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Coupon Coloring of Snark Graphs

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Abstract. A k-coupon coloring of a graph G is a k-coloring of G by colors $[k]$ $= \{1, 2, ..., k\}$ such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum integer k for which a k-coupon coloring exists is called the coupon coloring number of G, and it is denoted by $\chi_c(G)$. Every d-regular graph G has $\chi_c(G) \geq (1 - o(1))d/\log d$ as $d \to \infty$, and the proportion of d-regular graphs G for which $\chi_c(G) \leq (1 + o(1))d/\log d$ tends to 1 as $|V(G)| \to \infty$. Coupon coloring is known to be NP-complete for k-regular graphs, even when $k \geq 3$. Snarks form a subclass of cubic graphs that are non-Hamiltonian. This motivated us to focus on investigating coupon coloring specifically in the context of snark graphs.

Key words and Phrases: coupon coloring number, snark graph, regular graph.

1. INTRODUCTION

In this paper we consider the graphs that are simple, finite and undirected. Let $V(G)$ and $E(G)$ be the vertex set and edge set of the graph $G = (V, E)$ respectively. The neighbourhood of the vertex $x \in V(G)$, denoted as $\mathcal{N}(x)$ is the set of all vertices that are adjacent to x. The degree of a vertex x in a graph is the number of vertices adjacent to x. The minimum degree of a graph is denoted as $\delta(G)$. For the standard graph terminology notions, we in general follow [1, 2]. Graph coloring is one of the important and fertile area in the field of Graph Theory. Among various coloring problems, Chen et al.[3] introduced coupon coloring in the year 2015. Let G be a graph with no isolated vertices. A k-coupon coloring of G is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G, and is denoted by $\chi_c(G)$ [3]. The term coupon coloring is motivated by viewing colors as different types of coupons, where each vertex is required to obtain from its neighbors coupons of all types. If we imagine users $v_1, v_2, \ldots, v_n$

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each holding one bit of a k-bit message and being connected to certain other users, then a user can reconstruct the entire message from their contacts if and only if the graph of contacts has a k-coupon coloring. The task given a graph of contacts is to determine the coupon coloring number of the graph, to maximize the length of the message that can be transmitted [3]. Clearly, $\chi_c(G) \leq \delta(G)$ for any graph G. In a k-coloring c, a vertex v is considered to be a bad vertex if its neighborhood lacks vertices of every color from $[k]$ and it is clear that there are no bad vertices in a coupon coloring [4].

The coupon coloring number is also referred to as the total domatic number [5]. The concept of the total domatic number of a graph was introduced by Cockayne, Dawes, and Hedetniemi in [5]. A subset S of the vertex set $V(G)$ of a graph G is a total dominating set if every vertex of G is adjacent to at least one vertex from S. The maximum number of disjoint total dominating sets is called the total domatic number. In coupon coloring every color class must be a total dominating set in the graph. The coupon coloring number of cycle, wheel, unicyclic and bicyclic graph, complete graph, and complete k-partite graph were determined by Y Shi et al. in [4]. Additionally, coupon coloring has been examined in [6] and [7]. The coupon coloring number of a few binary products, including the lexicographic and cartesian products are studied in [8, 9]. From the viewpoint of complexity, the problem of deciding whether a graph G has a coupon coloring at most 3 is NP-complete, even when restricted to planar graphs of maximum degree 9. It is also NP-complete to determine whether the coupon coloring number of a bipartite planar graph with bounded maximum degree is at least 3.

Snark is a connected bridgeless non-Hamiltonian cubic graph with edge chromatic number four. Every cubic graph has an edge chromatic number of either three or four according to Vizing's theorem, hence a snark is associated with the specific case of four. The earliest and smallest known snark is the Petersen graph. In [10] Isaacs defined the Flower snark family and the Blanuša families. In 1989 John J. Watkins discovered the snark with 50 vertices known as Watkins snark [11]. The Szekers snark is also a snark with 50 vertices discovered by George Szekeres in the year 1973. For a study on snarks we cite the papers [12, 13, 14].

For every d-regular graph G, we have $\chi_c(G) \geq (1 - o(1))d/\log d$ as $d \to \infty$, and the proportion of d-regular graphs G for which $\chi_c(G) \leq (1+o(1))d/\log d$ tends to 1 as $|V(G)| \to \infty$ [3]. Also for every $k \geq 3$, it is NP-complete to decide whether $dt(G) \geq k$, where G is k-regular. Figure 1 shows a cubic graph with a coupon coloring number equal to 3.

This motivation led us to explore whether any cubic graphs exist with a coupon coloring number of 2, rather than 3. Specifically, we investigated the snark graph, a well-known cubic graph, to determine if its coupon coloring number is indeed 2 or 3. We examine the coupon coloring of certain infinite families of snark graphs in Section 2 and that of some finite families in Section 3.

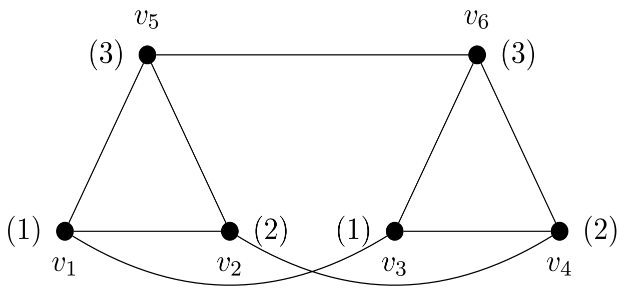

FIGURE 1. Cubic Graph with $\chi_c(G) = 3$

2. Coupon Coloring of Generalised Snarks

2.1. **Flower snark.**

Rufus Isaacs introduced the flower snark J_n in the year 1975 and the construction is as follows. Create n copies of the star graph on 4 vertices. Label each star's outer vertices B_i, C_i, and D_i, as well as the central vertex A_i. For $1 \leq i \leq n$, this generates a disconnected graph on $4n$ vertices with $3n$ edges (A_i–B_i, A_i–C_i, and A_i–D_i). Construct the n-cycle $(B_1...B_n)$ by adding n edges. Lastly, construct the $2n$-cycle $(C_1...C_nD_1...D_n)$ by adding $2n$ edges. By construction, the Flower snark J_n is a cubic graph with $6n$ edges and $4n$ vertices. Here, n must be odd for it to satisfy the necessary properties. Flower snark J_5 is shown in Figure 2.

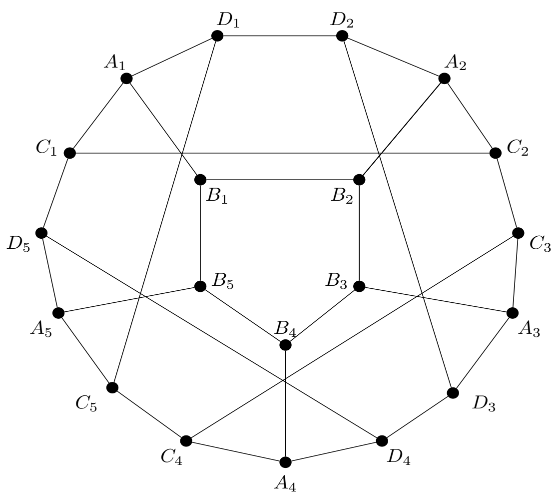

FIGURE 2. Flower Snark J_5

Theorem 2.1. *Let $G = J_n$, for odd n, be a flower snark with $4n$ vertices and $6n$ edges. Then $\chi_c(G) = 2$.*

Proof. Let $G = J_n$ be a flower snark with $4n$ vertices with vertex set $V(G) = \{A_i, B_i, C_i, D_i | 1 \leq i \leq n\}$. Since $\delta(G) = 3$, $\chi_c(G) \leq 3$. Assume that 3-coupon coloring exists. Let $c : V(G) \to [3]$ be the 3-coupon coloring. Consider B_1, the neighboring vertices of B_1 are A_1, B_2, B_n. Since $\chi_c(G) \leq 3$ and $c : V(G) \to [3]$ A_1 can get any one of the color from the set $\{1, 2, 3\}$. Without loss of generality assume that $c(B_2) = 1, c(B_n) = 2, c(A_1) = 3$. For A_n, the neighboring vertices are B_n, C_n and D_n. Since $c(B_n) = 2$, $c(C_n), c(D_n) \in \{1, 3\}$. Suppose $c(C_n) = 1$, then $c(D_n) = 3$, which is a contradiction since neighboring vertices of C_1 are A_1, D_n, C_2 and $c(A_1) = c(D_n) = 3$. Suppose $c(C_n) = 3$, then $c(D_n) = 1$ and we get a contradiction since neighboring vertices of D_1 are A_1, D_2, C_n and $c(A_1) = c(C_n) = 3$. Hence 3-coupon coloring is not possible. Now we prove 2-coupon coloring is possible. Define $c : V(G) \to [2]$ as follows.

Case 1: $\lfloor \frac{n}{2} \rfloor$ odd. Let $c(C_1) = c(D_1) = 2$,$c(B_1) = 1$, $c(A_i) = 1$ for all $i = 1, 2, \ldots, n$ and $c(B_i) = 2$ for all $i = 2, \ldots, n$. Define

$$c(C_i) = \begin{cases} 2 & \text{if } \lfloor \frac{i}{2} \rfloor \text{ is odd} \\ 1 & \text{if } \lfloor \frac{i}{2} \rfloor \text{ is even} \end{cases} \tag{1}$$

$$c(D_i) = \begin{cases} 1 & \text{if } \lfloor \frac{i}{2} \rfloor \text{ is odd} \\ 2 & \text{if } \lfloor \frac{i}{2} \rfloor \text{ is even.} \end{cases} \tag{2}$$

Consider an arbitrary vertex A_i for $i = \{2, \ldots, n\}$. From Equation 2.1 and 2.1, $c(C_i)$ and $c(D_i)$ are different for any i. Thus neighboring vertices of A_i has 2 colors. Neighboring vertices of A_1 are C_1, D_1, B_1 and $c(C_1) = c(D_1) = 2$ and $c(B_1) = 1$. For any B_i, $c(A_i) = 1$ for all i and $c(B_i) = 2$ for all $i = 2, \ldots, n$. Thus neighboring vertices of B_i also has 2 colors. Now for $i = 3, 4, \ldots, n-1$, consider the vertices C_i and D_i. Neighboring vertices of C_i are A_i, C_{i-1}, C_{i+1} and neighboring vertices of D_i are A_i, D_{i-1}, D_{i+1}. Then $c(C_{i-1})$ and $c(C_{i+1})$ will be different. Similarly in the case of D_i, C_1, D_1. The neighboring vertices of C_2 are C_1, C_3, A_2 and neighboring vertices of D_2 are D_1, D_3, A_2. Also $c(A_2) = c(A_2) = c(D_1) = 1$ and $c(C_1) = c(C_3) = c(D_1) = 2$. Similarly for C_n and D_n.

Case 2: $\lfloor \frac{n}{2} \rfloor$ even. Let $c(C_n) = c(D_n) = 2$ and $c(B_n) = 1$. $c(A_i) = 1$ for all $1 \leq i \leq n$ and $c(B_i) = 2$ for all $1 \leq i \leq n-1$. Now for $1 \leq i \leq n-1$,

$$c(C_i) = \begin{cases} 2 & \text{if } \lfloor \frac{i}{2} \rfloor \text{ is odd} \\ 1 & \text{if } \lfloor \frac{i}{2} \rfloor \text{ is even} \end{cases} \tag{3}$$

$$c(D_i) = \begin{cases} 1 & \text{if } \lfloor \frac{i}{2} \rfloor \text{ is odd} \\ 2 & \text{if } \lfloor \frac{i}{2} \rfloor \text{ is even.} \end{cases} \tag{4}$$

It is clear that c is a coupon coloring. Thus $\chi_c(G) = 2$. □

2.2. **Blanuša Snark.**

Watkins[15, 11] generalised the construction of First and Second Blanuša Snarks, defining two infinite families of generalised Blanuša Snarks. Let $B^1 = \{B^1_1, B^1_2,\ B^1_3, ...\}$ be the first family of generalised Blanuša Snarks and let $B^2 = \{B^2_1, B^2_2,\ B^2_3, ...\}$ be the second family of generalised Blanuša Snarks. The first member of B^1 is the First Blanuša Snark, while the first member of B^2 is the Second Blanuša Snark. In order to construct the first family, we consider the drawing of the Petersen Graph shown in the of Figure 3(a). In this figure, the leftmost drawing shows the first operand, and the next drawing shows the second operand. In order to construct B^1_1, remove edges u_1v_1 and u_2v_2 from the first operand, remove vertices z_1 and z_2 from the second operand, and add edges $u_1y_1, v_1x_1, u_2y_2, v_2x_2$. Then we obtain the graph B^1_1. Each member B^1_i of B^1, with $i > 1$, is obtained by the dot product $B^1_{i-1} \cdot P$, with B^1_{i-1} playing the role of P in the construction of B^1_1. Then assign the labels for B^1_i according to figure 3(b). The construction of family B^2 is similar to the construction of family B^1 [16].

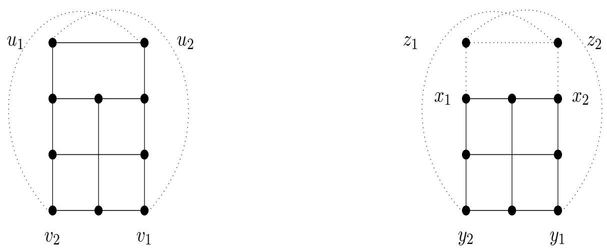

Fig.3(a)

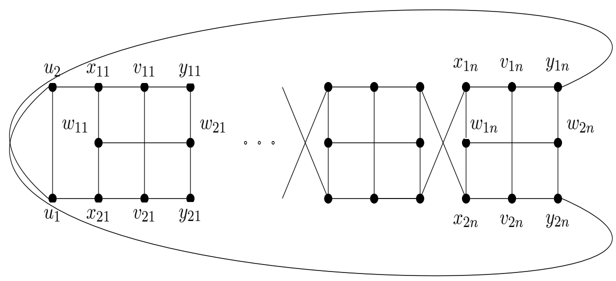

Fig.3(b)

FIGURE 3. (a)Basic Blocks of First Blanuša Construction, (b)Generalised Blanuša B^1_i of The First Family B^1

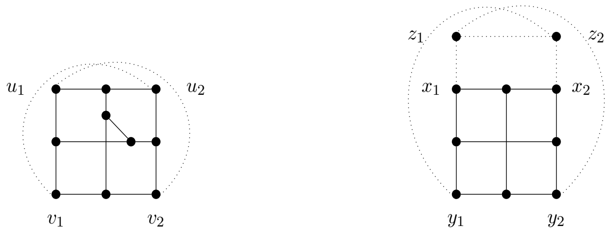

Fig.4(a)

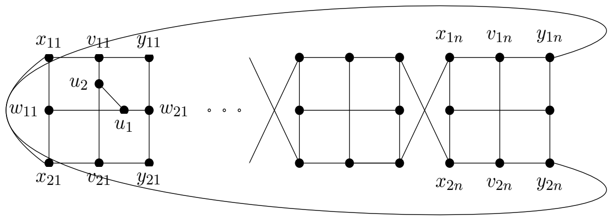

Fig.4(b)

FIGURE 4. (a)Basic Blocks of Second Blanuša Construction, (b)Generalised Blanuša B_i^2 of The Second Family B^2

Theorem 2.2. *Let B_i^1, $i \geq 1$ be the Generalised Blanuša snark of the first family B^1 of order $8i+10$. Then $\chi_c(B_i^1) = 2$.*

Proof. Let B_i^1 be the Generalised Blanuša snark of the first family B^1 of order $8n + 10$. Assume that the coupon coloring number of G is 3, then there exists $c : V(G) \to [3]$. Consider the vertex u_2. The neighbors of u_2 are x_{11}, u_1 and y_{2n}. Without loss of generality assume that $c(x_{11}) = 1$, $c(u_1) = 2$ and $c(y_{2n}) = 3$. The neighbors of w_{2n} are y_{2n}, y_{1n} and w_{12} and we have $c(y_{1n}), c(w_{1n}) \in \{1, 2\}$.
Case 1: $c(y_{1n}) = 1$. Then $c(w_{1n}) = 2$. After considering the neighboring vertices of u_1, we have $c(u_2), c(x_{21}) \in \{2, 3\}$.
Case 1.1: $c(u_2) = 2$. Thus we have $c(x_{21}) = 3$. Since $c(u_1) = 2$ and the neighbors of y_{1n} must have 3 colors. We have $c(v_{1n}), c(w_{2n}) \in \{1, 3\}$. When $c(v_{1n}) = 3$, we have a contradiction, thus $c(v_{1n}) = 1$ and hence $c(w_{2n}) = 1$. Now after considering the neighbors of y_{2n}, we have $c(v_{2n}) = 1$. This will lead to a contradiction.
Case 1.2: $c(u_2) = 3$. Then $c(x_{21}) = 2$ and we get $c(v_{1n}) = 3$ and $w_{2n} = 1$ when the vertex y_{1n} is taken into consideration. Finally we arrive at a contradiction when the neighboring vertices of y_{2n} are considered and $c(v_{2n}) = 2$.
Case 2: $c(y_{1n}) = 2$. Thus $c(w_{1n}) = 1$. Now we consider u_1. Then $c(u_2), c(x_{21}) \in \{1, 3\}$.
Case 2.1: $c(u_2) = 1$. Then $c(x_{21}) = 3$. After considering the vertex y_{1n}, we will arrive at a contradiction since either x_{1n} will become a bad vertex or v_{2n} will

become a bad vertex.
Case 2.2: $c(u_2) = 3$. Then we have $c(x_{21}) = 1$. Now consider the vertex y_{1n}, we have $c(v_{1n}) \in \{1, 3\}$. In both cases either x_{1n} will become bad vertex or v_{2n} will become a bad vertex. Thus we get contradiction in all cases. Hence $\chi_c(G) \leq 2$. Now it remains to show that $\chi_c(G) \geq 2$. Define $c : V(G) \to [2]$ as follows. $c(u_1) = c(u_2) = 1$, $c(x_{1i}) = c(v_{1i}) = c(y_{1i}) = 1$ for all $1 \leq i \leq n$ and $c(x_{2i}) = c(v_{2i}) = c(y_{2i}) = c(w_{ij}) = 2$ for all $1 \leq i \leq n$ and $j = 1, 2$. □

Theorem 2.3. *Let B_i^2, $i \geq 1$ be the Generalised Blanuša snark of the second family B^2 of order $8i + 10$. Then $\chi_c(B_i^2) = 2$.*

Proof. Let B_i^2 be the Generalised Blanuša snark of the second family B^2 of order $8i + 10$. We have $\chi_c(G) \leq 3$. Assume that 3-coupon coloring $c : V(G) \to [3]$ exists. Consider the vertex y_{1n}, the neighbors of y_{1n} are v_{1n}, w_{2n} and x_{21}. Without loss of generality assume that $c(v_{1n}) = 1$, $c(w_{2n}) = 2$ and $c(x_{21}) = 3$. Now after considering the vertex y_{2n}, we have $v_{2n} \in \{1, 3\}$. If $c(v_{2n}) = 1$, we arrive at a contradiction. Thus $c(v_{2n}) = 3$ and $c(x_{11}) = 1$. The neighborhood of w_{1n} consists of the vertices x_{1n}, x_{2n} and w_{2n}. Also $c(x_{1n}) \in \{1, 3\}$. Suppose $c(x_{1n}) = 3$ we can conclude that v_{1n} is a bad vertex. Thus $c(x_{1n}) = 1$ and hence $c(x_{2n}) = 3$. Since $c(x_{1n}) = 1$, $c(v_{2n}) = 3$, we get $c(y_{1n}) = 2$. Also after considering v_{2n}, we get $c(y_{2n}) = 2$ which leads to a contradiction. Thus $\chi_c(G) \leq 2$. Define $c : V(G) \to [2]$ as follows. $c(x_{1i}) = c(v_{1i}) = c(w_{1i}) = c(u_1) = 1$ for all $i = 1, 2, \ldots, n$. The remaining vertices can be colored using the color 2. Consider an arbitrary vertex x_{1i} where $1 \leq n$. $c(v_{1i}) = 1$ and $c(y_{2i}) = 2$ for all i. For any vertex v_{1i} and y_{1i}, $c(x_{1i}) = 1$, $c(y_{1i}) = 2$, $c(v_{1i}) = 1$, $c(w_{2i}) = 2$. Similarly for all vertices the neighborhood contains two colors. □

3. Coupon Coloring of some more Snark families

3.1. **Loupikene Snark.**

In 1976, Isaacs [17] introduced two families of Loupekine snarks. There are two Loupikene snarks on 22 vertices. One is the Loupikene snark and other is the Twisted Loupikene snark. Both snarks can be obtained from a basic block illustrated in Figure 5 consisting of vertices $\{a_i, b_, c_i, d_i, e_i, g_i | 1 \leq i \leq 3\}$. Along with these vertices a new vertex u is introduced and joined with the vertices d_1, d_2 and d_3. The edge set of loupikene snark is $\{c_1c_3, a_3f_3, b_3g_3, c_3e_2, a_2f_2, b_2g_2, c_2e_1, a_1f_1, b_1g_1, f_3g_1, f_3g_3, f_2g_3, g_2f_2, f_1g_2, f_1g_1, g_1f_3, g_3f_2, g_2f_1\}$. The edge set of Twisted Loupikene snark is $\{c_1c_3, a_3f_3, b_3g_3, f_2e_2, a_2f_2, b_2g_2, c_2e_1, a_1f_1, b_1g_1, f_3g_1, f_3g_3, f_2c_3, g_2f_2, f_1g_2, f_1g_1, g_1f_3, g_3f_2, g_2f_1\}$.

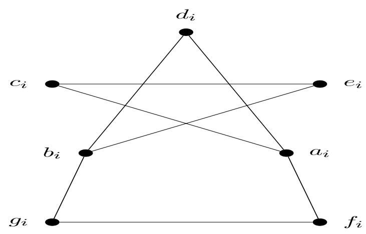

FIGURE 5. Basic Block B_i

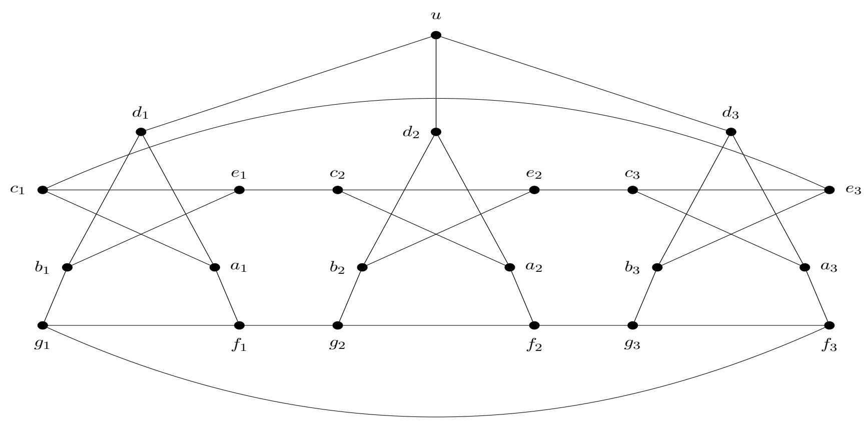

FIGURE 6. Loupikene Snark

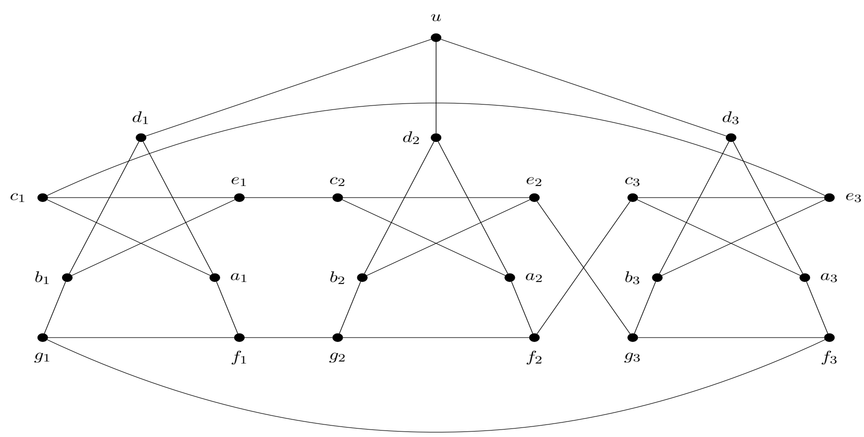

FIGURE 7. Twisted Loupikene Snark

Theorem 3.1. *Let G be a Loupikene snark on 22 vertices. Then $\chi_c(G) = 2$.*

Proof. Since $\delta(G) = 3$, $\chi_c(G) \leq 3$. Assume that a 3-coupon coloring $c : V(G) \to [3]$ exists. Consider the vertex f_1, the neighboring vertices are g_1, a_1, g_2. Without loss of generality assume that $c(g_1) = 1$, $c(a_1) = 2$ and $c(g_2) = 3$. After considering f_2 and then f_3, we get $c(a_2) = 1$, $c(g_3) = 2$ and $c(a_3) = 3$. Otherwise neighboring vertices of f_3 will get the same color. We already have $c(a_1) = 2$, hence the color of neighboring vertices of d_1 will be either 1 or 3. So b_1 will have the color 1 or 3. In both cases we get contradiction. Thus 3-coupon coloring is not possible. Define $c : V(G) \to [2]$ as follows. $c(a_i) = c(c_i) = 1$ and $c(b_i) = c(e_i) = 2$ for $i = 1, 3$, $c(f_i) = c(g_i) = 1$ for $i = 2, 3$, $c(d_1) = c(b_2) = c(e_2) = 1$ and $c(d_3) = c(a_2) = c(c_2) = c(d_2) = c(f_1) = c(g_1) = c(u) = 2$. Therefore $\chi_c(G) = 2$. □

Theorem 3.2. *Let G be a Twisted Loupikene snark on 22 vertices. Then $\chi_c(G) = 2$.*

Proof. Let G be a Twisted Loupikene snark on 22 vertices. We have $\chi_c(G) \leq 3$. Suppose there exists a 3-coupon coloring $c : V(G) \to [3]$. Consider the vertex c_1, the neighbors are a_1, e_1, e_3. Without loss of generality assume that $c(a_1) = 1, c(e_1) = 2$ and $c(e_3) = 3$. The neighboring vertices of c_3 are e_3, a_3, f_2. Since $c(e_3) = 3$ we have two cases, either $c(a_3) = 1$ or $c(a_3) = 2$.
Case1: $c(a_3) = 1$ and $c(f_2) = 2$. Consider the vertex c_2, again we have two cases. Either $c(a_2) = 1$ or $c(a_2) = 3$.
Case1.1: $c(e_2) = 1$ and $c(a_2) = 3$. Similarly when we consider the vertex d_1, we get $c(b_1) = 2$ or $c(b_1) = 3$. Here $c(b_1) = 3$ and $c(u) = 2$. Suppose $c(u) = 3$ and $c(b_1) = 2$. Then $c(u) = c(a_2) = 3$, which leads to a contradiction. Thus $c(b_1) = 3$ and $c(u) = 2$. After considering the neighboring vertices of d_3, y_3, d_2 and g_1, we get $c(b_3) = 3$, $c(f_3) = 2$, $c(b_2) = 1$ and $c(f_1) = 1$. This leads to a contradiction.
Case1.2: $c(a_2) = 1$ and $c(e_2) = 3$. Considering the vertex d_1, we have two cases. Either $c(b_1) = 2$ or $c(b_1) = 3$.
Case 1.2.1: $c(b_1) = 2$ and $c(u) = 3$. The neighboring vertices of d_2 are a_2, b_2 and u. We get $c(b_2) = 2$, which leads to a contradiction.
Case 1.2.2: $c(b_1) = 3$, $c(u) = 2$. Consider the vertex d_3. Since $c(a_3) = 1$, $c(u) = 2$, we get $c(b_3) = 3$. Also $c(b_3) = c(e_2) = 3$.
Case 2: $c(a_3) = 2$, $c(f_2) = 1$. The neighbors of the vertex c_2 are e_1, a_2 and e_2. Since $c(e_1) = 2$, we have $c(a_2) = 1$ or $c(a_2) = 3$.
Case 2.1: $c(a_2) = 1$, $c(e_2) = 3$.
Considering the vertex d_1, we get two cases $c(b_1) = 2$ or $c(b_1) = 3$.
Case 2.1.1: $c(b_1) = 2$. Since $c(a_3) = 2$, $c(u) = 3$ we have $c(b_3) = 1$. Also $c(f_3) = 2$. Thus g_1 will become a bad vertex.
Case 2.1.2: $c(b_1) = 3$, $c(u) = 2$. Suppose $c(b_1) = 3$, then $c(u) = 2$. Also $c(a_3) = 2$. As a result d_3 will become a bad vertex.
Case 2.2: $c(a_2) = 3$, $c(e_2) = 1$. The neighborhood of d_1 consists of the vertices a_1, b_1 and u. Here arises two cases $c(b_1) = 2$ or $c(b_1) = 3$. If $c(b_1) = 2$, then $c(u) = 3$ and we get d_2 as bad vertex. Similarly when $c(b_1) = 3$ we get d_3 as

a bad vertex. Thus 3-coupon coloring is not possible. Define $c : V(G) \to [2]$ as follows. $c(a_i) = c(c_i) = 1$ and $c(b_i) = c(e_i) = 2$ for $i = 1, 3$. For $i = 2, 3$ we define $c(f_i) = c(g_i) = 1$. Also we define $c(d_1) = c(b_2) = c(e_2) = 1$. Remaining vertices can be colored with 2. It can be easily seen that c is a coupon coloring. □

3.2. **Celmins-swart snark.**

The Celmins-swart snarks are the two snarks on 26 vertices and 39 edges. The vertices of Celmins-swart snark are $\{a_x, b_x, c_x, d_x, e_x, f_x,\ g_x, h_x, i_x, j_x,\ x = 1, 2\} \bigcup \{k, l, m, n, o, p\}$. Figure 8 and Figure 9 shows first Celmins-swart snark and second Celmins-swart snark.

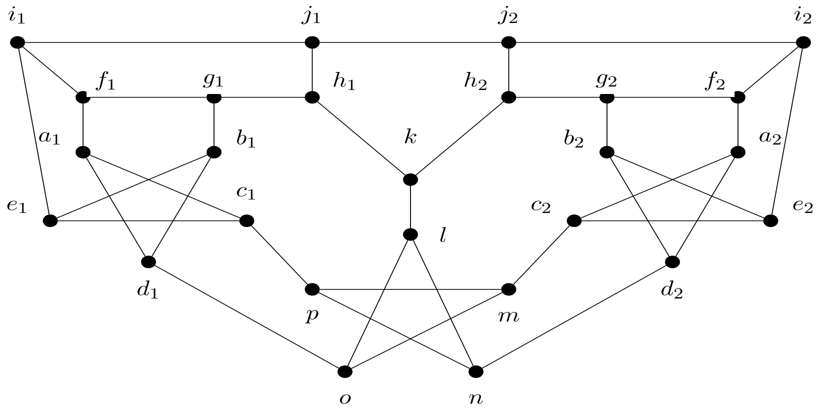

FIGURE 8. First Celmins-swart snark

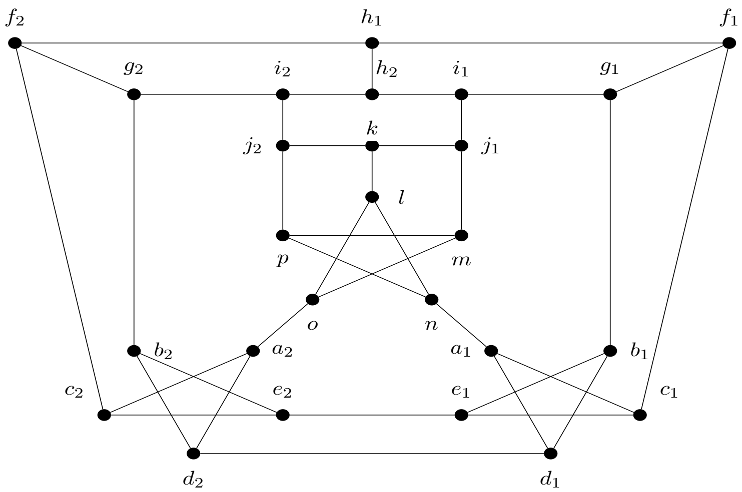

FIGURE 9. Second Celmins-swart snark

Theorem 3.3. *Let G be the first Celmins-swart snark on 26 vertices. Then $\chi_c(G) = 2$.*

Proof. Let G be the first Celmins swart snark on 26 vertices. We have $\chi_c(G) \leq 3$. Assume that a 3-coupon coloring $c : V(G) \to [3]$ exists. First consider the vertex g_1, the neighbors are f_1,b_1 and h_1. Without loss of generality assume that $c(f_1) = 1$, $c(b_1) = 2$ and $c(h_1) = 3$. Now for the neighboring vertices of i_1, we have $c(j_1) \in \{2, 3\}$.
Case 1: $c(j_1) = 2$
Then $c(e_1) = 3$. Since i_2 is the neighbor of j_2 and $c(j_1) = 2$. Also we have $c(i_2) \in \{1, 3\}$. If $c(i_2) = 1$, then $c(h_2) = 3$ which gives a contradiction. Thus $c(i_2) = 3$ and $c(h_2) = 1$. Now for the vertex g_2, $c(f_2)$, $c(b_2) \in \{2, 3\}$. $c(b_2) = 3$ will give rise to a contradiction. Thus $c(b_2) = 2$ and $c(f_2) = 3$. Similarly when we consider a_2, we get $c(c_2) = 1$ and $c(d_2) = 2$. Finally we consider the neighbors of the vertex n. Already we have $c(d_2) = 2$. So $c(l), c(p) \in \{1, 3\}$. Suppose $c(l) = 1$, we have $c(l) = c(h_2) = 1$ and if $c(l) = 3$ we have $c(l) = c(h_1) = 3$.
Case 2 : $c(j_1) = 3$
Then $c(e_1) = 2$. For the neighboring vertices of j_2, $c(i_2)$, $c(h_2) \in \{1, 2\}$.
Case 2.1 : $c(i_2) = 1$
Then $c(h_2) = 2$. Since f_2 and b_2 are neighbors of g_2, $c(f_2)$, $c(b_2) \in \{1, 3\}$. If $c(b_2) = 1$, we arrive at a contradiction. So $c(c_2) = 2$ and $c(d_2) = 3$. Finally consider the neighbors of the vertex n. We already have $c(d_2) = 3$, $c(l), c(p) \in \{1, 2\}$. When $c(l)$ takes any color from 1 or 2 we get a contradiction.
Case 2.2 : $c(i_2) = 2$
Then $c(h_2) = 1$. The vertices f_2, b_2 and h_2 are neighbors of g_2 and $c(h_2) = 1$. Thus $c(b_2), c(f_2) \in \{2, 3\}$. We get $c(b_2) = 3$ and $c(f_2) = 2$. Now consider the vertex a_2. Since $c(f_2) = 2$, we get $c(c_2) = 1$ and $c(d_2) = 3$. Finally when the vertex n is taken into consideration we arrive at a contradiction. Thus 3-coupon coloring is not possible. Therefore $\chi_c(G) \leq 2$. Define $c : V(G) \to [2]$ as follows. $c(a_x) = c(b_x) = c(c_x) = c(d_x) = c(e_x) = c(j_x) = 1$ for all $x = 1, 2$ and $c(k) = c(l) = 1$. For all other vertices we give the color 2. Thus neighbors of every vertex has 2 colors. Thus $\chi_c(G) = 2$. □

Theorem 3.4. *Let G be the second Celmins-swart snark on 26 vertices. Then $\chi_c(G) = 2$.*

Proof. Let G be the second Celmins swart snark on 26 vertices. Since $\delta(G) = 3$, then $\chi_c(G) \leq 3$. Assume that a 3-coupon coloring exists. Consider the vertex k, the neighboring vertices are j_2, j_1 and l. Without loss of generality assume that $c(j_2) = 1$, $c(j_1) = 2$ and $c(l) = 3$. For the vertex i_2 the neighbors are j_2, h_2 and g_2. We have $c(j_2) = 1$, then $c(h_2), c(g_2) \in \{2, 3\}$. If $c(h_2) = 2$ we get a contradiction. So $c(h_2) = 3$, thus $c(g_2) = 2$. Now we get $c(g_1) = 1$ when the vertex i_1 is taken into consideration. Also we get $c(h_1), c(c_1) \in \{2, 3\}$. Avoiding the contradictions we get $c(h_1) = 3$ and $c(c_1) = 2$. The vertices f_2 and f_1 are neighbors of h_1 and already $c(h_2) = 3$, then $c(f_2) \in \{1, 2\}$.

Case 1: $c(f_2) = 1$
Then $c(f_1) = 2$. Now for the vertex h_2, $c(i_2)$, $c(i_1) \in \{1, 2\}$. If $c(i_1) = 2$ we get a contradiction. Thus $c(i_2) = 2$ and $c(i_1) = 1$. Now two of the neighbors of the vertices g_2, g_1, e_1 and e_2 have 2 colors. From this we get $c(b_2) = 3$, $c(b_1) = 3$, $c(e_1) = 2$ and $c(e_2) = 1$. This gives us a contradiction.
Case 2 : $c(f_2) = 2$
Thus $c(f_1) = 1$. Following the same procedure of Case 1 we get $c(i_2) = 1$, $c(i_1) = 2$, $c(b_2) = 3$, $c(b_1) = 3$, $c(c_2) = 1$, $c(e_1) = 2$ and $c(e_2) = 1$. Finally consider the vertex a_2, $c(c_2) = 1$, $c(d_2) \in \{2, 3\}$. In both cases we get a contradiction. Thus 3-coupon coloring is not possible. Hence $\chi_c(G) \leq 2$. Define $c : V(G) \to [2]$ as follows. $c(a_2) = c(b_2) = c(c_2) = c(d_2) = c(e_2) = c(h_1) = c(f_1) = c(g_1) = c(h_2) = c(k) = c(l) = c(p) = c(n) = 1$. Remaining vertices can be colored with the color 2. □

3.3. **Double star snark.**

The Double-star snark $DS(n, r)$ is a snark with $n = 5$ and $r = 2$ on 30 vertices and 45 edges. Let $\{u_i, v_i, w_i, x_i, y_i, z_i\}$ be the vertices of $DS(n, r)$, where $1 \leq i \leq 5$. The link edges of $DS(n, r)$ forms $2n$-cycles with vertices $\{x_1, y_2, x_3, y_4, x_5, y_1, x_2, y_3, x_4, y_5, x_1\}$ and $\{w_1, z_3, w_5, z_2, w_4, z_1, w_3, z_5, w_2, z_4, w_1\}$.

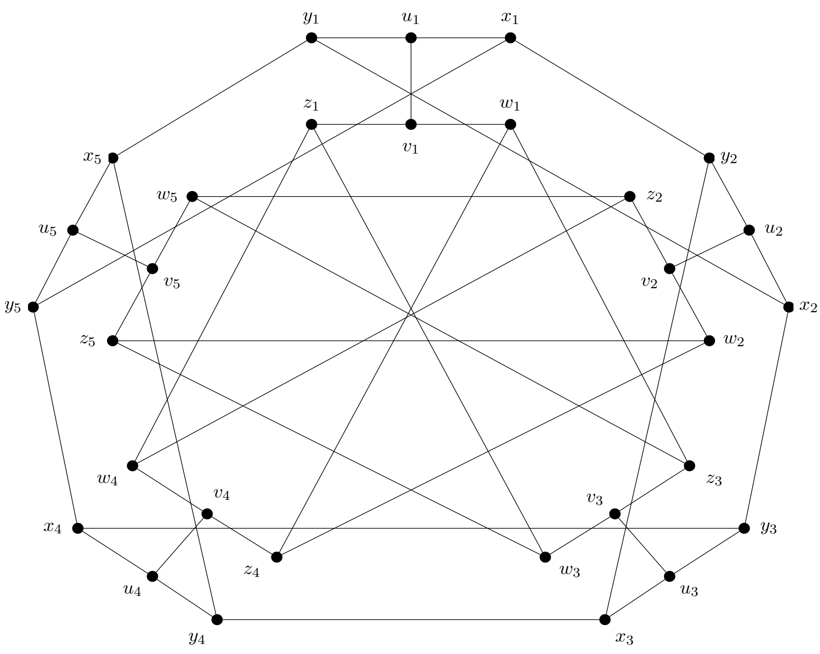

FIGURE 10. Double Star Snark

Theorem 3.5. *Let $G = DS(n,k)$ with $n = 5$ and $k = 2$ be a double star snark on 30 vertices. Then $\chi_c(G) = 2$.*

Proof. Let $G = DS(n,k)$ with $n = 5$ and $k = 2$ be a Double star snark on 30 vertices. We have $\chi_c(G) \leq 3$. Let us assume that a 3-coupon coloring $c : V(G) \to [3]$ exists. Consider the vertex y_1 the neighbors are u_1, x_5 and x_2. Without loss of generality assume that $c(u_1) = 1$, $c(x_2) = 2$ and $c(x_5) = 3$. One of the neighbors of x_1 is assigned with color 1. Thus $c(y_5) \in \{2,3\}$. When $c(y_5) = 3$ we get a contradiction. Similarly when we consider the neighbors of y_3, we get $c(u_3) = 1$ and $c(x_4) = 3$. Now we have $c(y_2) = 3$, $c(u_3) = 1$. Thus $c(y_4) = 2$. Since $c(x_4) = 3$, $c(x_1) \in \{1,2\}$. When $c(x_1) = 1$, we arrive at a contradiction. Thus $c(x_1) = 2$. Consider the vertex x_5, $c(u_5) = 1$ and $c(y_4) = 2$ and hence $c(y_1) = 3$. Similarly when the vertex y_2 is taken into consideration we will get a contradiction. Thus $\chi_c(G) \leq 2$. Define $c : V(G) \to [2]$ as follows. Let $c(u_i) = c(v_i) = 2$ for all $1 \leq i \leq 5$. Remaining vertices can be colored with 1. Thus $\chi_c(G) = 2$. □

3.4. **Watkins Snark.**

Watkins snark graph is a snark in graph theory, discovered by John J. Watkins[11] in 1989. Watkins snark has 50 vertices and 75 edges. The construction of Watkins snark includes 5 blocks B_l with vertices $\{a_l, b_l, c_l, d_l, e_l,\ f_l, g_l, h_l,\ i_l, j_l\}$ where $1 \leq l \leq 5$. The edge set of Watkins snark is $\{E(B_l)|1 \leq l \leq 5\} \bigcup \{c_1i_3, j_3a_4, c_4i_1, j_1a_2, c_2i_4, j_4a_5, c_5i_2, j_2a_3, c_3i_5, j_5a_1\}$.

Theorem 3.6. *Let G be Watkins snark on 50 vertices. Then $\chi_c(G) = 2$.*

Proof. Let G be Watkins snark on 50 vertices. Since $\delta(G) = 3$, $\chi_c(G) \leq 3$. Assume that a 3-coupon coloring exists. Consider the vertex a_1, the neighboring vertices are b_1, d_1, j_4. Without loss of generality $c(b_1) = 1$, $c(d_1) = 2$ and $c(j_4) = 3$. Neighboring vertices of g_1 are f_1, h_1, b_1. Since $c(b_1) = 1$, $c(f_1), c(h_1) \in \{2,3\}$. Suppose $c(h_1) = 2$ then d_1 and h_1 will have same color 2 and neighboring vertices e_1 are c_1,d_1 and h_1. Thus $c(h_1) = 3$, which implies $c(f_1) = 2$ and $c(c_1) = 1$. Now consider b_1 and its neighboring vertices. We have $c(a_1), c(g_1) \in \{2,3\}$. Since neighboring vertices of d_1 includes a_1 and f_1, we get a contradiction when $c(a_1) = 2$. Contradiction also arises when $c(a_1) = 3$. Thus 3-coupon coloring is not possible. Define $c : V(G) \to [2]$ as follows. For all $l = 1, 2, \ldots, 5$, $c(a_l) = c(b_l) = c(c_l) = c(i_l) = c(j_l) = 1$ and $c(d_l) = c(e_l) = c(f_l) = c(g_l) = c(h_l) = 2$. Consider an arbitrary vertex b_l where $l = 1, \ldots, 5$, $c(a_l) = c(c_l) = 1$ and $c(g_l) = 2$. In case of d_l, $c(a_l) = 1$, $c(e_l) = 2$. Similarly for every other vertex. Consequently, it is clear that c is a coupon coloring. Thus $\chi_c(G) = 2$. □

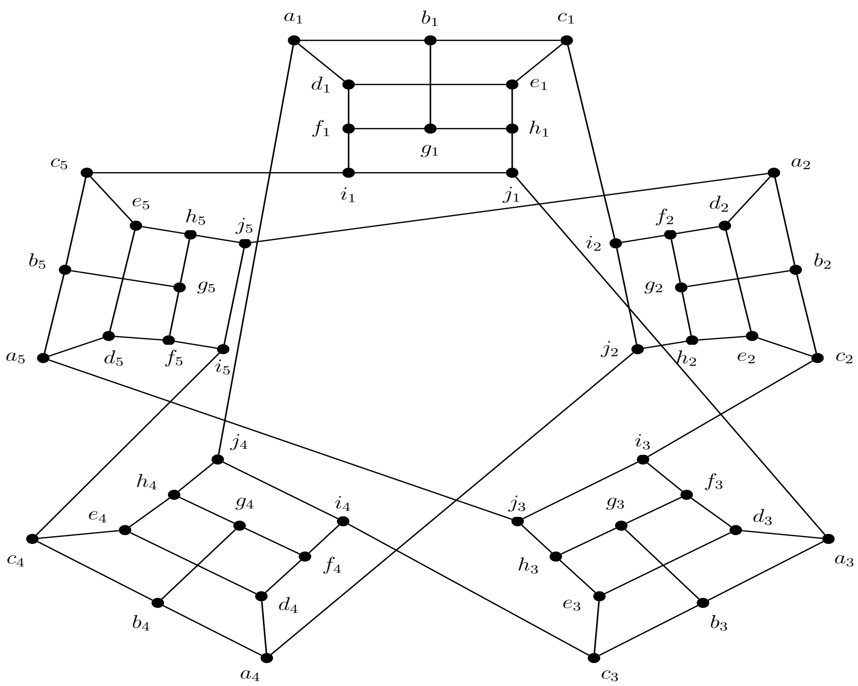

FIGURE 11. Watkins Snark

3.5. **Szekeres Snark.**

The Szekeres snark S_{50} is a snark with 50 vertices and 75 edges discovered by George Szekeres [18] in the year 1973. The construction of Szekeres snark has 5 blocks B_l with vertices $\{a_l, b_l, c_l, d_l, e_l, f_l, g_l, h_l, i_l, j_l\}$ where $1 \leq l \leq 5$. The edge set of Szekers snark is $\{E(B_l) | 1 \leq l \leq 5\} \bigcup \{c_1h_3, c_3h_5, c_5h_2, c_2h_4, c_4h_1\}$.

Theorem 3.7. *Let G be a Szekers snark with* 50 *vertices and* 75 *edges. Then* $\chi_c(G) = 2$.

Proof. Let G be a Szekers snark with 50 vertices and 75 edges. Since $\delta(G) = 3$, it follows that $\chi_c(G) \leq 3$. Suppose $\chi_c(G) = 3$, then a 3-coupon coloring $c : V(G) \to [3]$ exists. Consider the vertex j_1. Without loss of generality assume that $c(a_1) = 1$,$c(g_1) = 2$ and $c(d_1) = 3$. For the neighbours of i_1, $c(e_1), c(h_1) \in \{2, 3\}$.
Case 1: $c(e_1) = 3$
Thus $c(h_1) = 2$. Consider the vertex b_1, $c(c_1), c(f_1) \in \{2, 3\}$. If $c(c_1) = 3$, we get a contradiction since the neighborhood of d_1 has 2 vertex with same color. Therefore $c(c_1) = 2$ and $c(f_1) = 3$. Similarly after considering h_1, we get $c(i_1) = 1$. Now consider the neighboring vertices of a_1. This leads to a contradiction.
Case 2: $c(e_1) = 2$
Thus $c(h_1) = 3$. We get a contradiction similar to Case 1. Thus $\chi_c(G) \leq 2$.

Define $c : V(G) \to [2]$ as follows. $c(a_l) = c(b_l) = c(e_l) = c(f_l) = c(i_l) = 1$ and $c(c_l) = c(d_l) = c(g_l) = c(h_l) = c(j_l) = 2$ for all $l = 1, 2, \ldots, 5$. $\square$

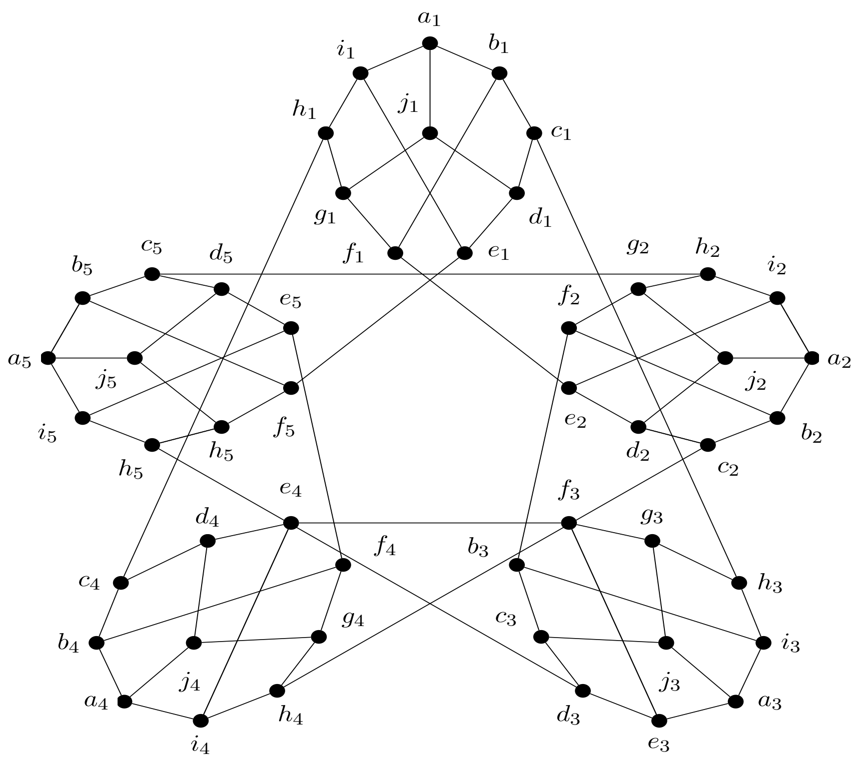

FIGURE 12. Szekeres Snark

4. CONCLUSION AND FUTURE SCOPE

In this paper, we explored that coupon coloring number of various finite and infinite families of snarks is 2 even though snark graphs are cubic. In future, the coupon coloring number of more snark families can be investigated.

Data Availability Statement. No new data were created or analyzed in this study.

Declarations. The authors declare no conflict of interest.

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