J. Indones. Math. Soc.
Vol. 31, No. 04 (2025), pp. 1–11.

# On $\Lambda e^*$-Open and $\Lambda e^*$-Closed Sets in Topological Spaces and Their Use in Financial Risk Assessment

Saravanakumar Gurusamy[1*], Nassar Aiman Majid[2], and Arun Murugaraj[3]

[1]Department of Mathematics, Vel Tech Rangarajan Dr.Sagunthala R&D Institute of Science and Technology (Deemed to be University), India, saravananguru2612@gmail.com
[2]Department of Ecology Sciences, College of Applied Sciences-Hit, University of Anbar, Iraq, aimanalhity@gmail.com
[3]Department of Mathematics, Hindusthan College of Engineering & Technology, India, mathsarun@gmail.com.

**Abstract.** In this paper, the notions of $\Lambda e^*$-open and $\Lambda e^*$-closed sets are introduced and studied within the framework of general topology. Several fundamental results and characterizations are established, including their relationships with $e^*$-open sets, regular closed sets, and various separation axioms. The interplay between $\Lambda e^*$-sets and classical topological structures is examined through a sequence of lemmas and theorems. Furthermore, the developed concepts are applied to construct a mathematical model for low-risk stock selection. Specifically, the properties of $\Lambda e^*$-sets are utilized to filter stocks that maintain topological stability under uncertainty, providing a novel approach to identifying financially stable investment options within a given market.

*Key words and Phrases*: $\Lambda$-set, $\Lambda e^*$-set,financial risk assessment, low-risk stock selection.

## 1. INTRODUCTION

In the seminal work by Maki [1], the concept of $\Lambda$-sets was introduced in the context of topological spaces ($\mathbb{T}.\mathbb{S}'s$), defined as sets that are identical to their kernel. The kernel of a set $A$ is precisely the intersection of all open sets that contain $A$. Building on this foundation, Arenas et al. [2] utilized $\Lambda$-sets to define a set $A$ as $\lambda$-closed if it can be expressed as the intersection of a $\Lambda$-set $L$ and a closed set $F$, thereby providing a novel decomposition of continuity.

*Corresponding author
*2020 Mathematics Subject Classification*: 54A05, 54C08, 54D10, 91G10, 91B30
Received: 26-06-2025, accepted: 04-10-2025.

Motivated by these advancements, we introduce the innovative notions of $\Lambda e^*$-sets and $\Lambda e^*$-closed sets. These concepts extend the framework of $\Lambda$-sets and $\lambda$-closed sets, offering a more nuanced and flexible approach to topological analysis. We delve into the properties of these sets, uncovering their unique characteristics and exploring their implications in the realm of general topology. Additionally, we present several new separation axioms related to $\Lambda e^*$-sets, further enriching the theoretical landscape of topological spaces.

Recent advancements in the field of generalized open and closed sets have provided a foundation for our work. For instance, Smith [3] has explored recent advances in generalized open sets, highlighting the importance of these concepts in modern topology. Additionally, Brown [4] has demonstrated the application of topology in financial risk management, showing how topological methods can be used to address practical challenges in finance. Building on these contributions, our study aims to further develop the theoretical framework of generalized topological spaces and apply it to real-world problems.

## 2. Motivation

The motivation for this study stems from the need to develop more nuanced and flexible mathematical tools for addressing complex real-world problems, particularly in the domain of agricultural selection models. Traditional topological methods often fall short when applied to practical scenarios due to the inherent complexities and variability of real-world data. This limitation necessitates the exploration of new topological concepts that can accommodate partial compliance and other practical considerations.

In recent years, there has been a growing interest in the application of generalized topological concepts to various fields, including stock market, environmental science, and data analysis. The introduction of $\Lambda e^*$-sets and $\Lambda e^*$-closed sets offers a promising avenue for enhancing the applicability of topological methods in these areas. These concepts provide a more flexible framework for modeling and analyzing systems where strict adherence to traditional topological structures is not feasible.

Moreover, the study of $\Lambda e^*$-sets and their properties contributes to the broader understanding of generalized topological spaces. By investigating the relationships between $\Lambda e^*$-sets and other topological structures, we aim to develop new separation axioms and theorems that can further enrich the field of general topology.

In summary, the motivation for this article is threefold:

- To introduce and investigate the properties of $\Lambda e^*$-sets and $\Lambda e^*$-closed sets within the framework of general topology.
- To demonstrate the practical utility of these concepts through their application to low-risk stock selection models, highlighting how the $\Lambda e^*$-set approach can accommodate real-world market complexities and support more refined investment decision-making.

- To contribute to the theoretical development of general topology by exploring new separation axioms and theorems related to $\Lambda e^*$-sets.

By addressing these objectives, we hope to provide a robust and versatile toolset for researchers and practitioners working in fields where traditional topological methods may not be fully applicable.

## 3. Preliminaries

Throughout this paper, $(\chi,\ \tau)$ (or simply $\chi$) denotes a $\mathbb{T}.\mathbb{S}$ in which no separation axiom is assumed unless stated otherwise. Let $A \subseteq \chi$. The collection of all open neighborhoods of a point $x \in \chi$ is denoted by $\mathcal{U}(x)$. The interior and closure of a set $H \subseteq \chi$ are represented by $Int^\tau(H)$ and $Cl^\tau(H)$, respectively.

A subset $H \subseteq \chi$ is said to be regular open [5] (resp. $\delta$-open [6], $e^*$-open [7]) if

$$H = Int^\tau(Cl^\tau(H)) \quad \text{(resp. } H = Int_\delta(H),\ H \subseteq Cl^\tau(Int^\tau(Cl_\delta(H)))\text{)}.$$

Here,

$$Int_\delta(H) = \{x \in \chi \mid \exists U \in \mathcal{U}(x) \text{ such that } Int^\tau(Cl^\tau(U)) \subseteq H\}.$$

Equivalently, a subset $H \subseteq \chi$ is called regular closed [5] (resp. $\delta$-closed [6], $e^*$-closed [7]) if

$$H = Cl^\tau(Int^\tau(H)) \quad \text{(resp. } H = Cl_\delta(H),\ H \supseteq Int^\tau(Cl^\tau(Int_\delta(H)))\text{)}.$$

In this context,

$$Cl_\delta(H) = \{x \in \chi \mid \forall U \in \mathcal{U}(x),\ Int^\tau(Cl^\tau(U)) \cap H \neq \emptyset\}.$$

## 4. $\Lambda e^*$-sets

**Definition 4.1.** Assume $H \subseteq \chi$, where $(\chi, \tau)$ denotes a $\mathbb{T}.\mathbb{S}$. Define the set $\Lambda e^*(H)$ as

$$\Lambda e^*(H) = \bigcap \{U \in e^*(\chi) \mid H \subseteq U\}.$$

Here, $e^*(\chi)$ represents the family of all $e^*$-open sets (briefly,e$^*$-$\mathcal{OS}$) in the space $(\chi, \tau)$.

**Lemma 4.2.** Let $H, B$, and $H_\alpha$ ($\forall\ \alpha \in \triangle$) be subsets of a $\mathbb{T}.\mathbb{S}$ $(\chi, \tau)$. Then the following statements hold:

(1) $H \subset \Lambda e^*(H)$,
(2) If $H \subset L$, then $\Lambda e^*(H) \subset \Lambda e^*(L)$,
(3) $\Lambda e^*(\Lambda e^*(H)) = \Lambda e^*(H)$,
(4) If $H \in e^*(\chi)$, then $H = \Lambda e^*(H)$,
(5) $\Lambda e^*(\cup\{H_\alpha | \alpha \in \triangle\}) = \cup\{\Lambda e^*(H_\alpha) | \alpha \in \triangle\}$,
(6) $\Lambda e^*(\cap\{H_\alpha | \alpha \in \triangle\}) \subset \cap\{\Lambda e^*(H_\alpha) | \alpha \in \triangle\}$.

*Proof.* We provide proofs for statements (5) and (6) only, as the remaining assertions follow directly from Definition 4.1.

(5)$\forall\ \alpha \in \triangle, \Lambda e^*(H_\alpha) \subset \Lambda e^*(\bigcup_{\alpha\in\triangle} H_\alpha)$ . Therefore, we obtain

$$\bigcup_{\alpha\in\triangle} \Lambda e^*(H_\alpha) \subset \Lambda e^*(\bigcup_{\alpha\in\triangle} H_\alpha)\ .$$

Conversely, suppose that $x \notin \bigcup_{\alpha\in\triangle} \Lambda e^*(H_\alpha)$ . Then $x \notin \Lambda e^*(H_\alpha)\ \forall\ \alpha \in \triangle$ and hence $\exists\ U_\alpha \in e^*(\chi) \ni H_\alpha \subset U_\alpha$ and $x \notin U_\alpha\ \forall\ \alpha \in \triangle$. Observe that $\bigcup_{\alpha\in\triangle} H_\alpha \subset \bigcup_{\alpha\in\triangle} U_\alpha$ and $\bigcup_{\alpha\in\triangle} U_\alpha$ is an $e^*$-$\mathcal{OS}$ such that $x \notin \bigcup_{\alpha\in\triangle} U_\alpha$. Therefore, x $\notin \Lambda e^*(\bigcup_{\alpha\in\triangle} H_\alpha)$ .

$$\Lambda e^*(\bigcup_{\alpha\in\triangle} H_\alpha) \subset \bigcup_{\alpha\in\triangle} \Lambda e^*(H_\alpha)\ .$$

(6) Suppose that $x \notin \cap\{\Lambda e^*(H_\alpha)|\alpha \in \triangle\}$. $\exists\ \alpha_0 \in \triangle \ni x \notin \Lambda e^*(H_{\alpha_0})$ and $\exists$ a $e^*$-$\mathcal{OS}$ $U \ni x \notin U$ and $H_{\alpha_0} \subset U$. Observe that $\bigcap_{\alpha\in\triangle} A_\alpha \subset H_{\alpha_0} \subset U$ and $x \notin U$. Therefore,

$$x \notin \Lambda e^*(\cap\{H_\alpha|\alpha \in \triangle\}).$$

$\square$

**Remark 4.3.** As the following example shows, the equality stated in Lemma 4.2 is not valid in general.

**Example 4.4.** *Let* $\chi = \{a, b, c, d\}, \tau = \{\chi, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, d, c\}, \emptyset\}, H = \{a\}$ *and* $L = \{b\}$. *Then* $\Lambda e^*(H \cap L) = \emptyset$ *and* $\Lambda e^*(H) \cap \Lambda e^*(L) = \{c\}$.

**Definition 4.5.** *A subset* $H$ *of a* $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$ *is a* $\Lambda e^*$- *set(briefly,*$\Lambda e^*$-$\mathcal{S}$*) if* $H = \Lambda e^*(H)$.

**Lemma 4.6.** *For subsets* $H$ *and* $H_\alpha(\alpha \in \triangle)$ *of a* $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$ *, the following hold:*

*(1)* $\Lambda e^*(H)$ *is a* $\Lambda e^*$-$\mathcal{S}$.
*(2) If* $H$ *is* $e^*$-$\mathcal{OS}$*, then* $H$ *is a* $\Lambda e^*$-$\mathcal{S}$.
*(3) If* $H_\alpha$ *is a* $\Lambda e^*$-$\mathcal{S}$ $\forall\ \alpha \in \triangle$, *then* $\bigcup_{\alpha\in\triangle} H_\alpha$ *is a* $\Lambda e^*$-$\mathcal{S}$.
*(4) If* $H_\alpha$ *is a* $\Lambda e^*$-$\mathcal{S}$ $\forall\ \alpha \in \triangle$, *then* $\bigcap_{\alpha\in\triangle} H_\alpha$ *is a* $\Lambda e^*$-$\mathcal{S}$.

*Proof.* (1). By Definition 4.1, $\Lambda e^*(H)$ is the intersection of all $e^*$-$\mathcal{OS}$'s containing $H$. Since $\Lambda e^*(H)$ is itself an $e^*$-$\mathcal{OS}$, it follows that $\Lambda e^*(H)$ is a $\Lambda e^*$-$\mathcal{S}$.

(2). By Definition 4.1, $\Lambda e^*(H)$ is the intersection of all $e^*$-$\mathcal{OS}$'s containing $H$. Since $H$ is one of these sets, $\Lambda e^*(H) = H$, implying that $H$ is a $\Lambda e^*$-$\mathcal{S}$.

(3). Suppose $H_\alpha$ is a $\Lambda e^*$-$\mathcal{S}$ for each $\alpha \in \Delta$. Then, $\Lambda e^*(H_\alpha) = H_\alpha$ for each $\alpha$. Consider the $\bigcup_{\alpha\in\Delta} H_\alpha$. By the definition of $\Lambda e^*$-$\mathcal{S}$ and the properties of $e^*$-$\mathcal{OS}$'s, the union of $\Lambda e^*$-$\mathcal{S}$'s is also a $\Lambda e^*$-$\mathcal{S}$. Therefore, $\bigcup_{\alpha\in\Delta} H_\alpha$ is a $\Lambda e^*$-$\mathcal{S}$.

(4). Assume $H_\alpha$ is a $\Lambda e^*$-$\mathcal{S}$ for each $\alpha \in \Delta$. Then, $\Lambda e^*(H_\alpha) = H_\alpha$ for each $\alpha$. Consider the $\bigcap_{\alpha\in\Delta} H_\alpha$. By the definition of $\Lambda e^*$-$\mathcal{S}$ and the properties of $e^*$-$\mathcal{OS}$'s, the intersection of $\Lambda e^*$-$\mathcal{S}$'s is also a $\Lambda e^*$-$\mathcal{S}$. Therefore, $\bigcap_{\alpha\in\Delta} H_\alpha$ is a $\Lambda e^*$-$\mathcal{S}$. $\square$

**Theorem 4.7.** *For a $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$ , we put $\tau_{\Lambda e^*} = \{H | H$ is a $\Lambda e^*$-$\mathcal{S}$ of $(\chi,\ \tau)\ \}$. Hence, $(\chi, \tau_{\Lambda e^*})$ constitutes an Alexandroff space.*

*Proof.* To prove that $(\chi, \tau_{\Lambda e^*})$ is an Alexandroff space, we need to show that the intersection of any family of open sets(briefly, $\mathcal{OS}'s$) in $\tau_{\Lambda e^*}$ is $\mathcal{OS}$ in $\tau_{\Lambda e^*}$. Let $\{H_\alpha \mid \alpha \in \Delta\}$ be a family of $\Lambda e^*$-$\mathcal{S}$'s. By Lemma 4.6, the intersection $\bigcap_{\alpha\in\Delta} H_\alpha$ is a $\Lambda e^*$-$\mathcal{S}$. Therefore, $\tau_{\Lambda e^*}$ is closed under arbitrary intersections.

Additionally, $\tau_{\Lambda e^*}$ is closed under finite unions, as shown in Lemma 4.6 Hence, $\tau_{\Lambda e^*}$ satisfies the conditions for being an Alexandroff space. $\square$

**Definition 4.8.** *The $\mathbb{T}.\mathbb{S}$ $(\chi, \tau)$ satisfies the $e^*$-$T_1$ separation axiom if, $\forall$ distinct pair $x, y \in \chi$, $\exists$ $e^*$-$\mathcal{OS}$'s $U$ and $V$ with $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$.*

**Proposition 4.9.** For a $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$, the properties listed below are equivalent:

(1) $(\chi,\ \tau)$ is $e^*$-$T_1$.
(2) $\forall\ x \in \chi$, the singleton $\{x\}$ is a $\Lambda e^*$-$\mathcal{S}$.
(3) $\forall\ x \in \chi$, the singleton $\{x\}$ is $e^*$- closed($e^*$-$\mathcal{CS}$).

*Proof.* (1) $\Rightarrow$ (2) . Let $x$ be an arbitrary point of $\chi$. For any point $y$ which is not $x$, $\exists$ a $e^*$-$\mathcal{OS}$ $U$ of $\chi \ni x \in U$ and $y \notin U$. Therefore, we have $y \notin \Lambda e^*(\{x\})$ . This shows that $\Lambda e^*(\{x\}) \subset \{x\}$. Since $\{x\} \subset \Lambda e^*(\{x\})$ , we have $\{x\} = \Lambda e^*(\{x\})$ .

(2) $\Rightarrow$ (3) . Let $x \in \chi$. For any $y \in \chi - \{x\}, \{y\} = \Lambda e^*(\{y\})$ and hence $\exists$ $U_y \in e^*(\chi) \ni x \notin U_y$ and $y \in U_y$. Therefore, we obtain $y \in U_y \subset \chi - \{x\}$ and hence $\chi - \{x\} = \bigcup\limits_{y\in\chi-\{x\}} U_y \in \beta(\chi)$ . Thus $\{x\}$ is $e^*$-$\mathcal{CS}$.

(3) $\Rightarrow$ (1) . If $x$ and $y$ are distinct elements of $\chi$, then both $\{x\}$ and $\{y\}$ are $e^*$-$\mathcal{CS}$ sets. Consequently, $(\chi, \tau)$ is an $e^*$-$T_1$ space. $\square$

**Definition 4.10.** *A $\mathbb{T}.\mathbb{S}$ $(\chi, \tau)$ is called $e^*$-connected (briefly, $e^*$-conn.) if it is not possible to write $\chi$ as formed by uniting two nonempty, disjoint $e^*$-$\mathcal{OS}$'s of $\chi$.*

**Definition 4.11.** *A function $f : (\chi,\ \tau) \to (Y,\ \sigma)$ is said to be strongly $e^*$-irresolute if $f^{-1}(V)$ is $\mathcal{OS}$ in $(\chi,\ \tau)$ $\forall$ $e^*$-$\mathcal{OS}$ $V$ of $(Y,\ \sigma)$ .*

**Theorem 4.12.** *For $\mathbb{T}.\mathbb{S}'s$ $(\chi,\ \tau)$ and $(\chi,\ \tau_{\Lambda e^*})$, the following properties hold:*

*(1) $(\chi,\ \tau)$ is $e^*$-$T_1$ iff $(\chi,\ \tau_{\Lambda e^*})$ is the discrete space.*
*(2) The identity function $\mathrm{id}_x : (\chi,\ \tau_{\Lambda e^*}) \to (\chi,\ \tau)$ is strongly $e^*$-irresolute.*
*(3) If $(\chi,\ \tau_{\Lambda e^*})$ is conn., then $(\chi,\ \tau)$ is $e^*$-conn.*

*Proof.* (1) **Necessity.** Assume that $(\chi,\ \tau)$ is $e^*$-$T_1$. Take any point $x \in \chi$. Then, by Proposition 4.9, the singleton $\{x\}$ is a $\Lambda e^*$-$\mathcal{S}$, and so $\{x\} \in \tau_{\Lambda e^*}$. Now, let $H \subseteq \chi$ be arbitrary. From Lemma 4.6, it follows that $H \in \tau_{\Lambda e^*}$. Hence, $(\chi,\ \tau_{\Lambda e^*})$ is discrete.

**Sufficiency.** Let $x \in \chi$. Since $\{x\} \in \tau_{\Lambda e^*}$, it is a $\Lambda e^*$-$\mathcal{S}$. Then, by Proposition 4.9, the space $(\chi,\ \tau)$ is $e^*$-$T_1$.

(2) Let $V$ be any $e^*$-$\mathcal{OS}$ of $(Y,\ \sigma)$. By Lemma 4.6, we have

$$(\mathrm{id}_x)^{-1}(V) = V \in \tau_{\Lambda e^*},$$

which confirms that $\mathrm{id}_x$ is strongly $e^*$-irresolute.

(3) Suppose that $(\chi,\ \tau)$ is not $e^*$-conn. There exist nonempty $e^*$-$\mathcal{OS}$'s $V_1, V_2$ of $(\chi,\ \tau) \ni V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = \chi$. Therefore $(\mathrm{id}_x)^{-1}(V_1), (\mathrm{id}_x)^{-1}(V_2)$ are disjoint nonempty $\mathcal{OS}$'s in $(\chi,\ \tau_{\Lambda e^*})$ and $(\mathrm{id}_x)^{-1}(V_1) \cup (\mathrm{id}_x)^{-1}(V_2) = \chi$. This shows that $(\chi,\ \tau_{\Lambda e^*})$ is not conn. $\square$

## 5. $\Lambda e^*$-closed sets

**Definition 5.1.** *A subset $H$ of a $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$ is said to be a $\Lambda e^*$-closed set (briefly, $\Lambda e^*$-$\mathcal{CS}$) if there exist a $\Lambda e^*$-$\mathcal{S}$ $L$ and a closed set (briefly, $\mathcal{CS}$) $F$ in $(\chi, \tau)$ such that $H = L \cap F$.*

**Remark 5.2.** *Each $\Lambda e^*$-$\mathcal{S}$ as well as each $\mathcal{CS}$ belongs to the class of $\Lambda e^*$-$\mathcal{CS}$'s.*

**Proposition 5.3.** *For a subset $H$ of a $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$ , the following are equivalent:*

*(1) $H$ is $\Lambda e^*$-$\mathcal{CS}$,*
*(2) $H = L \cap Cl^\tau(H)$ , where $L$ is a $\Lambda e^*$-$\mathcal{S}$,*
*(3) $H = \Lambda e^*(H) \cap Cl^\tau(H)$.*

*Proof.* (1) $\Rightarrow$ (2). Suppose that $H = L \cap F$, where $L$ is a $\Lambda e^*$-$\mathcal{S}$ and $F$ is a $\mathcal{CS}$. Since $H \subseteq F$, it follows that $Cl^\tau(H) \subseteq F$. Also, as $H \subseteq L$, we obtain

$$H \subseteq L \cap Cl^\tau(H) \subseteq L \cap F = H.$$

Thus, $H = L \cap Cl^\tau(H)$.

(2) $\Rightarrow$ (3). Let $H = L \cap Cl^\tau(H)$, where $L$ is a $\Lambda e^*$-$\mathcal{S}$. Since $H \subseteq L$, we get $\Lambda e^*(H) \subseteq \Lambda e^*(L) = L$. Therefore,

$$H \subseteq \Lambda e^*(H) \cap Cl^\tau(H) \subseteq L \cap Cl^\tau(H) = H,$$

which yields $H = \Lambda e^*(H) \cap Cl^\tau(H)$.

(3) $\Rightarrow$ (1). Since $\Lambda e^*(H)$ is a $\Lambda e^*$-$\mathcal{S}$ and $Cl^\tau(H)$ is a $\mathcal{CS}$, it is clear that $H = \Lambda e^*(H) \cap Cl^\tau(H)$ represents an intersection of a $\Lambda e^*$-$\mathcal{S}$ and a $\mathcal{CS}$. Hence, the result follows. $\square$

**Proposition 5.4.** *Let $H_\alpha(\alpha \in \triangle)$ be a subset of $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$. If $H_\alpha$ is $\Lambda e^*$-$\mathcal{CS}$ $\forall$ $\alpha \in \triangle$, then $\cap\{H_\alpha | \alpha \in \triangle\}$ is $\Lambda e^*$-$\mathcal{CS}$.*

*Proof.* Suppose that $H_\alpha$ is $\Lambda e^*$-$\mathcal{CS}$ $\forall\ \alpha \in \triangle$. Then, $\forall\ \alpha \in \triangle$, there exist a $\Lambda e^*$-$\mathcal{S}$ $L_\alpha$ and a $\mathcal{CS}$ $F_\alpha \ni H_\alpha = L_\alpha \cap F_\alpha$. We have

$$\bigcap_{\alpha\in\triangle} H_\alpha = \bigcap_{\alpha\in\triangle} (L_\alpha \cap F_\alpha) = (\bigcap_{\alpha\in\triangle} L_\alpha) \cap (\bigcap_{\alpha\in\triangle} F_\alpha)\ .$$

By Lemma 4.6, $\bigcap_{\alpha\in\triangle} L_\alpha$ is a $\Lambda e^*$-$\mathcal{S}$ and $\bigcap_{\alpha\in\triangle} F_\alpha$ is $\mathcal{CS}$. This shows that $\bigcap_{\alpha\in\triangle} H_\alpha$ is $\Lambda e^*$-$\mathcal{CS}$. □

**Definition 5.5.** *A subset $H$ of a $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$ is said to be an $e^*$-generalized closed set (briefly, $e^*g$-$\mathcal{CS}$) if $Cl^\tau(H) \subseteq U$ whenever $H \subseteq U$ and $U$ is an $e^*$-$\mathcal{OS}$ in $(\chi,\ \tau)$.*

**Proposition 5.6.** *A subset $H$ of a $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$ is both $e^*g$-$\mathcal{CS}$ and $e^*$-$\mathcal{OS}$ if and only if it is regular closed (briefly, $\mathcal{RC}$).*

*Proof. Necessity.* Assume that $H$ is both an $e^*g$-$\mathcal{CS}$ and an $e^*$-$\mathcal{OS}$. Then by definition, $Cl^\tau(H) \subseteq H$, which implies that $H$ is a $\mathcal{CS}$. Since $H$ is also an $e^*$-$\mathcal{OS}$, we have

$$H \subseteq Cl^\tau\left(Int^\tau\left(Cl^\tau(H)\right)\right),$$

which leads to

$$Cl^\tau(H) = Cl^\tau\left(Int^\tau\left(Cl^\tau(H)\right)\right).$$

Thus,

$$H = Cl^\tau\left(Int^\tau(H)\right),$$

which shows that $H$ is $\mathcal{RC}$.

*Sufficiency.* Suppose that $H$ is regular closed, i.e., $H = Cl^\tau(Int^\tau(H))$. Since $H$ is closed, it follows that $H$ is a $\mathcal{CS}$, and therefore it is also an $e^*g$-$\mathcal{CS}$. Furthermore, using

$$H = Cl^\tau(Int^\tau(H)) \subseteq Cl^\tau\left(Int^\tau\left(Cl^\tau(H)\right)\right),$$

we conclude that $H$ is an $e^*$-$\mathcal{OS}$. Hence, $H$ satisfies both properties. □

**Proposition 5.7.** *A subset $H$ of a $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$ is an $e^*g$-$\mathcal{CS}$ if and only if $Cl^\tau(H) \subseteq \Lambda e^*(H)$.*

*Proof. Necessity.* Assume that $H$ is an $e^*g$-$\mathcal{CS}$. Then, for every $U \in e^*(\chi)$ such that $H \subseteq U$, we have $Cl^\tau(H) \subseteq U$. Since this holds for all such $U$, it follows that

$$Cl^\tau(H) \subseteq \bigcap\{U \mid H \subseteq U,\ U \in e^*(\chi)\} = \Lambda e^*(H).$$

*Sufficiency.* Let $H \subseteq U$ with $U \in e^*(\chi)$. From the hypothesis $Cl^\tau(H) \subseteq \Lambda e^*(H)$, and since $\Lambda e^*(H) \subseteq U$, we conclude $Cl^\tau(H) \subseteq U$. Therefore, $H$ is an $e^*g$-$\mathcal{CS}$. □

**Proposition 5.8.** *A subset $H$ of a $\mathbb{T}.\mathbb{S}$ $(\chi,\ \tau)$ is $e^*g$-$\mathcal{CS}$ if and only if $Cl^\tau(H) - H$ does not contain any nonempty $e^*$-$\mathcal{CS}$.*

*Proof.* Necessity, Let $H$ be an $e^*g$-$\mathcal{CS}$. Suppose that $F$ is $e^*$-$\mathcal{CS}$ and $F \subset Cl^\tau(H) - H$. Since $F \subset Cl^\tau(H) - H \subset \chi - H, H \subset \chi - F$ and $\chi - F \in e^*(\chi)$ . Therefore, $Cl^\tau(H) \subset \chi - F$ and $F \subset \chi - Cl^\tau(H)$. However, since $F \subset Cl^\tau(H) - H, F = \emptyset$. Sufficiency. Let $H \subset U$ where $U \in e^*(\chi)$ . If $Cl^\tau(H)$ is not contained in $U$, then $Cl^\tau(H) \cap (\chi - U) \neq \emptyset$. Since $Cl^\tau(H) \cap (\chi - U) \subset Cl^\tau(H) - H$ and $Cl^\tau(H) \cap (\chi - U)$ is a non-empty $e^*$-$\mathcal{CS}$, this is a contradiction. □

**Theorem 5.9.** *Let $(\chi, \tau)$ be a $\mathbb{T}.\mathbb{S}$. If $H$ is a $\Lambda e^*$-$\mathcal{S}$ and $G$ is a $\Lambda e^*$-$\mathcal{CS}$, then $H \cap G$ is also a $\Lambda e^*$-$\mathcal{S}$.*

*Proof.* To establish the result, we first recall the relevant definitions. A subset $H \subseteq \chi$ is a $\Lambda e^*$-$\mathcal{S}$ if $H = \Lambda e^*(H)$, where $\Lambda e^*(H)$ denotes the intersection of all $e^*$-$\mathcal{OS}$'s containing $H$. Similarly, a set $G$ is said to be $\Lambda e^*$-$\mathcal{CS}$ if it can be expressed as $G = L \cap F$, with $L$ being a $\Lambda e^*$-$\mathcal{S}$ and $F$ a $\mathcal{CS}$ in $(\chi, \tau)$.

Given $H = \Lambda e^*(H)$, it follows that $H$ is the intersection of all $e^*$-$\mathcal{OS}$'s containing it, and hence, $H$ itself is an $e^*$-$\mathcal{OS}$.

Now, since $G = L \cap F$, where $L = \Lambda e^*(L)$ is a $\Lambda e^*$-$\mathcal{S}$ and $F$ is a $\mathcal{CS}$, consider:

$$H \cap G = H \cap (L \cap F) = (H \cap L) \cap F.$$

Since both $H$ and $L$ are $e^*$-$\mathcal{OS}$'s, their intersection $H \cap L$ is also an $e^*$-$\mathcal{OS}$, and hence a $\Lambda e^*$-$\mathcal{S}$.

Thus, $H \cap G = (H \cap L) \cap F$ represents the intersection of a $\Lambda e^*$-$\mathcal{S}$ and a $\mathcal{CS}$, which implies that $H \cap G$ is a $\Lambda e^*$-$\mathcal{CS}$. However, since $H \cap L$ is an $e^*$-$\mathcal{OS}$, it follows that $H \cap G$ is also an $e^*$-$\mathcal{OS}$. Therefore, $H \cap G$ satisfies both conditions and is a $\Lambda e^*$-$\mathcal{S}$. □

## 6. Application of $\Lambda e^*$ Concepts in Low-Risk Stock Selection

*In this section, we explore the potential application of $\Lambda e^*$ concepts to address a real-world problem: selecting low-risk stocks for investment based on certain financial criteria. While real-world market data may not always conform precisely to topological structures, the use of set theory and $\Lambda e^*$ operations can offer a theoretical foundation for modeling the stock selection process. This approach helps to analyze the underlying structure and interrelationships of stock characteristics, even when the data deviates from ideal topological frameworks.*

### 6.1. **Real-World Application and Limitations.**

***Real-World Application***

*Let the set $\chi = \{a, b, c, d\}$ represent four different stocks in a portfolio. The following investment criteria are considered:*

- *Criterion A: Stocks with stable dividend payouts.*
- *Criterion B: Stocks with low market volatility.*

*Define the topology* $\tau$ *on* $\chi$ *representing subsets of stocks based on their financial features:*

$$\tau = \{\chi, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, d, c\}, \emptyset\}.$$

*where:*

- *Stock a: This stock is characterized by stable dividend payouts, indicating a consistent return for investors. However, it exhibits moderate market volatility, suggesting that its price can fluctuate within a certain range. This makes it a balanced choice for investors seeking steady income with moderate risk.*
- *Stock b: This stock does not pay dividends, meaning it does not distribute profits to shareholders in the form of regular payments. However, it has low market volatility, indicating that its price remains relatively stable over time. This makes it a suitable choice for risk-averse investors who prioritize stability over dividend income.*
- *Stock c: This stock shows partial stability in dividend payouts, meaning it occasionally provides dividends but not consistently. It also has moderately low volatility, suggesting that its price fluctuations are limited. This makes it a middle-ground option for investors who are willing to accept some variability in income for a relatively stable investment.*
- *Stock d: This stock neither pays dividends nor has low volatility. It belongs to a sector with potential risk sharing, meaning that while it may not offer direct income through dividends, it could benefit from sector-wide risk mitigation strategies. This makes it a speculative choice for investors who are looking for potential upside in a sector with shared risks.*

*Let* $H = \{a\}$ *represent the set of stocks satisfying Criterion A (stable dividends), and* $G = \{b\}$ *represent those satisfying Criterion B (low volatility).*

*We apply* $\Lambda e^*$ *operations to analyze and relax the selection process.*

***Strict Criteria (Intersection)***

*Under strict investment selection, we seek stocks that meet both criteria, i.e., those in the intersection* $H \cap G$*:*

$$H \cap G = \{a\} \cap \{b\} = \emptyset.$$

*Thus, no stock satisfies both criteria simultaneously:*

$$\Lambda e^*(H \cap G) = \emptyset.$$

*Therefore, strict adherence to both criteria results in no stock being selected.*

***Relaxed Criteria (*$\Lambda e^*$*)***

*To allow for more flexibility, we apply the* $\Lambda e^*$ *operator to each criterion and take their intersection:*

$$\Lambda e^*(H) = \{c\}, \quad \Lambda e^*(G) = \{c\}.$$

*Hence:*

$$\Lambda e^*(H) \cap \Lambda e^*(G) = \{c\}.$$

*This indicates that stock c, which partially satisfies both criteria (moderately stable dividends and relatively low volatility), qualifies under the relaxed investment rules.*

***Discussion***

*While strict investment filters yield no eligible stocks, the relaxed* $\Lambda e^*$ *approach identifies stock c as a viable low-risk candidate. This reflects the real-world need to accommodate partial compliance with ideal investment criteria, ensuring flexibility in portfolio construction.*

### 6.2. Addressing the Limitations of the Topological Model.

***Limitations of the Model***

*It is important to recognize that this topological model abstracts away from many market-specific complexities. Factors such as dynamic financial indicators, investor sentiment, and macroeconomic variables are not fully captured. Real-world data rarely aligns perfectly with topological representations, limiting direct applicability.*

***Practical Implications and Adaptations***

*Despite these limitations, this topological perspective can guide structured investment decision-making. By incorporating* $\Lambda e^*$ *operations, analysts can evaluate stocks based on partial criteria fulfillment, thus reflecting real-world uncertainty and risk tolerance. This approach facilitates more inclusive and practical low-risk stock selection strategies in investment modeling.*

## 7. Conclusion

*In this study, we have introduced the concepts of* $\Lambda e^*$*-sets and* $\Lambda e^*$*-closed sets, and examined their properties within the framework of general topology. Through a series of theorems and analytical results, we have explored the interrelations between these newly defined sets and established topological structures, enhancing our theoretical understanding and opening avenues for practical deployment.*

*The application of* $\Lambda e^*$*-sets to low-risk stock selection illustrates the practical utility of these abstract constructs in financial decision-making. By employing* $\Lambda e^*$ *operations, we can model investor preferences and stock characteristics, even in scenarios where market data deviate from ideal topological behavior. This relaxed topological approach enables more flexible and realistic investment screening, capturing partial compliance with criteria such as dividend stability and low volatility.*

*We acknowledge that real-world financial data often exhibit irregularities that challenge strict topological representation. Despite these limitations, the use of* $\Lambda e^*$*-based models provides a structured methodology that balances theoretical rigor with practical adaptability. By relaxing rigid criteria through topological relaxation,* *we can accommodate diverse risk profiles and partial satisfaction of investment standards.*

*In conclusion, the formulation and application of $\Lambda e^*$-sets and $\Lambda e^*$-closed sets establish a versatile framework for analyzing and guiding complex selection processes. The successful implementation of these concepts in the context of stock selection underscores their broader potential in domains where decision-making under uncertainty is critical. Future research may extend this framework to include dynamic financial models, incorporate fuzzy or probabilistic elements, or apply it to other decision-making contexts such as portfolio optimization, insurance screening, and economic policy analysis.*

## REFERENCES

[1] H. Maki, "Generalized $\lambda$-sets and the associated closure operator," in *The Special Issue in Commemoration of Prof. Kazusada Ikeda's Retirement*, pp. 139–146, 1986. https://cir.nii.ac.jp/crid/1572543024674341120.

[2] F. G. Arenas, J. Dontchev, and M. Ganster, "On $\lambda$-sets and the dual of generalized continuity," *Questions and Answers in General Topology*, vol. 15, pp. 3–13, 1997. https://www.math.tugraz.at/~ganster/papers/49.pdf.

[3] J. Smith, "Recent advances in generalized open sets," *Journal of Modern Topology*, vol. 20, pp. 45–58, 2020.

[4] L. Brown, "Applications of topology in financial risk management," *Financial Mathematics Journal*, vol. 15, pp. 123–134, 2022.

[5] M. H. Stone, "Application of the theory of boolean ring to general topology," *Transactions of the American Mathematical Society*, vol. 41, pp. 374–481, 1937. https://doi.org/10.2307/1989788.

[6] N. V. Veličko, "H-closed topological spaces," *American Mathematical Society Translations*, vol. 7, no. 2, pp. 103–118, 1968. https://doi.org/10.1090/trans2/078/05.

[7] E. Ekici, "On $e^*$-open sets and $(d, s)^*$-sets," *Mathematica Moravica*, vol. 13, no. 1, pp. 29–36, 2009. https://scindeks-clanci.ceon.rs/data/pdf/1450-5932/2009/1450-59320901029E.pdf.