemma 2.3. [20] *"Group $\mathbb{Z}_{p^n}$ is always a cyclic group of order p^n."*

Theorem 2.4. [17] *"Let G be a finite cyclic group of order p^n where p is a prime. If H is a subgroup of order p^k, then degree of vertexH in Ψ-divisible graph of group G is $\tau(2k+1) + \lfloor \frac{1}{2}(\frac{2n+1}{2k+1} - 1)\rfloor - 2$ for all $k \in \{1, 2, \ldots, n\}$."*

3. Main Results

Lemma 3.1. *Let H be a cyclic subgroup of order p^k from the group G where p is prime, then $\Psi(H) = \frac{p^{2k+1}+1}{p+1}$.*

Proof. Given that H is a finite cyclic subgroup group of order p^k, then H contains exactly $\phi(p^i)$ elements of order p^i where $i = 0, 1, \ldots, k$. So, $\Psi(H) = \sum_{i=0}^{k} p^i \phi(p^i) = p^{2k} - p^{2k-1} + p^{2k-2} - \ldots + p^2 - p + 1 = \frac{p^{2k+1}+1}{p+1}$. $\square$

Lemma 3.2. *Let H and K be cyclic subgroups of a finite p-group G of orders p^{k_1} and p^{k_2}, respectively. Then*

$$\Psi(H) \mid \Psi(K) \quad \textit{if and only if} \quad 2k_1 + 1 \mid 2k_2 + 1.$$

Proof. Recall that for a cyclic subgroup of order p^m, we have $\Psi(p^m) = \frac{p^{2m+1}+1}{p+1}$. Hence $\Psi(H) \mid \Psi(K) \iff \frac{p^{2k_1+1}+1}{p+1} \mid \frac{p^{2k_2+1}+1}{p+1} \iff p^{2k_1+1} + 1 \mid p^{2k_2+1} + 1$.

We use the identity (valid for odd m, n) $\gcd(p^m + 1, p^n + 1) = p^{\gcd(m,n)} + 1$. Setting $m = 2k_1 + 1$ and $n = 2k_2 + 1$, both odd, gives $\gcd(p^{2k_1+1} + 1, p^{2k_2+1} + 1) = p^{\gcd(2k_1+1, 2k_2+1)} + 1$.

First, if $p^{2k_1+1} + 1 \mid p^{2k_2+1} + 1$, then the gcd equals the smaller number, so $p^{2k_1+1} + 1 = p^{\gcd(2k_1+1,2k_2+1)} + 1$, which implies $\gcd(2k_1 + 1, 2k_2 + 1) = 2k_1 + 1$, hence $2k_1 + 1 \mid 2k_2 + 1$.

Conversely, if $2k_1 + 1 \mid 2k_2 + 1$, write $2k_2 + 1 = s(2k_1 + 1)$ with s odd. Then $p^{2k_2+1}+1 = p^{(2k_1+1)s}+1$. Since s is odd, using $a^s+1 = (a+1)(a^{s-1}-a^{s-2}+\cdots-a+1)$ gives $p^{(2k_1+1)s}+1 = (p^{2k_1+1}+1)Q$ for some integer Q. Thus $p^{2k_1+1}+1 \mid p^{2k_2+1}+1$, and dividing by $p + 1$ yields $\Psi(H) = \frac{p^{2k_1+1}+1}{p+1} \mid \frac{p^{2k_2+1}+1}{p+1} = \Psi(K)$.

Therefore $\Psi(H) \mid \Psi(K) \iff 2k_1 + 1 \mid 2k_2 + 1$. $\square$

We examine the conditions under which the nontrivial subgroups of a group G cannot serve as extreme vertices in the Ψ-divisible graph of the finite group G.

Theorem 3.3. *Let H be a nontrivial cyclic subgroup of order p^k of the group G where p is the prime number. If $2k+1$ is divisible by two distinct primes, then the cyclic subgroup H cannot be an extreme vertex in the Ψ-divisible graph of the finite group G.*

Proof. Let p_1 and p_2 be two different prime divisors of $2k+1$, so both must be odd. It is given that H is a cyclic subgroup of order p^k, so there exists a unique non-trivial subgroup of each order p^i where $i = 1, \ldots, k$.

Taking $i = \frac{p_1-1}{2}$, there exists a unique subgroup H_1 of order $p^{\frac{p_1-1}{2}}$ from H such that $\Psi(H_1) = \frac{p^{p_1}+1}{p+1}$. Here $p_1|2k+1$, so $\Psi(H_1)|\Psi(H)$. Hence, H and H_1 are adjacent in the Ψ -divisible graph of the finite group G.

Similarly, take $i = \frac{p_2-1}{2}$, then there exists a unique subgroup H_2 of order $p^{\frac{p_2-1}{2}}$ from H such that $\Psi(H_2) = \frac{p^{p_2}+1}{p+1}$. Here $p_2|2k+1$, so $\Psi(H_2)|\Psi(H)$. Hence, H and H_2 are adjacent in the Ψ -divisible graph of the finite group G.

Here, H adjacent to both H_1 and H_2 however, they are not adjacent to each other because neither $\Psi(H_1)$ divides $\Psi(H_2)$ nor $\Psi(H_2)$ divides $\Psi(H_1)$. Thus, H cannot be an extreme vertex of the finite group G. $\square$

According to Theorem 3.3, if a nontrivial cyclic subgroup H of order p^k of a finite group G is an extreme vertex in the Ψ-divisible graph of the finite group G, then $2k+1$ must be a prime power.

Theorem 3.4. *If H is a non-trivial cyclic subgroup of order p^k from a finite group G and there exist two distinct cyclic subgroups H_1 and H_2 of order p^{k_1} and p^{k_2} that properly include H with the condition that $2k+1|2k_1+1$ and $2k+1|2k_2+1$. If $gcd(\frac{2k_1+1}{2k+1}, \frac{2k_2+1}{2k+1}) = 1$, then H cannot be an extreme vertex in the Ψ-divisible graph of the finite group G.*

Proof. The subgroup H is a proper subgroup of H_1 and H_2, then $k_1 > k$ and $k_2 > k$. So, $\frac{2k_1+1}{2k+1}$ and $\frac{2k_2+1}{2k+1}$ are integers greater than 1. By the fundamental theorem of arithmetic, every integer greater than 1 can be decomposed into prime factors. Using the concept $gcd(\frac{2k_1+1}{2k+1}, \frac{2k_2+1}{2k+1}) = 1$, there exist two distinct odd primes p_1 and p_2 such that $p_1|\frac{2k_1+1}{2k+1}$ and $p_2|\frac{2k_1+1}{2k+1}$

$\implies (2k+1)p_1|2k_1+1$ and $(2k+1)p_2|2k_2+1$.

It is given that H_1 is a cyclic subgroup of order p^{k_1}, so there exists a unique nontrivial cyclic subgroup of each order p^i where $i = 1, \ldots, k_1$. In addition, the subgroup of order p^j properly includes H when $j = k+1, k+2, \ldots, k_1$.

Taking $j = \frac{(2k+1)p_1-1}{2} \in \{k+1, k+2, \ldots, k_1\}$, there exists a unique subgroup H_3 of order $p^{\frac{(2k+1)p_1-1}{2}}$ from H_1 that properly includes H such that $\Psi(H)|\Psi(H_3)$. Hence, H and H_3 are adjacent in the Ψ -divisible graph of the finite group G.

It is given that H_2 is a cyclic subgroup of order p^{k_2}, so there exists a unique nontrivial cyclic subgroup of each order p^i where $i = 1, \dots, k_2$. In addition, the subgroup of order p^j properly includes H when $j = k+1, k+2, \dots, k_2$.

Taking $j = \frac{(2k+1)p_2-1}{2} \in \{k+1, k+2, \dots, k_2\}$, there exists a unique subgroup H_4 of order $p^{\frac{(2k+1)p_2-1}{2}}$ from H_2 that properly includes H such that $\Psi(H)|\Psi(H_4)$. Hence, H and H_4 are adjacent in the Ψ-divisible graph of the finite group G.

Here, H adjacent to both H_3 and H_4 nevertheless, they are not adjacent to each other because neither $\Psi(H_3)|\Psi(H_4)$ nor $\Psi(H_4)|\Psi(H_3)$. Thus, H cannot be an extreme vertex in the Ψ -divisible graph of the finite group G. $\square$

Corollary 3.5. If H is a nontrivial cyclic subgroup of order p^k from a finite group G and there exists a cyclic subgroup of order $p^{p_2k+\frac{p_2-1}{2}}$ that properly includes H where p_2 is an odd prime other than 3, then H cannot be an extreme vertex in the Ψ -divisible graph of the finite group G.

Proof. Take $k_1 = 3k+1$ and $k_2 = 5k+2$ such that $2k+1|2k_1+1$ and $2k+1|2k_2+1$. It is given that there exists a cyclic subgroup H_1 of order $p^{p_2k+\frac{p_2-1}{2}}$ that properly includes H where p_2 is the odd prime other than 3, then $k_1 \leq p_2k + \frac{p_2-1}{2}$ and $k_2 \leq p_2k + \frac{p_2-1}{2}$. So, there exist two cyclic subgroups of order p^{k_1} and p^{k_2} that properly includes H. Using the above theorem, H cannot be an extreme vertex in the Ψ -divisible graph of the finite group G. $\square$

Corollary 3.6. If H is a nontrivial cyclic subgroup of order p^k from a finite group $\mathbb{Z}_{p^n}$ and $k \leq \frac{n-2}{5}$, then H cannot be an extreme vertex in the Ψ -divisible graph of the finite group G.

Proof. We know that there exists a unique subgroup of order p^i where $i = 0, 1, \dots, n$ from the group $\mathbb{Z}_{p^n}$ and each subgroup of order p^i where $i = k+1, k+2, \dots, n$ that properly includes H. So, there exists a cyclic subgroup of order p^{5k+2} that properly includes H. Taking $p_2 = 5$ in above corollary 3.5, then H cannot be an extreme vertex in the Ψ -divisible graph of the finite group $\mathbb{Z}_{p^n}$. $\square$

According to theorems 3.3 ,3.4 and corollary 3.6, if H be a nontrivial subgroup of $\mathbb{Z}_{p^n}$ is extreme vertices of order p^k in the Ψ -divisible graph of the finite group $\mathbb{Z}_{p^n}$, then $2k+1$ is a prime power and $k > \frac{n-2}{5}$.

Theorem 3.7. *If H is a nontrivial subgroup of $\mathbb{Z}_{p^n}$ is an extreme vertex of order p^k in the Ψ-divisible graph of the finite group $\mathbb{Z}_{p^n}$ if and only if $2k+1$ is an odd prime power and $k > \frac{n-2}{5}$.*

Proof. It is given that $2k+1$ is an odd prime power, so there exists an odd prime q such that $2k+1 = q^s$. The list of divisors of q^s greater than 1 is $q, q^2, \dots, q^s$ only.

Take $k_i = (q^i - 1)/2$ and $k_j = (q^j - 1)/2$ where $1 \leq i < j \leq s$, then there exist two cyclic subgroups H_{k_i} and H_{k_j} of order p^{k_i} and p^{k_j} such that $H_{k_i} \leqslant H_{k_j}$. So, $\Psi(H_{k_i}) = \frac{(p^{q^i}+1)}{p+1}$ and $\Psi(H_{k_j}) = \frac{(p^{q^j}+1)}{p+1}$.
Also, $q^i|q^j \implies 2k_i+1|2k_j+1$.

Hence, by Lemmas 2.2 and 3.2, we get $\Psi(H_{k_i})|\Psi(H_{k_j})$ and $H_{k_i} \leqslant H_{k_j}$. Therefore, H_{k_i} and H_{k_j} are adjacent to each other. If we take $j = s$, then $H = H_{k_s}$. Hence, we conclude that $H_{k_1}, H_{k_2}, \ldots, H_{k_{s-1}}$ are subgroups of H that are adjacent to H and also adjacent to each other.

Now, we determine all subgroups L of order p^l where $k < l \leq n$ from $\mathbb{Z}_{p^n}$ that properly include H and $\Psi(H)|\Psi(L)$, so $2k+1|2l+1$. If $2k+1 = q^s$ where q is an odd prime, then possible value for $2l+1$ is $3(2k+1)$ otherwise $k > \frac{n-2}{5}$ fails.

Now two cases arise on the basis of value l exists or not.

Case 1: If no such l exists.

In this case H is adjacent to $H_{k_1}, H_{k_2}, \ldots, H_{k_{s-1}}$ and they are also adjacent to each other. Hence, H is an extreme vertex in the Ψ -divisible graph of the finite group $\mathbb{Z}_{p^n}$.

Case 2: If $l = 3k+1$

So, we have a subgroup L of order p^{3k+1} which includes subgroups H and $\Psi(H)|\Psi(L)$. Therefore L includes subgroups H_{k_i} and $\Psi(H_{k_i})|\Psi(L)$ because proper subgroup H_{k_i} of H is adjacent to H in the Ψ -divisible graph of the finite group $\mathbb{Z}_{p^n}$ where $i = 1, 2, \ldots, s-1$. Hence, H is an extreme vertex in the Ψ -divisible graph of the finite group $\mathbb{Z}_{p^n}$. $\square$

4. Conclusion

In this article, we obtain the necessary and sufficient conditions for a vertex of the finite cyclic group $\mathbb{Z}_{p^n}$ to be an extreme vertex in the Ψ -divisible graph under:

H be a non-trivial subgroup of $\mathbb{Z}_{p^n}$ is an extreme vertex of order p^k in the Ψ -divisible graph of the finite group $\mathbb{Z}_{p^n}$ if and only if $2k+1$ is an odd prime power and $k > \frac{n-2}{5}$.

5. Further Scope of Research

The result of the extreme vertices of Ψ -divisible graph of group $\mathbb{Z}_{p^n}$ reported in this article may be extended to the group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.

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Data availability : No data was used for the research described in the article.

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