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The Non-Normal Cyclic Subgroup Graph Associated with Quasidihedral Groups of Order 16

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Abstract. Let G be a group and H be a non-normal cyclic subgroup of G. The non-normal cyclic subgroup graph, denoted as $\Gamma_H^{NN}(G)$, is defined as a directed graph with vertex set elements of G such that for two distinct elements x and y in G, x is the initial vertex and y is the terminal vertex of an edge if their product is in H. In 2020, the idea of the non-normal subgroup graph, which is the extension of the subgroup graph was developed. This research identifies the non-normal cyclic subgroup graphs connected to order 16 quasidihedral groups. Firstly, the Groups, Algorithms and Programming (GAP) software is used to identify the non-normal cyclic subgroups. The definition of $\Gamma_H^{NN}(G)$ is then used to calculate the adjacency and direction of the vertices. Lastly, Maple 2016 software will be used to visualize these graphs.

Key words and Phrases: group theory, graph theory, subgroup graph, quasidihedral group.

1. INTRODUCTION

In recent years, extensive research has been conducted on graphs related to groups. In real-life applications, graphs are essential in various fields, especially in timetable scheduling and route optimization, such as in Google Maps. In timetable scheduling, nodes represent time slots, classrooms, or teachers, and edges show

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possible assignments. Algorithms help to create schedules without conflicts, ensuring resources like rooms and teachers are not double-booked. Similarly, Google Maps can represent as graph where locations as vertices and roads as edges, with distances or travel times as weights. The application uses algorithms to find the fastest or shortest route, adjusting for traffic in real-time. These applications show how graphs help solve complex problems efficiently [1].

This study extends the idea of subgroup graphs to the graph of non-normal cyclic subgroups. If a subgroup graph H of a group G has the same left and right cosets, it is categorized as a normal subgroup. Most researchers focus on an undirected graphs, including the study on undirected graph on the data complexity of regular trails [2]. However, there are some research in the directed graph which are related to group theory. For instance, [3] defined an order graph which is a new representation of finite groups. Next, the subgroup graph was introduced by Anderson et al. [4]. According to Anderson et al. [4], the subgroup graph, denoted as $\Gamma_H(G)$ is a graph which consists of a vertex set of group G such that x is the starting point while y is the endpoint of the directed graph if and only if $xy \in H$, where H is a subgroup of G. The subgroup graph is then extended to a non-normal subgroup graph, which has been defined by Rahin et al. [5]. Then, Rahin et al. introduced the non-normal subgroup graphs of some dihedral groups in [5] and generalized quaternion groups in [6]. There are some research on graphs that apply into groups such as [7] that study about coprime graph of generalized quaternion group. In 2024, Rahin [8] found the general form of the non-normal cyclic subgroup graph for dihedral and generalized quaternion groups. Hence, this paper will explore on the non-normal cyclic subgroup graph of quasidihedral groups of order 16, with the goal of preparing for future work, where a subsequent paper will focus in determining the general form of the non-normal cyclic subgroup graph for the quasidihedral groups.

A potential application of the non-normal cyclic subgroup graphs in cryptography is cryptographic hash functions which lies in their ability to generate complex, collision-resistant hash values. The structure of the graph can be used to define a hashing algorithm where inputs are mapped to elements of the graph, and the hash value is derived from specific subgroup relations. Due to the unpredictability and intricate subgroup interactions within the graph, even small changes in the input would produce vastly different hash outputs, a key property of secure hash functions. This makes the graph-based hashing a promising approach for applications such as digital signatures, blockchain security, and password storage.

This paper will discuss on the non-normal subgroup graphs of quasidihedral groups of order 16 with the help of Groups, Algorithms and Programming software (GAP). GAP software is very helpful in defining the normal and non-normal subgroups of the groups. Futhermore, Maple 2016 software has been used to visualize the graphs based on the subgroups obtained from GAP software.

2. LITERATURE REVIEW

This section contains several definitions related to subgroup graphs.

Definition 2.1. ***Subgroup graph*** [9] *Let G be a group and H is a subgroup of G. $\Gamma_H(G)$ denotes as the subgroup graph which is a directed graph with the vertex set G, and two distinct elements x and y are adjacent if and only if xy $\in$ H.*

Definition 2.2. ***Non-normal cyclic subgroup graph*** [8] *Let $\Gamma_H^{NN}(G)$ be the non-normal cyclic subgroup graph of the non-normal cyclic subgroup of a group G, denotes as H. Then, $\Gamma_H^{NN}(G)$ is a directed graph with vertex set G such that x is the initial vertex and y is the terminal vertex of an edge if any only if x $\neq$ y and xy $\in$ H, denoted by x $\rightarrow$ y.*

In 2024, Rahin [8] introduced the non-normal cyclic subgroup graph and studied on the non-normal cyclic subgroup graph for dihedral and generalized quaternion groups. Apart from that, the general form of the graph of those two groups is determined together with the general form of the incidence matrix for the non-normal cyclic subgroup graphs.

The quasidihedral group, which is denoted as QD_{2^n}, is a non-abelian group of order 2^n where $n \in \mathbb{N}$ and $n \geq 4$. The group presentation of quasidihedral groups is given as:

$$QD_{2^n} = \langle a, b | a^{2^{n-1}} = b^2 = 1, bab = a^{2^{n-2}-1} \rangle. \tag{1}$$

Recently, many researchers have conducted studies on quasidihedral groups, particularly focusing on the graphs of groups. For example, Pahil Muhidin [10] constructed the prime order and composite order Cayley graphs of quasidihedral groups. In addition, the concept of commuting graphs has been extended by defining the generalized commuting graph of dihedral, quasidihedral and semi-dihedral groups [11]. Other researchers have also investigated the conjugacy class graph of quasidihedral and generalized quaternion groups using some topological indices [12].

Next, Proposition 2.3 and 2.4, as determined in [8], are stated as follows.

Proposition 2.3. [8] *Let $\Gamma_H^{NN}(D_{2n})$ be the dihedral groups of order 2n, $D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$. Then,*

$\Gamma_H^{NN}(D_{2n}) = \{\cup_{i=1}^{k} \Gamma_H(D_{2n})\} \cup K_2 \cup K_2$ with 3n-2 edges, where $n = 2k+2, k \in \mathbb{Z}^+$.

Proposition 2.4. [8] *Let $\Gamma_H^{NN}(Q_{4n})$ be the generalized quaternion groups of order 4n,$Q_{4n} = \langle a, b | a^n = b^2, a^{2n} = b^4 = 1, bab = a^{-1} \rangle$. Then,*

$\Gamma_H^{NN}(Q_{4n}) = \{\Gamma_H(Q_{4n})\} \cup K_4 \cup K_4$ with 14n-4 edges, where $n = 2k+2, k \in \mathbb{Z}^+$.

These propositions which include the general forms of the non-normal cyclic subgroup graph for the dihedral and generalized quaternion groups will be used to determine some graph properties, which are graph coloring and chromatic number of the non-normal cyclic subgroup graph for dihedral group of order 16, and generalized quaternion group of order 16, to compare with the graph properties for the graph of quasidihedral group of order 16, in the next section.

3. RESULTS AND DISCUSSION

The non-normal cyclic subgroup graph corresponding to the quasidihedral groups of order 16 is identified in this section. The methodology is illustrated in Figure 1.

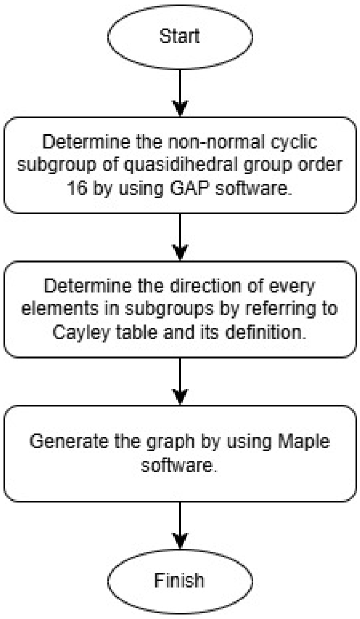


FIGURE 1. Research methodology flowchart

The quasidihedral groups of order 16, with its elements is $QD_{16} = \{1, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$ with $a^8 = b^2 = 1, bab = a^3$. The non-normal cyclic subgroups of QD_{16} are $\langle b\rangle$, $\langle a^{-1}ba\rangle$, $\langle a^{-2}ba^2\rangle$, $\langle b^{-1}a^{-1}bab\rangle$, $\langle b, aba\rangle$, $\langle a^2b^{-1}, aba^{-1}\rangle$, $\langle ba^{-1}\rangle$ and $\langle ba\rangle$. The GAP code to obtain the non-normal cyclic subgroups of QD_{16} are as shown in Figure 2:

```
gap> F:=FreeGroup("a","b");
<free group on the generators [ a, b ]>
gap> a:=F.1;b:=F.2;
a
b
gap> G:=F/[a^8,b^2,b*a*b*a^5];
<fp group on the generators [ a, b ]>
gap> a:=G.1;b:=G.2;
a
b
gap> Order(G);
16
gap> e:=Elements(G);
[ <identity ...>, <[ [ 1, -2, 1, -1, 1, -1, 1, -1 ], [ 2, 3 ] ]|a^-15>, b,
  <[ [ 1, -2, 1, -1, 1, -1, 1, -1 ], [ 2, 2 ] ]|a^-10>,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 4 ] ]|a^20>, a^-15*b,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 3 ] ]|a^15>, <[ [ 1, 1, 1, 1, 1, 1, 1, 2 ]
]|a^5>, a^10*b,
  a^20*b, <[ [ 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 2 ] ]|a^10>, a^-5*b, a^5*b,
  <[ [ 1, -2, 1, -1, 1, -1, 1, -1 ] ]|a^-5>, a^-10*b, a^15*b ]
gap> AllSubgroups(G);
[ Group([  ]), Group([ a^20 ]), Group([ b ]), Group([ a^-1*b*a ]), Group([ a^-
2*b*a^2 ]),
  Group([ b^-1*a^-1*b*a*b ]), Group(<fp, no generators known>), Group([ b, a*b*a
]),
  Group([ a^2*b^-1, a*b*a^-1 ]), Group([ b*a^-1 ]), Group([ b*a ]), Group(<fp, no
generators known>)
    , Group(<fp, no generators known>), Group(<fp, no generators known>), Group([
a, b ]) ]
gap> N:=NormalSubgroups(G);
[ Group(<fp, no generators known>), Group(<fp, no generators known>),
  Group(<fp, no generators known>), Group(<fp, no generators known>),
  Group(<fp, no generators known>), Group(<fp, no generators known>),
  Group(<fp, no generators known>) ]
gap> Elements(N);
[ Group([  ]), Group([ a, b ]), Group([ a ]), Group([ a^-2, b ]), Group([ a^-2,
b*a^-1 ]),
  Group([ a^-2 ]), Group([ a^-4 ]) ]
```

FIGURE 2. GAP coding to obtain the non-normal cyclic subgroups of QD_{16}

Based on Figure 2, the non-normal cyclic subgroups that have been found using GAP software can be simplified as shown in Table 1. In addition, their elements and their orders are shown in Table 1 too.

Table 1: The elements and order of the non-normal cyclic subgroups of QD_{16}

Subgroup generated by GAP software	Simplified subgroup	Elements	Order
$\langle b \rangle$	$\langle b \rangle$	$\{1, b\}$	2
$\langle a^{-1}ba \rangle$	$\langle a^2 b \rangle$	$\{1, a^2 b\}$	2
$\langle a^{-2}ba^2 \rangle$	$\langle a^4 b \rangle$	$\{1, a^4 b\}$	2
$\langle b^{-1}a^{-1}bab \rangle$	$\langle a^6 b \rangle$	$\{1, a^6 b\}$	2
$\langle b, aba \rangle$	$\langle b, a^4 b \rangle$	$\{1, b, a^4, a^4 b\}$	4
$\langle a^2 b^{-1}, aba^{-1} \rangle$	$\langle a^2 b, a^6 b \rangle$	$\{1, a^2 b, a^6 b, a^4\}$	4

Subgroup generated by GAP software	Simplified subgroup	Elements	Order
$\langle ba^{-1}\rangle$	$\langle a^5b\rangle$	$\{1, a^5b\}$	2
$\langle ba\rangle$	$\langle a^3b\rangle$	$\{1, a^3b\}$	2

After all the non-normal cyclic subgroup of QD_{16} are identified, determination on the direction of elements in subgroups is shown in following propositions.

Proposition 3.1. *Let $\Gamma^{NN}_{H_1}(QD_{16})$ be the non-normal cyclic subgroup graph of QD_{16}, where $H_1 = \langle b\rangle = \{1, b\}$. Then, $\Gamma^{NN}_{H_1}(QD_{16})$ is as shown in Figure 3.*

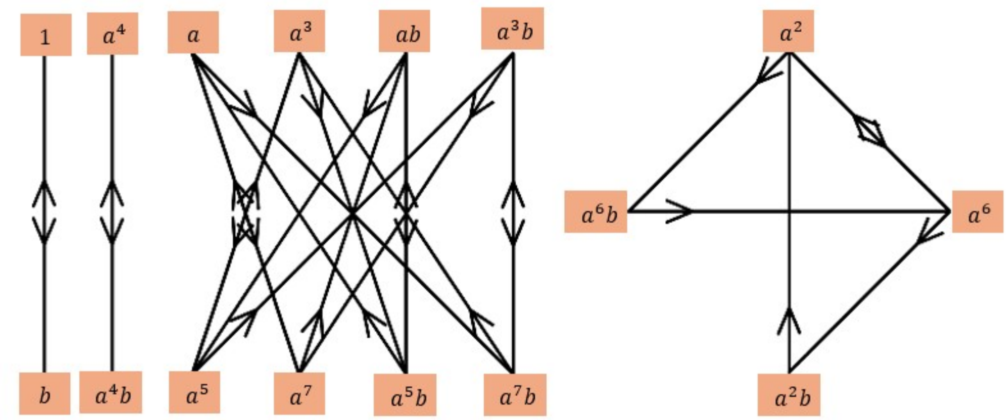


FIGURE 3. $\Gamma^{NN}_{H_1}(QD_{16})$

Proof. Let $H_1 = \langle b\rangle = \{1, b\}$. The product of the elements of QD_{16} and the directed edges are determined by using the Cayley table shown as follows.

TABLE 2. Cayley table of QD_{16}

	1	$\mathbf{a^4}$	**a**	$\mathbf{a^2}$	$\mathbf{a^3}$	$\mathbf{a^5}$	$\mathbf{a^6}$	$\mathbf{a^7}$	**b**	**ab**	$\mathbf{a^2b}$	$\mathbf{a^3b}$	$\mathbf{a^4b}$	$\mathbf{a^5b}$	$\mathbf{a^6b}$	$\mathbf{a^7b}$
1	1	a^4	a	a^2	a^3	a^5	a^6	a^7	b	ab	a^2b	a^3b	a^4b	a^5b	a^6b	a^7b
$\mathbf{a^4}$	a^4	1	a^5	a^6	a^7	a	a^2	a^3	a^4b	a^5b	a^6b	a^7b	b	ab	a^2b	a^3b
a	a	a^5	a^2	a^3	a^4	a^6	a^7	1	ab	a^2b	a^3b	a^4b	a^5b	a^6b	a^7b	b
$\mathbf{a^2}$	a^2	a^6	a^3	a^4	a^5	a^7	1	a	a^2b	a^3b	a^4b	a^5b	a^6b	a^7b	b	ab
$\mathbf{a^3}$	a^3	a^7	a^4	a^5	a^6	1	a	a^2	a^3b	a^4b	a^5b	a^6b	a^7b	b	ab	a^2b
$\mathbf{a^5}$	a^5	a	a^6	a^7	1	a^2	a^3	a^4	a^5b	a^6b	a^7b	b	ab	a^2b	a^3b	a^4b
$\mathbf{a^6}$	a^6	a^2	a^7	1	a	a^3	a^4	a^5	a^6b	a^7b	b	ab	a^2b	a^3b	a^4b	a^5b
$\mathbf{a^7}$	a^7	a^3	1	a	a^2	a^4	a^5	a^6	a^7b	b	ab	a^2b	a^3b	a^4b	a^5b	a^6b
b	b	a^4b	a^3b	a^6b	ab	a^7b	a^2b	a^5b	1	a^3	a^6	a	a^4	a^7	a^2	a^5
ab	ab	a^5b	a^4b	a^7b	a^2b	b	a^3b	a^6b	a	a^4	a^7	a^2	a^5	1	a^3	a^6
$\mathbf{a^2b}$	a^2b	a^6b	a^5b	b	a^3b	ab	a^4b	a^7b	a^2	a^5	1	a^3	a^6	a	a^4	a^7
$\mathbf{a^3b}$	a^3b	a^7b	a^6b	ab	a^4b	a^2b	a^5b	b	a^3	a^6	a	a^4	a^7	a^2	a^5	1
$\mathbf{a^4b}$	a^4b	b	a^7b	a^2b	a^5b	a^3b	a^6b	ab	a^4	a^7	a^2	a^5	1	a^3	a^6	a
$\mathbf{a^5b}$	a^5b	ab	b	a^3b	a^6b	a^4b	a^7b	a^2b	a^5	1	a^3	a^6	a	a^4	a^7	a^2
$\mathbf{a^6b}$	a^6b	a^2b	ab	a^4b	a^7b	a^5b	b	a^3b	a^6	a	a^4	a^7	a^2	a^5	1	a^3
$\mathbf{a^7b}$	a^7b	a^3b	a^2b	a^5b	b	a^6b	ab	a^4b	a^7	a^2	a^5	1	a^3	a^6	a	a^4

To determine the directed edges for the graph, each group element is represented as a vertex. By examining Table 2, when two elements from a row and a column are multiplied, the result shaded in the table indicates a directed edge from vertex of row to vertex of column. This process is repeated for all possible pairs of group elements, and the corresponding directed edges are included in the graph.

By Definition 2.2, $x \to y$ if and only if $x \neq y$ and $xy \in H$. Thus,

$a^5b \to a$ since $a^5b \cdot a = b \in H_1$, $\quad a \to a^7b$ since $a \cdot a^7b = b \in H_1$,
$a^2b \to a^2$ since $a^2b \cdot a^2 = b \in H_1$, $\quad a^7 \to a$ since $a^7 \cdot a = 1 \in H_1$,
$a^7b \to a^3$ since $a^7b \cdot a^3 = b \in H_1$, $\quad a^6 \to a^2$ since $a^6 \cdot a^2 = 1 \in H_1$,
$ab \to a^5$ since $ab \cdot a^5 = b \in H_1$, $\quad a^5 \to a^3$ since $a^5 \cdot a^3 = 1 \in H_1$,
$a^6b \to a^6$ since $a^6b \cdot a^6 = b \in H_1$, $\quad a^3 \to a^5$ since $a^3 \cdot a^5 = 1 \in H_1$,
$a^3b \to a^7$ since $a^3b \cdot a^7 = b \in H_1$, $\quad a^2 \to a^6$ since $a^2 \cdot a^6 = 1 \in H_1$,
$a^7 \to ab$ since $a^7 \cdot ab = b \in H_1$, $\quad a \to a^7$ since $a \cdot a^7 = 1 \in H_1$,
$a^6 \to a^2b$ since $a6 \cdot a^2b = b \in H_1$, $\quad a^5b \to ab$ since $a^5b \cdot ab = 1 \in H_1$,
$a^5 \to a^3b$ since $a^5 \cdot a^3b = b \in H_1$, $\quad a^7b \to a^3b$ since $a^7b \cdot a^3b = 1 \in H_1$,
$a^3 \to a^5b$ since $a^3 \cdot a^5b = b \in H_1$, $\quad ab \to a^5b$ since $ab \cdot a^5b = 1 \in H_1$,
$a^2 \to a^6b$ since $a^2 \cdot a^6b = b \in H_1$, $\quad a^3b \to a^7b$ since $a^3b \cdot a^7b = 1 \in H_1$,
$b \to 1$ and $1 \to b$ since $b \cdot 1 = 1 \cdot b = b \in H_1$,

$a^4b \to a^4$ and $a^4 \to a^4b$ since $a^4b \centerdot a^4 = a^4 \centerdot a^4b = b \in H_1$.
Therefore, $\Gamma^{NN}_{H_1}(QD_{16})$ is the graph shown in Figure 3. $\square$

Proposition 3.2. *Let $\Gamma^{NN}_{H_2}(QD_{16})$ be the non-normal cyclic subgroup graph of QD_{16}, where $H_2 = \langle a^2b \rangle = \{1, a^2b\}$. Then, $\Gamma^{NN}_{H_2}(QD_{16})$ is as shown in Figure 4.*

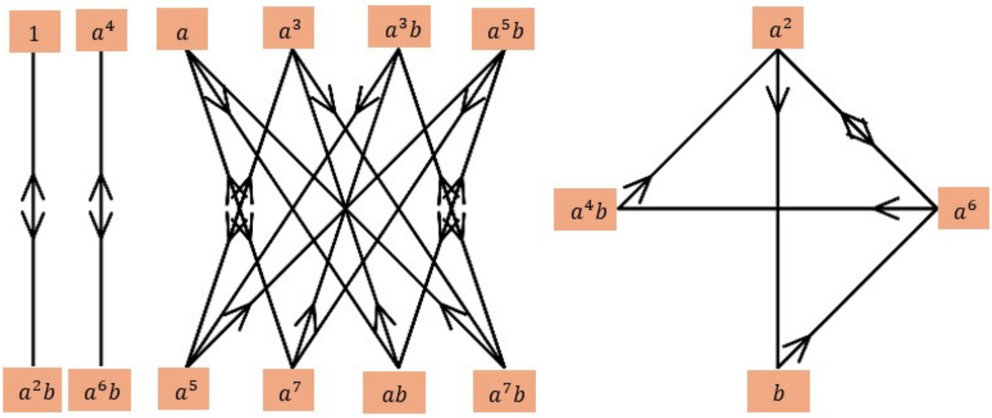


FIGURE 4. $\Gamma^{NN}_{H_2}(QD_{16})$

Proof. Let $H_2 = \langle a^2b \rangle = \{1, a^2b\}$. The product of the elements of QD_{16} and the directed edges are determined by using the Cayley table shown in Table 2.

By the Definition 2.2,

$a^7 \to a$ since $a^7 \centerdot a = 1 \in H_2$,	$a^5b \to a^7$ since $a^5b \centerdot a^7 = a^2b \in H_2$,
$a^7b \to a$ since $a^7b \centerdot a = a^2b \in H_2$,	$a^2 \to b$ since $a^2 \centerdot b = a^2b \in H_2$,
$a^6 \to a^2$ since $a^6 \centerdot a^2 = 1 \in H_2$,	$a \to ab$ since $a \centerdot ab = a^2b \in H_2$,
$a^4b \to a^2$ since $a^4b \centerdot a^2 = a^2b \in H_2$,	$a^5b \to ab$ since $a^5b \centerdot ab = 1 \in H_2$,
$a^5 \to a^3$ since $a^5 \centerdot a^3 = 1 \in H_2$,	$a^7 \to a^3b$ since $a^7 \centerdot a^3b = a^2b \in H_2$,
$ab \to a^3$ since $ab \centerdot a^3 = a^2b \in H_2$,	$a^7b \to a^3b$ since $a^7b \centerdot a^3b = 1 \in H_2$,
$a^3 \to a^5$ since $a^3 \centerdot a^5 = 1 \in H_2$,	$a^6 \to a^4b$ since $a^6 \centerdot a^4b = a^2b \in H_2$,
$a^3b \to a^5$ since $a^3b \centerdot a^5 = a^2b \in H_2$,	$a^5 \to a^5b$ since $a^5 \centerdot a^5b = a^2b \in H_2$,
$a^2 \to a^6$ since $a^2 \centerdot a^6 = 1 \in H_2$,	$ab \to a^5b$ since $ab \centerdot a^5b = 1 \in H_2$,
$b \to a^6$ since $b \centerdot a^6 = a^2b \in H_2$,	$a^3 \to a^7b$ since $a^3 \centerdot a^7b = a^2b \in H_2$,
$a \to a^7$ since $a \centerdot a^7 = 1 \in H_2$,	$a^3b \to a^7b$ since $a^3b \centerdot a^7b = 1 \in H_2$,

$a^2b \to 1$ and $1 \to a^2b$ since $a^2b \centerdot 1 = 1 \centerdot a^2b = a^2b \in H_2$,
$a^6b \to a^4$ and $a^4 \to a^6b$ since $a^6b \centerdot a^4 = a^4 \centerdot a^6b = a^2b \in H_2$.
Therefore, $\Gamma^{NN}_{H_2}(QD_{16})$ is the graph shown in Figure 4. $\square$

Proposition 3.3. *Let $\Gamma^{NN}_{H_3}(QD_{16})$ be the non-normal cyclic subgroup graph of QD_{16}, where $H_3 = \langle a^4b\rangle = \{1, a^4b\}$. Then, $\Gamma^{NN}_{H_3}(QD_{16})$ is as shown in Figure 5.*

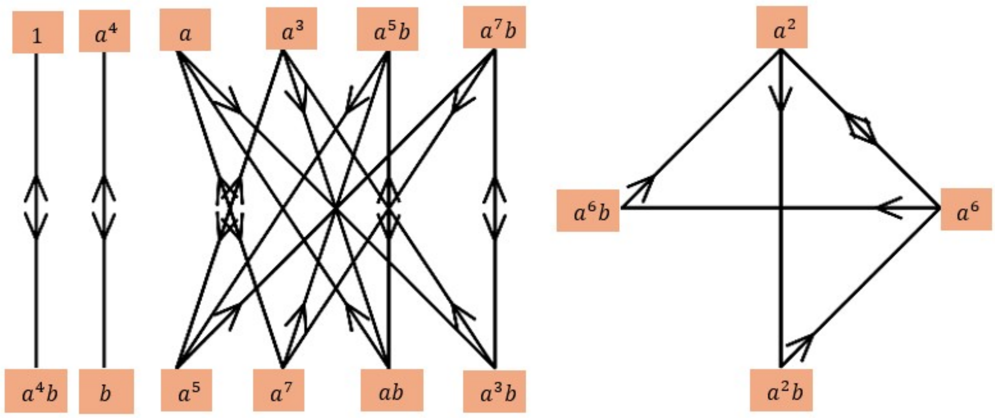


FIGURE 5. $\Gamma^{NN}_{H_3}(QD_{16})$

Proof. Let $H_3 = \langle a^4b\rangle = \{1, a^4b\}$. The product of the elements of QD_{16} and the directed edges are determined by using the Cayley table shown in Table 2.

By the Definition 2.2,

$a^7 \to a$ since $a^7 \centerdot a = 1 \in H_3$,
$a^7b \to a^7$ since $a^7b \centerdot a^7 = a^4b \in H_3$,
$ab \to a$ since $ab \centerdot a = a^4b \in H_3$,
$a^3 \to ab$ since $a^3 \centerdot ab = a^4b \in H_3$,
$a^6 \to a^2$ since $a^6 \centerdot a^2 = 1 \in H_3$,
$a^5b \to ab$ since $a^5b \centerdot ab = 1 \in H_3$,
$a^6b \to a^2$ since $a^6b \centerdot a^2 = a^4b \in H_3$,
$a^2 \to a^2b$ since $a^2 \centerdot a^2b = a^4b \in H_3$,
$a^5 \to a^3$ since $a^5 \centerdot a^3 = 1 \in H_3$,
$a \to a^3b$ since $a \centerdot a^3b = a^4b \in H_3$,
$a^3b \to a^3$ since $a^3b \centerdot a^3 = a^4b \in H_3$,
$a^7b \to a^3b$ since $a^7b \centerdot a^3b = 1 \in H_3$,
$a^3 \to a^5$ since $a^3 \centerdot a^5 = 1 \in H_3$,
$a^7 \to a^5b$ since $a^7 \centerdot a^5b = a^4b \in H_3$,
$a^5b \to a^5$ since $a^5b \centerdot a^5 = a^4b \in H_3$,
$ab \to a^5b$ since $ab \centerdot a^5b = 1 \in H_3$,
$a^2 \to a^6$ since $a^2 \centerdot a^6 = 1 \in H_3$,
$a^6 \to a^6b$ since $a^6 \centerdot a^6b = a^4b \in H_3$,
$a^2b \to a^6$ since $a^2b \centerdot a^6 = a^4b \in H_3$,
$a^5 \to a^7b$ since $a^5 \centerdot a^7b = a^4b \in H_3$,
$a \to a^7$ since $a \centerdot a^7 = 1 \in H_3$,
$a^3b \to a^7b$ since $a^3b \centerdot a^7b = 1 \in H_3$,
$a^4b \to 1$ and $1 \to a^4b$ since $a^4b \centerdot 1 = 1 \centerdot a^4b = a^4b \in H_3$,
$b \to a^4$ and $a^4 \to b$ since $b \centerdot a^4 = a^4 \centerdot b = a^4b \in H_3$.

Therefore, $\Gamma^{NN}_{H_3}(QD_{16})$ is the graph shown in Figure 5. □

Proposition 3.4. *Let $\Gamma^{NN}_{H_4}(QD_{16})$ be the non-normal cyclic subgroup graph of QD_{16}, where $H_4 = \langle a^6b\rangle = \{1, a^6b\}$. Then, $\Gamma^{NN}_{H_4}(QD_{16})$ is as shown in Figure 6.*

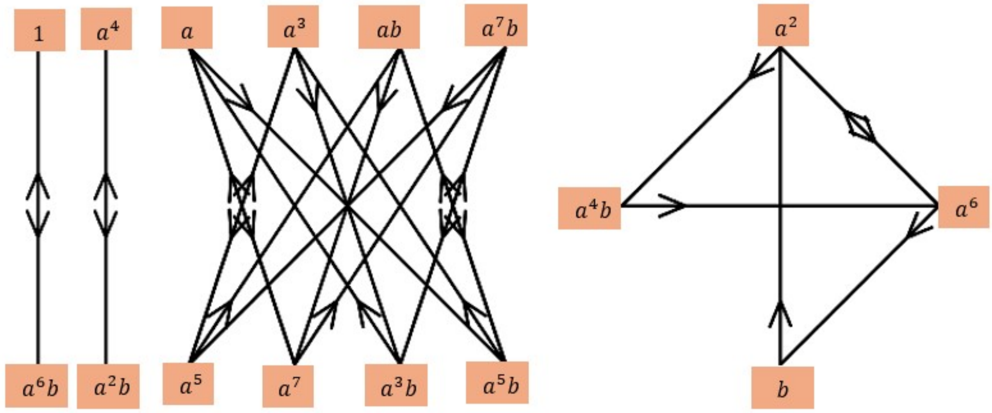


FIGURE 6. $\Gamma^{NN}_{H_4}(QD_{16})$

Proof. Let $H_4 = \langle a^6b \rangle = \{1, a^6b\}$. The product of the elements of QD_{16} and the directed edges are determined by using the Cayley table shown in Table 2.

By the Definition 2.2,

$a^7 \to a$ since $a^7 \centerdot a = 1 \in H_4$, $ab \to a^7$ since $ab \centerdot a^7 = a^6b \in H_4$,
$a^3b \to a$ since $a^3b \centerdot a = a^6b \in H_4$, $a^6 \to b$ since $a^6 \centerdot b = a^6b \in H_4$,
$a^6 \to a^2$ since $a^6 \centerdot a^2 = 1 \in H_4$, $a^5 \to ab$ since $a^5 \centerdot ab = a^6b \in H_4$,
$b \to a^2$ since $b \centerdot a^2 = a^6b \in H_4$, $a^5b \to ab$ since $a^5b \centerdot ab = 1 \in H_4$,
$a^5 \to a^3$ since $a^5 \centerdot a^3 = 1 \in H_4$, $a^3 \to a^3b$ since $a^3 \centerdot a^3b = a^6b \in H_4$,
$a^5b \to a^3$ since $a^5b \centerdot a^3 = a^6b \in H_4$, $a^7b \to a^3b$ since $a^7b \centerdot a^3b = 1 \in H_4$,
$a^3 \to a^5$ since $a^3 \centerdot a^5 = 1 \in H_4$, $a^2 \to a^4b$ since $a^2 \centerdot a^4b = a^6b \in H_4$,
$a^7b \to a^5$ since $a^7b \centerdot a^5 = a^6b \in H_4$, $a \to a^5b$ since $a \centerdot a^5b = a^6b \in H_4$,
$a^2 \to a^6$ since $a^2 \centerdot a^6 = 1 \in H_4$, $ab \to a^5b$ since $ab \centerdot a^5b = 1 \in H_4$,
$a^4b \to a^6$ since $a^4b \centerdot a^6 = a^6b \in H_4$, $a^7 \to a^7b$ since $a^7 \centerdot a^7b = a^6b \in H_4$,
$a \to a^7$ since $a \centerdot a^7 = 1 \in H_4$, $a^3b \to a^7b$ since $a^3b \centerdot a^7b = 1 \in H_4$,
$a^6b \to 1$ and $1 \to a^6b$ since $a^6b \centerdot 1 = 1 \centerdot a^6b = a^6b \in H_4$,
$a^2b \to a^4$ and $a^4 \to a^2b$ since $a^2b \centerdot a^4 = a^4 \centerdot a^2b = a^6b \in H_4$.
Therefore, $\Gamma^{NN}_{H_4}(QD_{16})$ is the graph shown in Figure 6. □

Proposition 3.5. *Let $\Gamma^{NN}_{H_5}(QD_{16})$ be the non-normal cyclic subgroup graph of QD_{16}, where $H_5 = \langle b, a^4b \rangle = \{1, b, a^4, a^4b\}$. Then, $\Gamma^{NN}_{H_5}(QD_{16})$ is as shown in Figure 7.*

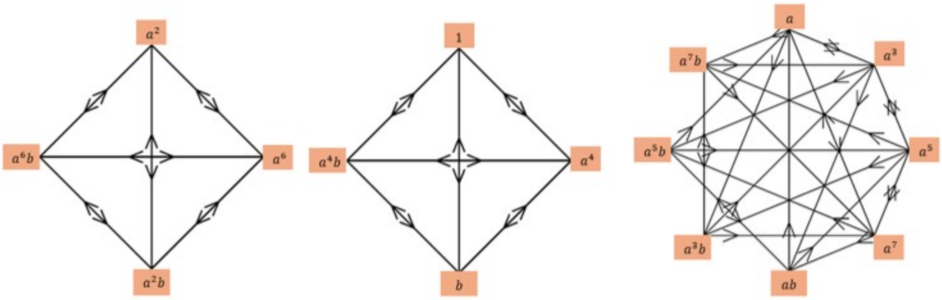


FIGURE 7. $\Gamma^{NN}_{H_5}(QD_{16})$

Proof. Let $H_5 = \langle b, a^4b\rangle = \{1, b, a^4, a^4b\}$. The product of the elements of QD_{16} and the directed edges are determined by using the Cayley table shown in Table 2.

By the Definition 2.2,

$a^3 \to a$ since $a^3 \,.\, a = a^4 \in H_5$, $\quad$ $a^3b \to a^7$ since $a^3b \,.\, a^7 = b \in H_5$,
$a^7 \to a$ since $a^7 \,.\, a = 1 \in H_5$, $\quad$ $a^7b \to a^7$ since $a^7b \,.\, a^7 = a^4b \in H_5$,
$ab \to a$ since $ab \,.\, a = a^4b \in H_5$, $\quad$ $a^3 \to ab$ since $a^3 \,.\, ab = a^4b \in H_5$,
$a^5b \to a$ since $a^5b \,.\, a = b \in H_5$, $\quad$ $a^7 \to ab$ since $a^7 \,.\, ab = b \in H_5$,
$a \to a^3$ since $a \,.\, a^3 = a^4 \in H_5$, $\quad$ $a^5b \to ab$ since $a^5b \,.\, ab = 1 \in H_5$,
$a^5 \to a^3$ since $a^5 \,.\, a^3 = 1 \in H_5$, $\quad$ $a \to a^3b$ since $a \,.\, a^3b = a^4b \in H_5$,
$a^3b \to a^3$ since $a^3b \,.\, a^3 = a^4b \in H_5$, $\quad$ $a^5 \to a^3b$ since $a^5 \,.\, a^3b = b \in H_5$,
$a^7b \to a^3$ since $a^7b \,.\, a^3 = b \in H_5$, $\quad$ $a^7b \to a^3b$ since $a^7b \,.\, a^3b = 1 \in H_5$,
$a^3 \to a^5$ since $a^3 \,.\, a^5 = 1 \in H_5$, $\quad$ $a^3 \to a^5b$ since $a^3 \,.\, a^5b = b \in H_5$,
$a^7 \to a^5$ since $a^7 \,.\, a^5 = a^4 \in H_5$, $\quad$ $a^7 \to a^5b$ since $a^7 \,.\, a^5b = a^4b \in H_5$,
$ab \to a^5$ since $ab \,.\, a^5 = b \in H_5$, $\quad$ $ab \to a^5b$ since $ab \,.\, a^5b = 1 \in H_5$,
$a^5b \to a^5$ since $a^5b \,.\, a^5 = a^4b \in H_5$, $\quad$ $a \to a^7b$ since $a \,.\, a^7b = b \in H_5$,
$a \to a^7$ since $a \,.\, a^7 = 1 \in H_5$, $\quad$ $a^5 \to a^7b$ since $a^5 \,.\, a^7b = a^4b \in H_5$,
$a^5 \to a^7$ since $a^5 \,.\, a^7 = a^4 \in H_5$, $\quad$ $a^3b \to a^7b$ since $a^3b \,.\, a^7b = 1 \in H_5$,
$a^2 \to a^6b$ since $a^2 \,.\, a^6b = b \in H_5$, $\quad$ $a^2 \to a^2b$ since $a^2 \,.\, a^2b = a^4b \in H_5$,
$a^6b \to a^2$ since $a^6b \,.\, a^2 = a^4b \in H_5$, $\quad$ $a^2b \to a^2$ since $a^2b \,.\, a^2 = b \in H_5$,
$a^6 \to a^2b$ since $a^6 \,.\, a^2b = b \in H_5$, $\quad$ $a^6 \to a^6b$ since $a^6 \,.\, a^6b = a^4b \in H_5$,
$a^2b \to a^6$ since $a^2b \,.\, a^6 = a^4b \in H_5$, $\quad$ $a^6b \to a^6$ since $a^6b \,.\, a^6 = b \in H_5$,
$a^2 \to a^6$ and $a^6 \to a^2$ since $a^2 \,.\, a^6 = a^6 \,.\, a^2 = 1 \in H_5$,
$b \to 1$ and $1 \to b$ since $b \,.\, 1 = 1 \,.\, b = b \in H_5$,
$a^4b \to a^4$ and $a^4 \to a^4b$ since $a^4b \,.\, a^4 = a^4 \,.\, a^4b = b \in H_5$,
$1 \to a^4$ and $a^4 \to 1$ since $1 \,.\, a^4 = a^4 \,.\, 1 = a^4 \in H_5$,
$b \to a^4b$ and $a^4b \to b$ since $b \,.\, a^4b = a^4b \,.\, b = a^4 \in H_5$,
$a^6b \to a^2b$ and $a^2b \to a^6b$ since $a^6b \,.\, a^2b = a^2b \,.\, a^6b = a^4 \in H_5$,
$1 \to a^4b$ and $a^4b \to 1$ since $1 \,.\, a^4b = a^4b \,.\, 1 = a^4b \in H_5$,

$b \to a^4$ and $a^4 \to b$ since $b \cdot a^4 = a^4 \cdot b = a^4b \in H_5$.

Therefore, $\Gamma^{NN}_{H_5}(QD_{16})$ is the graph shown in Figure 7. □

Proposition 3.6. *Let $\Gamma^{NN}_{H_6}(QD_{16})$ be the non-normal cyclic subgroup graph of QD_{16}, where $H_6 = \langle a^2b, a^6b\rangle = \{1, a^2b, a^6b, a^4\}$. Then, $\Gamma^{NN}_{H_6}(QD_{16})$ is as shown in Figure 8.*

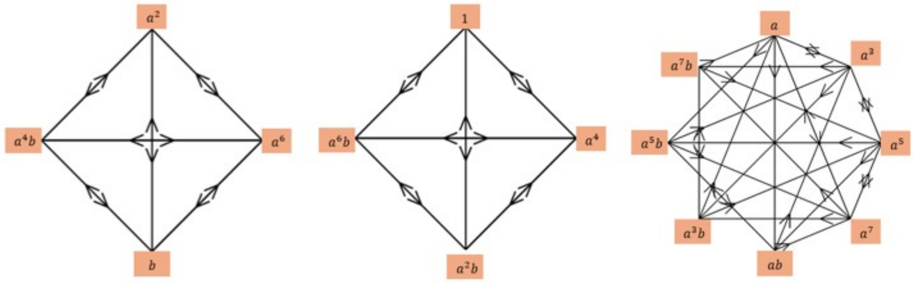


FIGURE 8. $\Gamma^{NN}_{H_6}(QD_{16})$

Proof. Let $H_6 = \langle a^2b, a^6b\rangle = \{1, a^2b, a^6b, a^4\}$. The product of the elements of QD_{16} and the directed edges are determined by using the Cayley table shown in Table 2.

By the Definition 2.2,

$a^3 \to a$ since $a^3 \cdot a = a^4 \in H_6$,
$a^7 \to a$ since $a^7 \cdot a = 1 \in H_6$,
$a^3b \to a$ since $a^3b \cdot a = a^6b \in H_6$,
$a^7b \to a$ since $a^7b \cdot a = a^2b \in H_6$,
$a \to a^3$ since $a \cdot a^3 = a^4 \in H_6$,
$a^5 \to a^3$ since $a^5 \cdot a^3 = 1 \in H_6$,
$ab \to a^3$ since $ab \cdot a^3 = a^2b \in H_6$,
$a^5b \to a^3$ since $a^5b \cdot a^3 = a^6b \in H_6$,
$a^3 \to a^5$ since $a^3 \cdot a^5 = 1 \in H_6$,
$a^7 \to a^5$ since $a^7 \cdot a^5 = a^4 \in H_6$,
$a^3b \to a^5$ since $a^3b \cdot a^5 = a^2b \in H_6$,
$a^7b \to a^5$ since $a^7b \cdot a^5 = a^6b \in H_6$,
$a \to a^7$ since $a \cdot a^7 = 1 \in H_6$,
$a^5 \to a^7$ since $a^5 \cdot a^7 = a^4 \in H_6$,
$a^4b \to a^2$ since $a^4b \cdot a^2 = a^2b \in H_6$,
$a^2 \to a^4b$ since $a^2 \cdot a^4b = a^6b \in H_6$,
$b \to a^6$ since $b \cdot a^6 = a^2b \in H_6$,
$a^6 \to b$ since $a^6 \cdot b = a^6b \in H_6$,
$ab \to a^7$ since $ab \cdot a^7 = a^6b \in H_6$,
$a^5b \to a^7$ since $a^5b \cdot a^7 = a^2b \in H_6$,
$a \to ab$ since $a \cdot ab = a^2b \in H_6$,
$a^5 \to ab$ since $a^5 \cdot ab = a^6b \in H_6$,
$a^5b \to ab$ since $a^5b \cdot ab = 1 \in H_6$,
$a^3 \to a^3b$ since $a^3 \cdot a^3b = a^6b \in H_6$,
$a^7 \to a^3b$ since $a^7 \cdot a^3b = a^2b \in H_6$,
$a^7b \to a^3b$ since $a^7b \cdot a^3b = 1 \in H_6$,
$a \to a^5b$ since $a \cdot a^5b = a^6b \in H_6$,
$a^5 \to a^5b$ since $a^5 \cdot a^5b = a^2b \in H_6$,
$ab \to a^5b$ since $ab \cdot a^5b = 1 \in H_6$,
$a^3 \to a^7b$ since $a^3 \cdot a^7b = a^2b \in H_6$,
$a^7 \to a^7b$ since $a^7 \cdot a^7b = a^6 \in H_6$,
$a^3b \to a^7b$ since $a^3b \cdot a^7b = 1 \in H_6$,
$b \to a^2$ since $b \cdot a^2 = a^6b \in H_6$,
$a^2 \to b$ since $a^2 \cdot b = a^2b \in H_6$,
$a^6 \to a^4b$ since $a^6 \cdot a^4b = a^2b \in H_6$,
$a^4b \to a^6$ since $a^4b \cdot a^6 = a^6b \in H_6$,
$a^2 \to a^6$ and $a^6 \to a^2$ since $a^2 \cdot a^6 = a^6 \cdot a^2 = 1 \in H_6$,

$a^2b \to 1$ and $1 \to a^2b$ since $a^2b \centerdot 1 = 1 \centerdot a^2b = a^2b \in H_6$,
$a^6b \to a^4$ and $a^4 \to a^6b$ since $a^6b \centerdot a^4 = a^4 \centerdot a^6b = a^2b \in H_6$,
$1 \to a^6b$ and $a^6b \to 1$ since $1 \centerdot a^6b = a^6b \centerdot 1 = a^6b \in H_6$,
$a^4 \to a^2b$ and $a^2b \to a^4$ since $a^4 \centerdot a^2b = a^2b \centerdot a^4 = a^6b \in H_6$,
$1 \to a^4$ and $a^4 \to 1$ since $1 \centerdot a^4 = a^4 \centerdot 1 = a^4 \in H_6$,
$b \to a^4b$ and $a^4b \to b$ since $b \centerdot a^4b = a^4b \centerdot b = a^4 \in H_6$,
$a^6b \to a^2b$ and $a^2b \to a^6b$ since $a^6b \centerdot a^2b = a^2b \centerdot a^6b = a^4 \in H_6$.
Therefore, $\Gamma^{NN}_{H_6}(QD_{16})$ is the graph shown in Figure 8. □

Proposition 3.7. *Let $\Gamma^{NN}_{H_7}(QD_{16})$ be the non-normal cyclic subgroup graph of QD_{16}, where $H_7 = \langle a^5b\rangle = \{1, a^5b, a^4, ab\}$. Then, $\Gamma^{NN}_{H_7}(QD_{16})$ is as shown in Figure 9.*

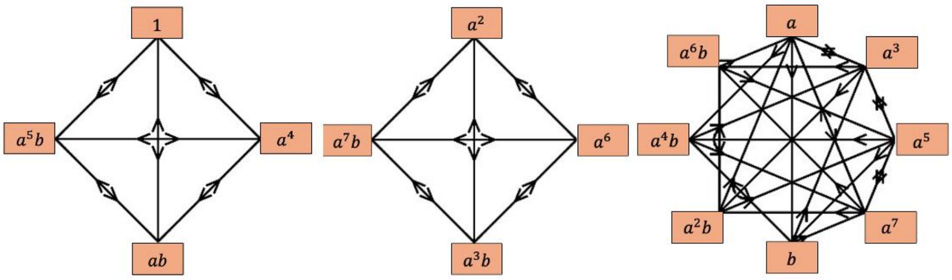


FIGURE 9. $\Gamma^{NN}_{H_7}(QD_{16})$

Proof. Let $H_7 = \langle a^5b\rangle = \{1, a^5b, a^4, ab\}$. The product of the elements of QD_{16} and the directed edges are determined by using the Cayley table shown in Table 2.

By the Definition 2.2,

$a \to b$ since $a \centerdot b = ab \in H_7$, $a \to a^4b$ since $a \centerdot a^4b = a^5b \in H_7$,
$b \to a^7$ since $b \centerdot a^7 = a^5b \in H_7$, $b \to a^3$ since $b \centerdot a^3 = ab \in H_7$,
$a^7 \to a^2b$ since $a^7 \centerdot a^2b = ab \in H_7$, $a^7 \to a^6b$ since $a^7 \centerdot a^6b = a^5b \in H_7$,
$a^5 \to b$ since $a^5 \centerdot b = a^5b \in H_7$, $a^5 \to a^4b$ since $a^5 \centerdot a^4b = ab \in H_7$,
$a^2b \to a$ since $a^2b \centerdot a = a^5b \in H_7$, $a^2b \to a^5$ since $a^2b \centerdot a^5 = ab \in H_7$,
$a^4b \to a^7$ since $a^4b \centerdot a^7 = ab \in H_7$, $a^4b \to a^3$ since $a^4b \centerdot a^3 = a^5b \in H_7$,
$a^2 \to a^3b$ since $a^2 \centerdot a^3b = a^5b \in H_7$, $a^2 \to a^7b$ since $a^2 \centerdot a^7b = ab \in H_7$,
$a^3b \to a^2$ since $a^3b \centerdot a^2 = ab \in H_7$, $a^3 \to a^2b$ since $a^3 \centerdot a^2b = a^5b \in H_7$,
$a^3 \to a^6b$ since $a^3 \centerdot a^6b = ab \in H_7$, $a^6b \to a$ since $a^6b \centerdot a = ab \in H_7$,
$a^6b \to a^5$ since $a^6b \centerdot a^5 = 1 \in H_7$, $a^7b \to a^2$ since $a^7b \centerdot a^2 = 1 \in H_7$,
$a^6 \to a^7b$ since $a^6 \centerdot a^7b = a^5b \in H_7$, $a^7b \to a^6$ since $a^7b \centerdot a^6 = ab \in H_7$,
$1 \to a^4$ and $a^4 \to 1$ since $1 \centerdot a^4 = a^4 \centerdot 1 = a^4 \in H_7$,

$1 \to ab$ and $ab \to 1$ since $1 \centerdot ab = ab \centerdot 1 = ab \in H_7$,
$1 \to a^5b$ and $a^5b \to 1$ since $1 \centerdot a^5b = a^5b \centerdot 1 = a^5b \in H_7$,
$a \to a^7$ and $a^7 \to a$ since $a \centerdot a^7 = a^7 \centerdot a = 1 \in H_7$,
$a \to a^3$ and $a^3 \to a$ since $a \centerdot a^3 = a^3 \centerdot a = a^4 \in H_7$,
$b \to a^4b$ and $a^4b \to b$ since $b \centerdot a^4b = a^4b \centerdot b = a^4 \in H_7$,
$a^6 \to a^2$ and $a^2 \to a^6$ since $a^6 \centerdot a^2 = a^2 \centerdot a^6 = 1 \in H_7$,
$a^6 \to a^3b$ and $a^3b \to a^6$ since $a^6 \centerdot a^3b = a^3b \centerdot a^6 = a^5b \in H_7$,
$a^4 \to ab$ and $ab \to a^4$ since $a^4 \centerdot ab = ab \centerdot a^4 = a^5b \in H_7$,
$a^4 \to a^5b$ and $a^5b \to a^4$ since $a^4 \centerdot a^5b = a^5b \centerdot a^4 = ab \in H_7$,
$ab \to a^5b$ and $a^5b \to ab$ since $ab \centerdot a^5b = a^5b \centerdot ab = 1 \in H_7$,
$a^7 \to a^5$ and $a^5 \to a^7$ since $a^7 \centerdot a^5 = a^5 \centerdot a^7 = a^4 \in H_7$,
$a^5 \to a^3$ and $a^3 \to a^5$ since $a^5 \centerdot a^3 = a^3 \centerdot a^5 = 1 \in H_7$,
$a^2b \to a^6b$ and $a^6b \to a^2b$ since $a^2b \centerdot a^6b = a^6b \centerdot a^2b = a^4 \in H_7$,
$a^3b \to a^7b$ and $a^7b \to a^3b$ since $a^3b \centerdot a^7b = a^7b \centerdot a^3b = 1 \in H_7$.
Therefore, $\Gamma^{NN}_{H_7}(QD_{16})$ is the graph shown in Figure 9. □

Proposition 3.8. *Let $\Gamma^{NN}_{H_8}(QD_{16})$ be the non-normal cyclic subgroup graph of QD_{16}, where $H_8 = \langle a^3b \rangle = \{1, a^3b, a^4, a^7b\}$. Then, $\Gamma^{NN}_{H_8}(QD_{16})$ is as shown in Figure 10.*

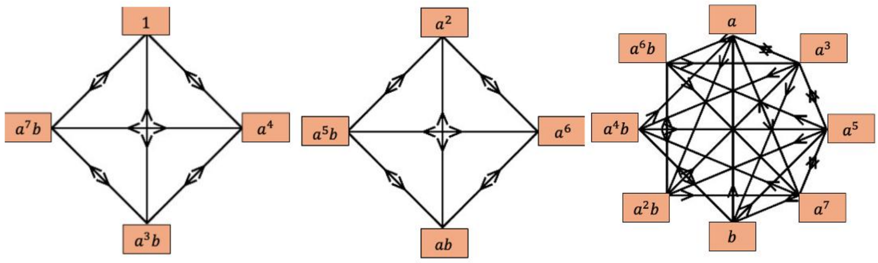


FIGURE 10. $\Gamma^{NN}_{H_8}(QD_{16})$

Proof. Let $H_8 = \langle a^3b \rangle = \{1, a^3b, a^4, a^7b\}$. The product of the elements of QD_{16} and the directed edges are determined by using the Cayley table shown in Table 2.

By the Definition 2.2,
$a \to a^2b$ since $a \centerdot a^2b = a^3b \in H_8$,
$a \to a^6b$ since $a \centerdot a^6b = a^7b \in H_8$,
$b \to a$ since $b \centerdot a = a^3b \in H_8$,
$b \to a^5$ since $b \centerdot a^5 = a^7b \in H_8$,
$a^7 \to b$ since $a^7 \centerdot b = a^7b \in H_8$,
$a^7 \to a^4b$ since $a^7 \centerdot a^4b = a^3b \in H_8$,
$a^5 \to a^2b$ since $a^5 \centerdot a^2b = a^7b \in H_8$,
$a^5 \to a^6b$ since $a^5 \centerdot a^6b = a^3b \in H_8$,
$a^2b \to a^7$ since $a^2b \centerdot a^7 = a^7b \in H_8$,
$a^2b \to a^3$ since $a^2b \centerdot a^3 = a^3b \in H_8$,
$a^4b \to a$ since $a^4b \centerdot a = a^7b \in H_8$,
$a^4b \to a^5$ since $a^4b \centerdot a^5 = a^3b \in H_8$,

$a^3 \to b$ since $a^3 \centerdot b = a^3b \in H_8$, $a^3 \to a^4b$ since $a^3 \centerdot a^4b = a^7b \in H_8$,
$a^6b \to a^7$ since $a^6b \centerdot a^7 = a^3b \in H_8$, $a^6b \to a^3$ since $a^6b \centerdot a^3 = a^7b \in H_8$,
$a^6 \to ab$ since $a^6 \centerdot ab = a^7b \in H_8$, $ab \to a^6$ since $ab \centerdot a^6 = a^3b \in H_8$,
$a^6 \to a^5b$ since $a^6 \centerdot a^5b = a^3b \in H_8$, $a^5b \to a^6$ since $a^5b \centerdot a^6 = a^7b \in H_8$,
$ab \to a^2$ since $ab \centerdot a^2 = a^7b \in H_8$, $a^2 \to ab$ since $a^2 \centerdot ab = a^3b \in H_8$,
$a^2 \to a^5b$ since $a^2 \centerdot a^5b = a^7b \in H_8$, $a^5b \to a^2$ since $a^5b \centerdot a^2 = a^3b \in H_8$,
$1 \to a^4$ and $a^4 \to 1$ since $1 \centerdot a^4 = a^4 \centerdot 1 = a^4 \in H_8$,
$1 \to a^3b$ and $a^3b \to 1$ since $1 \centerdot a^3b = a^3b \centerdot 1 = a^3b \in H_8$,
$1 \to a^7b$ and $a^7b \to 1$ since $1 \centerdot a^7b = a^7b \centerdot 1 = a^7b \in H_8$,
$a \to a^7$ and $a^7 \to a$ since $a \centerdot a^7 = a^7 \centerdot a = 1 \in H_8$,
$a \to a^3$ and $a^3 \to a$ since $a \centerdot a^3 = a^3 \centerdot a = a^4 \in H_8$,
$b \to a^4b$ and $a^4b \to b$ since $b \centerdot a^4b = a^4b \centerdot b = a^4 \in H_8$,
$a^6 \to a^2$ and $a^2 \to a^6$ since $a^6 \centerdot a^2 = a^2 \centerdot a^6 = 1 \in H_8$,
$a^4 \to a^3b$ and $a^3b \to a^4$ since $a^4 \centerdot a^3b = a^3b \centerdot a^4 = a^7b \in H_8$,
$a^4 \to a^7b$ and $a^7b \to a^4$ since $a^4 \centerdot a^7b = a^7b \centerdot a^4 = a^3b \in H_8$,
$ab \to a^5b$ and $a^5b \to ab$ since $ab \centerdot a^5b = a^5b \centerdot ab = 1 \in H_8$,
$a^7 \to a^5$ and $a^5 \to a^7$ since $a^7 \centerdot a^5 = a^5 \centerdot a^7 = a^4 \in H_8$,
$a^5 \to a^3$ and $a^3 \to a^5$ since $a^5 \centerdot a^3 = a^3 \centerdot a^5 = 1 \in H_8$,
$a^2b \to a^6b$ and $a^6b \to a^2b$ since $a^2b \centerdot a^6b = a^6b \centerdot a^2b = a^4 \in H_8$,
$a^3b \to a^7b$ and $a^7b \to a^3b$ since $a^3b \centerdot a^7b = a^7b \centerdot a^3b = 1 \in H_8$.
Therefore, $\Gamma^{NN}_{H_8}(QD_{16})$ is the graph shown in Figure 10. □

Based on Propositions 3.1 to 3.8, the non-normal cyclic subgroup graph of the quasidihedral groups of order 16 is observed to be the union of some connected graphs. Since the results are limited to QD_{16}, hence its general form has not yet been found.

Next, based on Propositions 2.3 and 2.4, as found by Rahin [8], are used to study some graph properties of the dihedral and generalized quaternion groups of order 16. The graph properties that will be investigated are the maximal and minimal cyclic subgroups, graph coloring, and chromatic number, which are presented in Table 3. This comparison highlights key differences between the non-normal cyclic subgroup graph of dihedral and generalized quaternion group of order 16.

Once all the non-normal cyclic subgroup graph associated with quasidihedral groups of order 16 have been identified, in Proposition 3.1 - 3.8 an analysis of their graph properties can be conducted. Table 4 and Table 5 present some graph properties which are the graph coloring, chromatic number and identification of whether the subgroup is minimal or maximal.

Table 3. Graph properties of the non-normal cyclic subgroup graph associated with dihedral and generalized quaternion groups of order 16

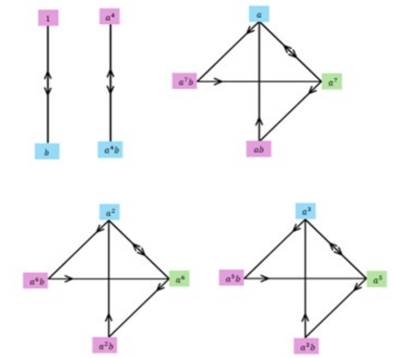
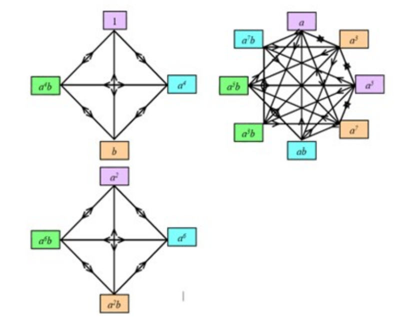
Subgroup	Maximal or minimal	Chromatic number	Graph coloring
$\langle b\rangle$	Minimal	3	
$\langle b\rangle$	Minimal	4	

Table 4. Graph properties of the non-normal cyclic subgroup graph of $\langle b\rangle, \langle a^2b\rangle$ and $\langle a^4b\rangle$ associated with quasidihedral group of order 16

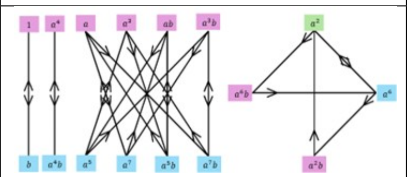
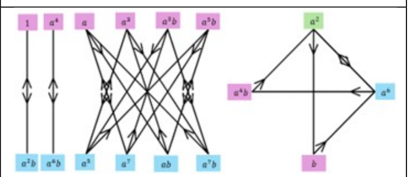
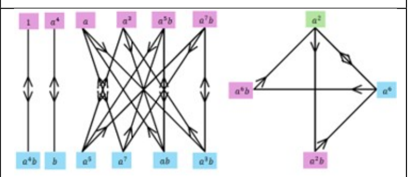
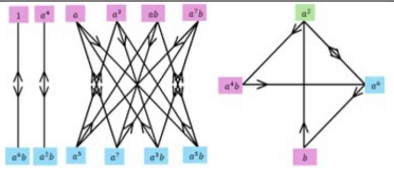
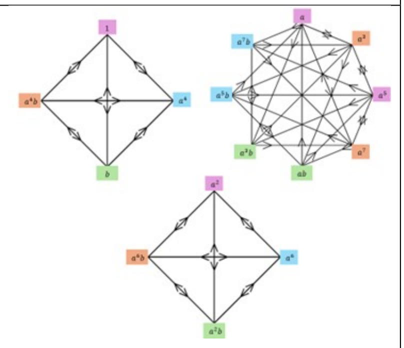
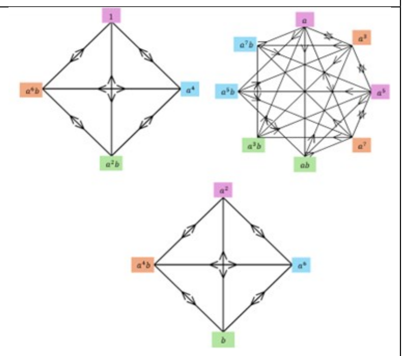
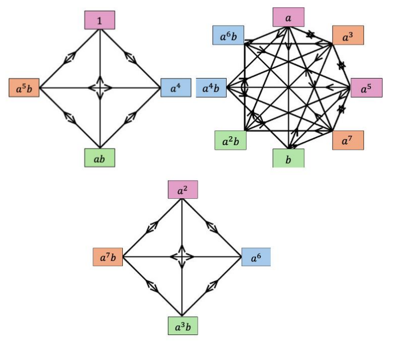
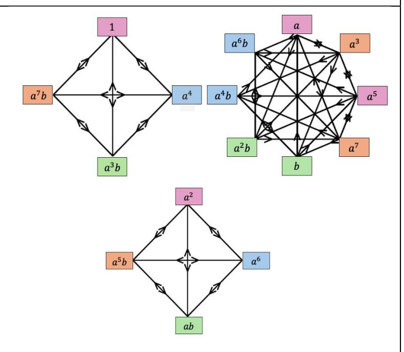
Subgroup	Maximal or minimal	Chromatic number	Graph coloring
$\langle b\rangle$	Minimal	3	
$\langle a^2b\rangle$	Minimal	3	
$\langle a^4b\rangle$	Minimal	3	

TABLE 5. Graph properties of the non-normal cyclic subgroup graph of $\langle a^6b\rangle, \langle b, a^4b\rangle, \langle a^2b, a^6b\rangle, \langle a^5b\rangle$ and $\langle a^3b\rangle$ associated with quasidihedral group of order 16

Subgroup	Maximal or minimal	Chromatic number	Graph coloring
$\langle a^6b\rangle$	Minimal	3	
$\langle b, a^4b\rangle$	Maximal	4	
$\langle a^2b, a^6b\rangle$	Maximal	4	
$\langle a^5b\rangle$	Minimal	4	
$\langle a^3b\rangle$	Minimal	4	

4. CONCLUSION

This research has focused on determining the non-normal cyclic subgroup graphs for a quasidihedral group, specifically QD_{16}, where $n = 4$. The non-normal cyclic subgroup graph of the quasidihedral group of order 16 can be viewed as a union of several connected graphs. Its chromatic number is either 3 or 4, which is consistent with the chromatic numbers of the corresponding graphs in the dihedral and generalized quaternion groups. For future studies, the non-normal cyclic subgroup graphs for quasidihedral groups of other orders can be explored using a similar approach. In subsequent research, the general form of these graphs could also be established. Additionally, future work could investigate the topological indices of the non-normal cyclic subgroup graph of quasidihedral groups.

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REFERENCES

[1] B. Mondal and K. De, "An overview applications of graph theory in real field," *International Journal of Scientific Research in Computer Science, Engineering and Information Technology*, vol. 2, no. 5, pp. 751–759, 2017. https://www.academia.edu/34939310/An_Overview_Applications_of_Graph_Theory_in_Real_Field.

[2] W. Martens and T. Popp, "The complexity of regular trail and simple path queries on undirected graphs," in *Proceedings of the 41st ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems*, pp. 165–174, 2022. https://doi.org/10.1145/3517804.3524149.

[3] B. N. Al-Hasanat, A. S. Al-Hasanat, and J. Ma'an, "Order graph: A new representation of finite groups," *International Journal of Mathematics and Computer Science*, vol. 14, no. 4, pp. 809–819, 2019. https://future-in-tech.net/14.4/R-Hasanat.pdf.

[4] D. F. Anderson, J. Fasteen, and J. D. LaGrange, "The subgroup graph of a group," *Arabian Journal of Mathematics*, vol. 1, no. 1, pp. 17–27, 2012. https://link.springer.com/article/10.1007/s40065-012-0018-1.

[5] N. F. Rahin, N. H. Sarmin, and S. Ilangovan, "The non-normal subgroup graph of some dihedral groups," *ASM Science Journal*, vol. 13, pp. 1–7, 2020. https://www.akademisains.gov.my/asmsj/?mdocs-file=4862.

[6] N. F. Rahin, N. Sarmin, and S. Ilangovan, "The non-normal subgroup graph for some generalised quaternion groups.," *Southeast Asian Bulletin of Mathematics*, vol. 44, no. 6, 2020. http://www.seams-bull-math.ynu.edu.cn/downloadfile.jsp?filemenu=_202006&filename=10_44(6).pdf.

[7] N. Nurhabibah, I. G. A. W. Wardhana, and N. W. Switrayni, "Numerical invariants of coprime graph of a generalized quaternion group," *Journal of the Indonesian Mathematical Society*, pp. 36–44, 2023. https://doi.org/10.22342/jims.29.1.1245.36-44.

[8] N. F. Rahin, *The non-normal cyclic subgroup graph for dihedral and generalized quaternion groups and its Moore-Penrose inverse matrix*. PhD thesis, Universiti Teknologi Malaysia, 2024.

[9] F. Kakeri and A. Erfanian, "The complement of subgroup graph of a group," *Journal of Prime Research in Mathematics*, vol. 11, no. 1, pp. 55–60, 2015. https://jprm.sms.edu.pk/index.php/jprm/article/view/108.

[10] O. Pahil Muhidin, *Prime order and composite order Cayley graphs of generalised quaternion group and quasi-dihedral group*. PhD thesis, Universiti Teknologi Malaysia, 2020.

[11] M. A. El-Sanfaz, N. H. Sarmin, and S. N. A. Zamri, "Generalized commuting graph of dihedral, semi-dihedral and quasi-dihedral groups," *Malaysian Journal of Fundamental and Applied Sciences*, vol. 17, no. 6, pp. 711–719, 2021.

[12] V. K. Bhat and S. K. Sharma, "Zagreb and wiener indices of the conjugacy class graph of the quasi-dihedral and generalized quaternion groups," *Appl. Math*, vol. 16, no. 5, pp. 815–822, 2022. http://naturalspublishing.com/files/published/vn979fk1211hi3.pdf.