J. Indones. Math. Soc.
Vol. 31, No. 04 (2025), pp. 1–11.

Some Properties of Cartesian Product of Non-Coprime Graph Associated with Finite Group

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Abstract. This paper investigates several properties of the Cartesian product of two non-coprime graphs associated with finite groups. Specifically, we focus on key numerical invariants, namely the domination number, independence number, and diameter. The non-coprime graph associated with finite group G is constructed with the vertex set $G \setminus \{e\}$ and connects two distinct vertices if and only if their orders are not coprime. Using this construction, we investigate the Cartesian products of non-coprime graphs associated with various types of groups, including nilpotent groups, dihedral groups, and p-groups. We derive several new results, including exact expressions for the domination number, independence number, and diameter of these Cartesian products.

Key words and Phrases: finite group, non-coprime graph, domination number, independence number, diameter.

1. INTRODUCTION

In recent years, numerous studies have examined graphs arising as representations of a group. Some of these studies include commuting graphs [1] and non-commuting graphs [2], cycle graphs [3], and other graphs of a group. In addition to representations of a group, research on graphs as representations of a ring has also developed. Some studies related to this include zero-divisor graphs [4], prime graphs [5], and Jacobson graphs [6].

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2020 Mathematics Subject Classification: 05C25, 05C69, 05C76
Received: 12-06-2025, accepted: 04-10-2025.

In 2014, Ma et al. [7] introduced a new concept known as the coprime graph. For any finite group G, the coprime graph over G is a simple graph with the set of vertex consisting of all elements of G. An edge exists between two distinct vertices if and only if their orders are coprime. This definition has motivated several studies on its properties, one of which examines the coprime graph associated with the generalized quaternion group [8].

In 2016, Mansoori et al. [9] introduced the complementary concept of the non-coprime graph. The non-coprime graph related to a finite group G is constructed using $G \setminus \{e\}$ as the vertex set and two distinct vertices x and y are adjacent if and only if the greatest common divisor of their orders is not equal to 1. This graph is denoted by Π_G.

Furthermore, graph theory introduces operations that combine two graphs. One of them is the Cartesian product, as described in [10]. Several characteristics of the Cartesian product associated with non-coprime graphs have been explained in [9] and [11]. Therefore, this research will discuss some other properties of the Cartesian product involving non-coprime graphs over finite groups. These properties are the domination number, independence number, and diameter. In this paper, the terminologies related to graph theory refer to [10] and that related to group theory refers to [12].

This research significantly enhances our understanding of the interplay between group theory and graph theory. By investigating the Cartesian product of non-coprime graphs of finite groups, it demonstrates how essential graph parameters—such as the domination number, independence number, and diameter—are determined by the groups' underlying algebraic structures. Moreover, the findings extend earlier studies on coprime and non-coprime graphs and reveal potential applications in network analysis, cryptography, and the modeling of complex systems. Overall, this work establishes a robust theoretical framework that lays the groundwork for further exploration in both pure and applied discrete mathematics.

2. PRELIMENARIES

We begin by recalling the definition of the non-coprime graph of a finite group.

Definition 2.1 ([9]). *The non-coprime graph associated with G, represented by Π_G, consists of the vertex set $G \setminus \{e\}$, where two distinct vertices x and y are connected by an edge if and only if their orders are non-coprime.*

Figure 1 and Figure 2 show the non-coprime graph associated with group $\mathbb{Z}_4$ and group S_3, respectively.

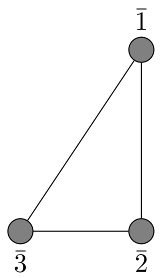

FIGURE 1. Non-coprime graph $\Pi_{\mathbb{Z}_4}$

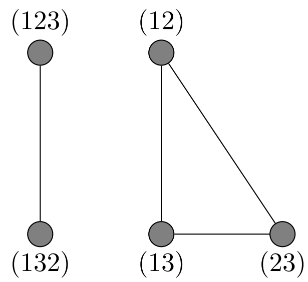

FIGURE 2. Non-coprime graph Π_{S_3}

The following propositions and theorems are previous results of the non-coprime graph associated with finite groups.

Proposition 2.2 ([9]). *The non-coprime graph Π_G is a complete graph if and only if G is a p-group.*

Proposition 2.3 ([9]). *If G is an abelian group, then diam$(\Pi_G) \leq 2$ and Π_G is connected.*

Theorem 2.4 ([9]). *If G is a nilpotent group, then the domination number of Π_G is $\gamma(\Pi_G) = 1$.*

Theorem 2.5 ([9]). *The independence number of Π_G is $\alpha(\Pi_G) = |\pi(G)|$, where $\pi(G)$ denotes the set of prime divisors of the order of the group G.*

Proposition 2.6 ([9]). *Let D_{2n} denote a dihedral group with $n \geq 4$. If n is a prime power and odd, then $\Pi_{D_{2n}}$ contains two complete components K_n and K_{n-1}.*

In Definition 2.7, the Cartesian product of two graphs is stated.

Definition 2.7 ([10]). *Let $\mathcal{G}_1 = (V(\mathcal{G}_1), E(\mathcal{G}_1))$ and $\mathcal{G}_2 = (V(\mathcal{G}_2), E(\mathcal{G}_2))$ be some graphs. The cartesian product of $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined as a graph whose vertex set is $V(\mathcal{G}_1)\times V(\mathcal{G}_2)$ in which two distinct vertices (x_1, y_1) and (x_2, y_2) are connected by an edge if and only if either $x_1 = x_2$ and $y_1y_2 \in E(\mathcal{G}_2)$, or $y_1 = y_2$ and $x_1x_2 \in E(\mathcal{G}_1)$.*

For example, the Cartesian product of $\Pi_{\mathbb{Z}_4}$ and Π_{S_3} is shown in Figure 3.

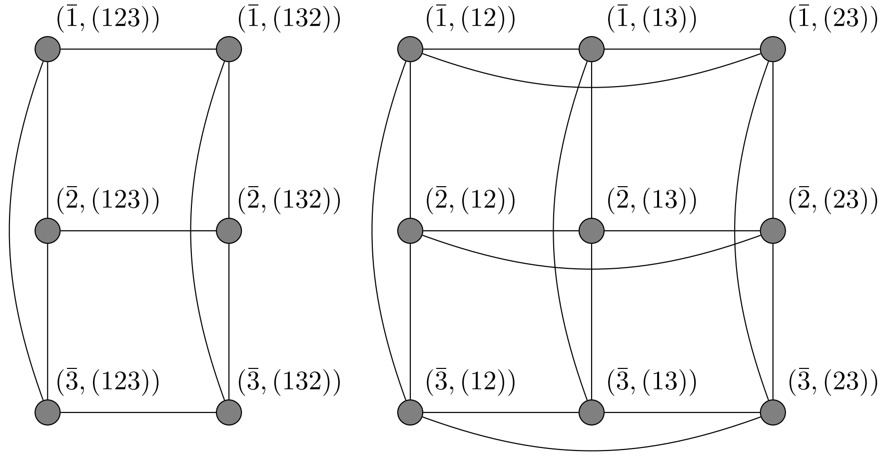

FIGURE 3. Cartesian Product $\Pi_{\mathbb{Z}_4} \times \Pi_{S_3}$

In this article, the following assumptions are used unless otherwise stated. Let Π_G and Π_H be non-coprime graphs of group G and group H respectively. Suppose that $V(\Pi_G) = \{x_1, x_2, \ldots, x_n\}$ and $V(\Pi_H) = \{y_1, y_2, \ldots, y_m\}$ for some $n, m \in \mathbb{N}$ and $m \geq n$. We form a partition $P = \{V_1, V_2, \ldots, V_n\}$ on $V(\Pi_G \times \Pi_H)$, where

$$V_i = \{(x_i, y) \mid y \in V(\Pi_H)\} \text{ for every } i = 1, 2, \ldots, n.$$

Let Π_{V_i} be the induced subgraph of $\Pi_G \times \Pi_H$ on V_i for each $i = 1, 2, \ldots, n$. Then, we have Π_{V_i} isomorphic to Π_H.

Let $\mathcal{G}$ be a graph. A subset $D \subset V(\mathcal{G})$ is called a dominating set of $\mathcal{G}$ if every vertex in $V(\mathcal{G}) \setminus D$ is adjacent to at least one vertex in D. Furthermore, the domination number of $\mathcal{G}$, denoted by $\gamma(\mathcal{G})$, is the size of the smallest dominating set in the graph. The independent set of a graph $\mathcal{G}$ is the subset $A \subseteq V(\mathcal{G})$ in which there are no two adjacent vertices. Denoted by $\alpha(\mathcal{G})$, the independence number of $\mathcal{G}$ is defined as the number of vertices in its largest independent set. The diameter of $\mathcal{G}$, written as $diam(\mathcal{G})$, is defined as the greatest distance between any pair of vertices in $V(\mathcal{G})$.

3. MAIN RESULTS

In this chapter, we discuss various properties of the Cartesian product of two non-coprime graphs associated with finite groups related to domination number, independence number, and diameter.

Several properties related to the domination number of non-coprime graphs over finite groups, such as nilpotent groups, dihedral groups, p-groups, and so on, have been presented in [9]. The domination number for the Cartesian product of two non-coprime graphs over a nilpotent group is given in Theorem 3.2. However, the following lemma is given before proving Theorem 3.2.

Lemma 3.1. *Given a non-coprime graph Π_G with domination number $\gamma(\Pi_G) = 1$ and a non-coprime graph Π_H with domination number $\gamma(\Pi_H) = 1$. The domination number for the Cartesian product graph $\Pi_G \times \Pi_H$ is*

$$\gamma(\Pi_G \times \Pi_H) = \min\{|V(\Pi_G)|, |V(\Pi_H)|\}.$$

Proof. Given non-coprime graphs Π_G with $\gamma(\Pi_G) = 1$ and Π_H with $\gamma(\Pi_H) = 1$. Therefore, $\gamma(\Pi_{V_i}) = 1$. Let $A_i = \{(x_i, y_i)\}$ be the minimum dominating set of Π_{V_i} for each $i = 1, 2, \ldots, n$. We form $A = \bigcup_{i=1}^{n} A_i$. Let $u \in V(\Pi_G \times \Pi_H) \setminus A$ be an arbitrary point with $u = (x_s, y_t)$ where $1 \leq s \leq n$ and $1 \leq t \leq m$. This implies that $u \in V_s = V(\Pi_{V_s})$, and hence there exists $a_s \in A_s \subset A \subset V(\Pi_G \times \Pi_H)$ such that $ua_s \in E(\Pi_G \times \Pi_H)$. Therefore, A is a dominating set of $\Pi_G \times \Pi_H$.

Furthermore, consider any set $B \subset V(\Pi_G \times \Pi_H)$ with $|B| = k < n = |A|$. Since $k < n$, there exists a partition V_i such that for each $v \in V_i$, $v \notin B$. On the other hand, as $k < n < m$, there exists a point $v_i \in V_i \subset V(\Pi_G \times \Pi_H) \setminus B$, such that for each $b \in B$, $v_i b \notin E(\Pi_G \times \Pi_H)$. Thus, B is not a dominating set, which shows that A is the minimum dominating set of $\Pi_G \times \Pi_H$. In other words,

$$\gamma(\Pi_G \times \Pi_H) = |A| = n = \min\{|V(\Pi_G)|, |V(\Pi_H)|\}.$$

□

Theorem 3.2. *Let G and H be some nilpotent groups. The domination number for the Cartesian product $\Pi_G \times \Pi_H$ is equal to* $\min\{|V(\Pi_G)|, |V(\Pi_H)|\}$.

Proof. Given nilpotent groups G and H, by Theorem 5.(i) in [9], the domination number of Π_G and Π_H are $\gamma(\Pi_G) = 1$ and $\gamma(\Pi_H) = 1$. According to Lemma 3.1, we obtain $\gamma(\Pi_G \times \Pi_H) = \min\{|V(\Pi_G)|, |V(\Pi_H)|\}$. □

The following theorem provides the domination number for the Cartesian product of non-coprime graphs over a nilpotent group and non-coprime graphs over a dihedral group.

Theorem 3.3. *Let G be a nilpotent group and D_{2n} be a dihedral group with n being an odd prime power. The domination number for the Cartesian product $\Pi_G \times \Pi_{D_{2n}}$ is equal to*

$$\min\{|V(\Pi_G)|, n\} + \min\{|V(\Pi_{D_{2n}})|, n-1\}.$$

Proof. Based on Proposition 8.(i) [9], the non-coprime graph $\Pi_{D_{2n}}$ consists of two complete components, namely K_n and K_{n-1}. Since G is a nilpotent group, we have $\gamma(\Pi_G) = 1$, so that Π_G is a connected graph. As a result, the Cartesian product $\Pi_G \times \Pi_{D_{2n}}$ consists of two components, namely the Cartesian products $C_1 = \Pi_G \times K_n$ and $C_2 = \Pi_G \times K_{n-1}$. We now compute the domination number of each component separately. For the component C_1, since $\gamma(K_n) = 1$ and based on Lemma 3.1, we obtain $\gamma(C_1) = \min\{|V(\Pi_G)|, n\}$. Using the same method, we obtain $\gamma(C_2) = \min\{|V(\Pi_G)|, n-1\}$. Therefore,

$$\begin{aligned}\gamma(\Pi_G \times \Pi_{D_{2n}}) &= \gamma(C_1) + \gamma(C_2) \\ &= \min\{|V(\Pi_G)|, n\} + \min\{|V(\Pi_G)|, n-1\}.\end{aligned}$$

□

The domination number for the Cartesian product of two non-coprime graphs over the dihedral group is given in Theorem 3.4 below.

Theorem 3.4. *Given the dihedral groups D_{2n} and D_{2m} where n and m are odd prime power. The domination number for the Cartesian product $\Pi_{D_{2n}} \times \Pi_{D_{2m}}$ is equal to $4n-2$ if $n < m$ and equal to $4n-3$ if $n = m$.*

Proof. Based on Proposition 8.(i) in [9], the non-coprime graph $\Pi_{D_{2n}}$ consists of two complete components, namely K_n and K_{n-1}, and the non-coprime graph $\Pi_{D_{2m}}$ consists of two complete components, namely K_m and K_{m-1}. As a result, the Cartesian product $\Pi_{D_{2n}} \times \Pi_{D_{2m}}$ consists of four components, namely the Cartesian products $C_1 = K_n \times K_m$, $C_2 = K_n \times K_{m-1}$, $C_3 = K_{n-1} \times K_m$, and $C_4 = K_{n-1} \times K_{m-1}$. Then, it is observed that

$$\gamma(K_n) = \gamma(K_{n-1}) = \gamma(K_m) = \gamma(K_{m-1}) = 1.$$

As a result, based on Lemma 3.1, we obtain $\gamma(C_1) = \min\{n, m\}$, $\gamma(C_2) = \min\{n, m-1\}$, $\gamma(C_3) = \min\{n-1, m\}$, and $C_4 = \min\{n-1, m-1\}$. Therefore, for $n < m$, we have,

$$\begin{aligned}\gamma(\Pi_{D_{2n}} \times \Pi_{D_{2m}}) =& \gamma(C_1) + \gamma(C_2) + \gamma(C_3) + \gamma(C_4) \\ =& \min\{n, m\} + \min\{n, m-1\} + \min\{n-1, m\} \\ & + \min\{n-1, m-1\} \\ =& n + n + n - 1 + n - 1 \\ =& 4n - 2.\end{aligned}$$

Meanwhile, for $n = m$, we obtain

$$\begin{aligned}\gamma(\Pi_{D_{2n}} \times \Pi_{D_{2m}}) =& \gamma(C_1) + \gamma(C_2) + \gamma(C_3) + \gamma(C_4) \\ =& \min\{n, m\} + \min\{n, m-1\} + \min\{n-1, m\} \\ & + \min\{n-1, m-1\} \\ =& n + n - 1 + n - 1 + n - 1 \\ =& 4n - 3.\end{aligned}$$

□

The following theorems explain the independence number for the Cartesian product of two non-coprime graphs.

Theorem 3.5. *Let p and q be prime numbers. If G and H are a p-group and a q-group respectively, then*

$$\alpha(\Pi_G \times \Pi_H) = \min\{|V(\Pi_G)|, |V(\Pi_H)|\}.$$

Proof. Suppose G is a p-group and H is a q-group. Based on Theorem 6 in [9] we obtain $\alpha(\Pi_G) = 1$ and $\alpha(\Pi_H) = 1$. Without loss of generality, suppose $\min\{|V(\Pi_G)|, |V(\Pi_H)|\} = |V(\Pi_G)| = n$. Thus, we get $\alpha(\Pi_{V_i}) = 1$ for every $i = 1, 2, \ldots, n$. Suppose that $A_i = \{(x_i, a_i) \mid a_i \in V(\Pi_H)\}$ is the maximum independent set of Π_{V_i} and $A = \cup_{i=1}^{n} A_i$ where $a_i \neq a_j$ for each $i \neq j$. For any $u, v \in A$ with $u \neq v$, suppose $u = (x_i, a_i)$ and $v = (x_j, a_j)$. Thus, $x_i \neq x_j$ and $a_i \neq a_j$. By the definition of the Cartesian product, we get $uv \notin E(\Pi_G \times \Pi_H)$. Therefore, A is an independent set of $\Pi_G \times \Pi_H$. Take an arbitrary set $B \subset V(\Pi_G \times \Pi_H)$ with $|B| > |A|$. Consequently, there are $b_1, b_2 \in B$ with $b_1, b_2 \in V_i$ for some i. We have $b_1 b_2 \in E(\Pi_G \times \Pi_H)$, because $\alpha(\Pi_{V_i}) = 1$. This means that B is not an independent set, so that A is the maximum independent set of $\Pi_G \times \Pi_H$. Hence,

$$\alpha(\Pi_G \times \Pi_H) = |A| = n = \min\{|V(\Pi_G)|, |V(\Pi_H)|\}.$$

□

Theorem 3.6. *Let G be a p-group and H be an arbitrary group with the number of distinct prime divisors of $|H|$ being 2. Suppose k is the number of distinct maximum independent sets in Π_H that do not intersect. Then,*

(i) $\alpha(\Pi_G \times \Pi_H) = 2|V(\Pi_G)|$, *if* $|V(\Pi_G)| \leq k$,
(ii) $\alpha(\Pi_G \times \Pi_H) = \min\{k + |V(\Pi_G)|, |V(\Pi_H)|\}$, *if* $|V(\Pi_G)| > k$.

Proof. Let G be a p-group and H be an arbitrary group with the number of distinct prime divisors of H being 2. Based on Theorem 6 in [9] obtained $\alpha(\Pi_H) = 2$. Suppose that all distinct maximum independent sets in Π_H that do not intersect are $B_1, B_2, \ldots, B_k$. Consider the following two cases.

(i) First case, for $|V(\Pi_G)| \leq k$. For all $i = 1, 2, \ldots, n$ the sets $A_i = \{(x_i, b) \mid b \in B_i\}$ are the maximum independent set in Π_{V_i}. Define $A = \cup_{i=1}^{n} A_i$. Note that, B_i for all $i = 1, 2, \ldots, n$ are the distinct maximum independent sets in Π_H that do not intersect, then A is maximum independent set in $\Pi_G \times \Pi_H$. Therefore,

$$\alpha(\Pi_G \times \Pi_H) = |A| = \sum_{i=1}^{n} |A_i| = \sum_{i=1}^{n} |B_i| = 2n = 2|V(\Pi_G)|.$$

(ii) Second case, for $|V(\Pi_G)| > k$. For all $i = 1, 2, \ldots, k$ the set $A_i = \{(x_i, b) \mid b \in B_i\}$ are the maximum independent set in Π_{V_i}. Define

$$C_j = \{(x_j, y_{t_j}) \mid y_{t_j} \in V(\Pi_H) \setminus \cup_{i=1}^{k} B_i\}$$

for all $k < j \leq n$ and $C = \cup_{j=k+1}^{n} C_j$ is defined by the condition $y_{t_j} \neq y_{t_s}$ for each $j \neq s$. Note that, B_i for all $i = 1, 2, \ldots, k$ are the distinct maximum independent sets in Π_H that do not intersect, then $\cup_{i=1}^{k} A_i$ is independent set in $\Pi_G \times \Pi_H$. By the definition of the Cartesian product, C is the independent set in $\Pi_G \times \Pi_H$. Therefore $A = \cup_{i=1}^{k} A_i \cup C$ is the independent set in $\Pi_G \times \Pi_H$. Consider that, $\left|\bigcup_{i=1}^{k} A_i\right| = k \cdot \alpha(\Pi_H) = 2k$. Then, we have two following cases.

(a) If $|V(\Pi_H)| - 2k \geq n - k$, then $|C| = n - k$. Therefore,

$$|A| = \left|\bigcup_{i=1}^{k} A_i\right| + |C| = 2k + n - k = k + |V(\Pi_G)|.$$

Furthermore, take arbitrary set $D \subseteq V(\Pi_G \times \Pi_H)$ with $|D| > |A|$. If there is $E \subset D$ such that $E \subset \Pi_{V_i}$ for some $i \in \{1, 2, \ldots, n\}$ and E is not independent set, then D is not independent set. If not, it means that the set $E_i \subset D$ such that $E_i \subset \Pi_{V_i}$ for all $i \in \{1, 2, \ldots, n\}$ is an independent set. The number of E_i with $|E_i| = 2$ for some $i \in \{1, 2, \ldots, n\}$ must be more than k. Hence there exist $(x_i, y), (x_j, y) \in D$ with $i \neq j$ such that (x_i, y) adjacent to (x_j, y). Thus, D is not an independent set, and hence A is the maximum independent set.

(b) If $|V(\Pi_H)| - 2k < n - k$, then $|C| = |V(\Pi_H)| - 2k$. Therefore,

$$|A| = \left|\bigcup_{i=1}^{k} A_i\right| + |C| = 2k + |V(\Pi_H)| - 2k = |V(\Pi_H)|.$$

Furthermore, take arbitrary set $D \subseteq V(\Pi_G \times \Pi_H)$ with $|D| > |A|$. Thus, there exist $(x_i, y), (x_j, y) \in D$ with $i \neq j$ such that (x_i, y) adjacent to (x_j, y). Hence, D is not an independent set, so that A is the maximum independent set.

Based on these two cases, we have $\alpha(\Pi_G \times \Pi_H) = \min\{k + |V(\Pi_G)|, |V(\Pi_H)|\}$. □

In Lemma 3.7, we give the independence numbers for non-coprime graph of dihedral group D_{2n} with $n = p^k$.

Lemma 3.7. *Let D_{2n} be a dihedral group. If $n = p^k$, for some odd prime p and $k \in \mathbb{N}$, then $\alpha(\Pi_{D_{2n}}) = 2$ with the number of distinct maximum independent sets that do not intersect is $n - 1$.*

Proof. Based on Proposition 8 in [9], the non-coprime graph $\Pi_{D_{2n}}$ contains two complete components K_n and K_{n-1}. Thus, $\alpha(\Pi_{D_{2n}}) = 2$ and the maximum independent set are $\{a, b\}$ with $a \in K_n$ and $b \in K_{n-1}$. Hence, the number of distinct maximum independent sets in $\Pi_{D_{2n}}$ that do not intersect is $n - 1$. □

By Theorem 3.6 and Lemma 3.7, we derive the corollary below.

Corollary 3.8. *Let G be a p-group and D_{2n} be a dihedral group in which n is an odd prime power. Then,*

(i) $\alpha(\Pi_G \times \Pi_H) = 2|V(\Pi_G)|$, *if* $|V(\Pi_G)| \leq n - 1$,
(ii) $\alpha(\Pi_G \times \Pi_H) = 2n - 1$, *if* $|V(\Pi_G)| > n - 1$.

Proof. Consider G to be a p-group and D_{2n} to be a dihedral group in which n is an odd prime power. The number of distinct prime divisors of $|D_{2n}|$ is 2. Furthermore, based on Lemma 3.7 the number of distinct maximum independent sets that do not intersect is $n-1$.

(i) By Theorem 3.6 we get $\alpha(\Pi_G \times \Pi_H) = 2|V(\Pi_G)|$ for $|V(\Pi_G)| \leq n-1$.
(ii) By Theorem 3.6 we have $\alpha(\Pi_G \times \Pi_H) = \min\{n-1+|V(\Pi_G)|, |V(\Pi_{D_{2n}})|\}$. Note that $n-1+|V(\Pi_G)| > n-1+n-1 = 2n-2$. Since $n \in \mathbb{N}$, the smallest value of $n-1+|V(\Pi_G)|$ is $2n-1$. Thus, $\alpha(\Pi_G \times \Pi_H) = \min\{2n-1, 2n-1\} = 2n-1$.

□

For the last discussion in this chapter, we examine the diameter of the Cartesian product of non-coprime graph associated with finite groups, specifically the p-group and the cyclic group. In the following proposition, we give the diameter of the Cartesian product of two non-coprime graph associated with the p-group.

Proposition 3.9. *Let G and H be a p-group and a q-group, respectively. Then, diam*$(\Pi_G \times \Pi_H) = 2$.

Proof. Consider G and H to be a p-group and a q-group, respectively, and by Proposition 4 [9] both Π_G and Π_H are complete graphs. Given $(a,b),(c,d) \in V(\Pi_G \times \Pi_H)$. If (a,b) and (c,d) are adjacent, then $d((a,b),(c,d)) = 1$. Otherwise, since both Π_G and Π_H are complete graphs, we have a path $(a,b)-(c,b)-(c,d)$. Moreover, $d((a,b),(c,d)) = 2$. Thus, $\mathrm{diam}(\Pi_G \times \Pi_H) = 2$. □

In the theorem below, we compute the diameter of Cartesian product of non-coprime graph related to a p-group and non-coprime graph related to a cyclic group that is not a p-group.

Theorem 3.10. *Let p and q be prime numbers. If G is a p-group and H is a cyclic group that is not a q-group, then diam*$(\Pi_G \times \Pi_H) = 3$.

Proof. Suppose G is a p-group and H is a cyclic group that is not q-group. By Proposition 4 [9], Π_G is a complete graph. Since H is a cyclic group, there is $y \in H$ such that $\langle y \rangle = H$. Hence, vertex y adjacent to every vertex of graph Π_H.

Take two arbitrary vertices $(a,b),(c,d) \in V(\Pi_G \times \Pi_H)$. If $(a,b),(c,d) \in V_i$ for some $i = 1,2,\ldots,n$, by Proposition 1 [9] we have

$$d((a,b),(c,d)) \leq \mathrm{diam}(\Pi_{V_i}) = \mathrm{diam}(\Pi_H) \leq 2.$$

If $(a,b) \in V_i$ and $(c,d) \in V_j$ for some $1 \leq i,j \leq n$ with $i \neq j$, the analysis is divided into the following three cases

(i) If (a,b) is adjacent to (c,d), then $d((a,b),(c,d)) = 1$.
(ii) If (a,b) is not adjacent to (c,d) and $bd \in E(\Pi_H)$, then we can find a path $(a,b)-(c,b)-(c,d)$. Thus, $d((a,b),(c,d)) = 2$.
(iii) If (a,b) is not adjacent to (c,d) and $bd \notin E(\Pi_H)$, then we can find a path $(a,b)-(a,y)-(c,y)-(c,d)$. Thus, $d((a,b),(c,d)) = 3$.

Hence, $\mathrm{diam}(\Pi_G \times \Pi_H) = 3$. □

We also determine the diameter for the Cartesian product of two non-coprime graphs related to the cyclic group but not a p-group.

Theorem 3.11. *Let G and H be two cyclic groups. If G and H are not p-group, then diam*$(\Pi_G \times \Pi_H) = 4$.

Proof. Suppose $G = \langle g \rangle$ and $H = \langle h \rangle$ for some $g \in G$ and $h \in H$. Thus, g is adjacent to every vertex in Π_G and h is adjacent to every vertex in Π_H. Let $V(\Pi_G) = \{x_i, x_2, \ldots, x_n\}$ and $V(\Pi_H) = \{y_1, y_2, \ldots, y_m\}$. We form $P = \{V_1, V_2, \ldots, V_n\}$ and $Q = \{U_1, U_2, \ldots, U_m\}$ as partitions on $V(\Pi_G \times \Pi_H)$ with

$$V_i = \{(x_i, y) \mid y \in V(\Pi_H)\} \text{ for every } i = 1, 2, \ldots, n$$

and

$$U_j = \{(x, y_j) \mid x \in V(\Pi_G)\} \text{ for every } j = 1, 2, \ldots, m,$$

respectively.

Furthermore, for every $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, the subgraphs Π_{V_i} and Π_{U_j} are induced subgraphs of $\Pi_G \times \Pi_H$ by V_i and U_j, respectively. By the definition of Cartesian product, we have that Π_{V_i} is isomorphic to Π_H and Π_{U_j} isomorphic to Π_G. Take arbitrary two vertices $(a, b), (c, d) \in V(\Pi_G \times \Pi_H)$. For $a = c$, $(a, b), (c, d) \in V_i$ holds, for some $i \in \{1, 2, \ldots, n\}$. By Proposition 1 [9], we obtain

$$d((a, b), (c, d)) \leq \text{diam}(\Pi_{V_i}) = \text{diam}(\Pi_H) \leq 2.$$

For $b = d$, $(a, b), (c, d) \in U_j$ holds for some $j \in \{1, 2, \ldots, m\}$. Therefore,

$$d((a, b), (c, d)) \leq \text{diam}(\Pi_{U_j}) = \text{diam}(\Pi_G) \leq 2.$$

For $a \neq c$ and $b \neq d$, consider the following cases.

(i) If (a, b) is not adjacent to (c, d) and $ac \in E(\Pi_G)$ or $bd \in E(\Pi_H)$, then we can find one of three paths, i.e. $(a, b) - (c, b) - (c, d)$, $(a, b) - (c, b) - (c, h) - (c, d)$, and $(a, b) - (a, d) - (g, d) - (c, d)$. Thus, $d((a, b), (c, d)) \leq 3$.

(ii) If (a, b) is not adjacent to (c, d), $ac \notin E(\Pi_G)$, and $bd \notin E(\Pi_H)$, then we can find a path $(a, b) - (a, h) - (a, d) - (g, d) - (c, d)$. Thus, $d((a, b), (c, d)) = 4$.

Hence, $\text{diam}(\Pi_G \times \Pi_H) = 4$. □

4. CONCLUDING REMARKS

In this paper, we have explored several properties of the Cartesian product of two non-coprime graphs over finite groups, specifically focusing on the domination number, independence number, and diameter. Through various lemmas and theorems, we have derived new results, expanding the current understanding of non-coprime graphs, particularly for nilpotent groups, dihedral groups, and p-groups. These results enrich the ongoing research in algebraic graph theory by establishing new structural properties of non-coprime graphs under Cartesian products.

Acknowledgement. The authors express their sincere gratitude to the anonymous referees for their valuable comments and constructive suggestions, which have significantly improved the quality and clarity of this manuscript.

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