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Graceful Vit Labeling: A New Approach and Its Applications to Graphs

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Abstract. Consider an undirected, simple graph $G = (V(G), E(G))$. A graceful labeling of graph G is an injective function $f : V(G) \to \{0, 1, 2, \ldots, |E(G)|\}$ such that the induced edge labels, defined by $f^*(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$, are all distinct. In this paper, we introduce a new type of graceful labeling, called **graceful vit labeling**. A graceful labeling f of graph G is called a graceful vit labeling if the vertex weight function $w_f : V(G) \to \mathbb{N}$, defined by $w_f(v) = f(v) + \sum_{uv \in E(G)} f^*(uv)$ for every $v \in V(G)$, assigns pairwise distinct weights to all vertices in graph G. In other words, no two vertices have the same sum of their own label and the labels of all edges incident to them. In this paper we present various examples of graphs that admit graceful vit labeling and explores structural properties and necessary conditions for their existence.

Key words and Phrases: graph, labeling, graceful graph, graceful vit graph.

1. INTRODUCTION

Graph theory, as discussed in [1], has experienced significant growth, particularly in the area of graph labeling. Gallian's survey [2] highlights the progress that has been made in this field. One of the labeling methods that has emerged is graceful labeling. In 1966, Rosa [3] introduced the concept of graceful labeling, originally referred to as β-valuation. This method of labeling was subsequently termed graceful labeling by Golomb [4].

In a graph G with vertex set $V(G)$ and edge set $E(G)$, a graceful labeling f is an injective function $f : V(G) \to \{0, 1, \ldots, |E(G)|\}$, such that if we define $w_f : E(G) \to \{1, 2, \ldots, E(G)\}$, where $w_f(ab) = |f(a) - f(b)|$ for every $ab \in E(G)$, then w_f is a bijective function. A graph is classified as a graceful graph when a labeling of this type is possible.

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Beutner and Harborth [5] established the criteria that fully determine whether a complete graph can be gracefully labeled. Further insights into which graphs are graceful have been provided by Eshghi [6], Zhou [7], and Dhami [8]. Kaneria et al. [9] explored graceful labelings in binary trees and regular trees. Additionally, Aldred et al. [10] conducted computational studies on the number of graceful labelings in path graphs. Gross and Yellen [11] compiled a list of graphs proven to be graceful, while Drake and Redl [12] examined graphs that are not graceful. Research on graceful labelings has continued to develop in various classes of graphs arising from algebraic structures. Recent studies have focused particularly on graphs constructed from groups. For instance, Sehgal et al. investigated graceful labelings of power graphs of the group $\mathbb{Z}_{2^{k-1}} \times \mathbb{Z}_4$ [13], and further extended their work to prime index graphs of the group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ [14]. In addition, Kumari et al. explored super graceful labelings of comaximal subgroup graphs derived from the cyclic group [15]. These studies demonstrate a growing interest in establishing graceful labelings on graphs motivated by algebraic properties.

Ahmed et al. [16, 17] introduced a novel labeling technique called graceful antimagic labeling. In this scheme, a injective function $f : V(G) \to \{0, 1, \ldots, |E(G)|\}$ (graceful labeling) assigns different values to the vertices of a graph G, where the labels are taken from the set $\{0, 1, 2, \ldots, |E(G)|\}$. The corresponding edge labeling f^*, determined by $f^*(ab) = |f(a) - f(b)|$ for each edge $ab \in E(G)$, must fulfill the following conditions: for any two different edges,

(1) the edge labels must be distinct, meaning that f is a graceful labeling; and
(2) $w_{f^*}(a) \neq w_{f^*}(b)$ for any two different vertices $a, b \in V(G)$, where

$$w_{f^*}(a) = \sum_{ab \in E(G)} f^*(ab).$$

This implies that f^* is also an antimagic labeling. In this case, the labeling f is called a graceful antimagic labeling.

Inspired by the concepts of graceful labeling and vertex irregular total labeling described by Ahmad and Bača [18], we define a new labeling, called graceful vit labeling. A graceful labeling f on graph $G = (V(G), E(G))$ is called as a graceful vit labeling if for any $x, y \in V(G)$, where $x \neq y$,

$$f(x) + \sum_{ux \in E(G)} (|f(x) - f(u)|) \neq f(y) + \sum_{vy \in E(G)} (|f(y) - f(v)|).$$

In this case, to simplify the notation, given a graceful labeling f on a graph G, the induced edge label from the graceful weight is denoted by $f^*(ab) = |f(a) - f(b)|$ for every $ab \in E(G)$. Then, the graceful vit weight of each vertex $u \in V(G)$ is defined as $w_f(u) = f(u) + \sum_{uv \in E(G)} f^*(uv)$. Thus, a graceful labeling f on a graph G is called a graceful vit labeling if for every two different vertices $a, b \in V(G)$, it holds that $w_f(a) \neq w_f(b)$. Furthermore, a graph G is called a **graceful vit graph** if it admits a graceful vit labeling. A graceful vit labeling extends the idea of graceful labeling by adding the condition that the vertex weight values for any two distinct vertices must also be distinct. Although this condition is similar to that in graceful antimagic labeling, the key difference lies in how vertex weights are defined. In graceful vit labeling, the weight of a vertex is calculated by summing the vertex's graceful label and the weights of all edges incident to that vertex, derived from graceful labeling. This definition of vertex weight aligns with the definition of vertex weight in total irregular vertex labeling. Apparently, not all graceful labels are also graceful vit, as shown in Figure 1. The labeling shown in the first graph meets the criteria for a graceful labeling but does not qualify as a graceful vit labeling. In contrast, the second graph exhibits a labeling that satisfies the graceful vit condition. To enhance clarity, the corresponding edge labels are displayed, and the vertex weights (indicated in red) are also presented.

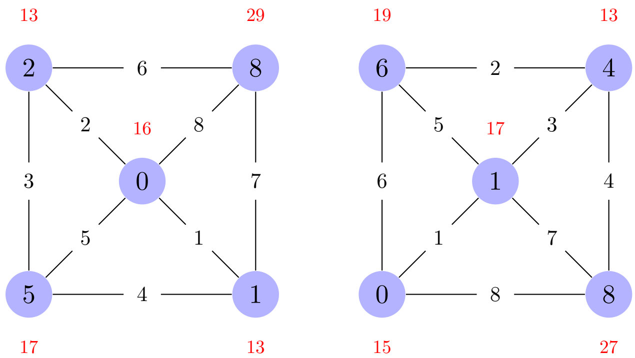

FIGURE 1. A Graceful Graph That is Not Graceful Vit (Left) and A Graceful Vit Graph (Right)

This study focuses on examining the concept of graceful vitness—the characteristic of admitting a graceful vit labeling—across a variety of graph classes. Specifically, we examine the graceful vitness of nearly complete graphs, certain families of trees, cycles, and wheel graphs. By focusing on these graph structures, we aim to deepen the understanding of how graceful vit labeling can be applied and identify the conditions under which it is possible.

This investigation builds on the foundational ideas of graceful, graceful antimagic, and vertex irregular total labelings discussed in previous works. By integrating these concepts, we extend the study of labeling techniques and provide a new perspective on their applicability to well-known graph families. Our findings contribute to the growing body of literature on graph labeling and uncover new connections between graph structure and labeling properties.

2. MAIN RESULTS

This section opens with a fundamental result concerning graceful vit labelings of complete graphs. The subsequent theorem establishes criteria for a complete graph to possess a graceful vit labeling, depending on how many vertices it contains.

Theorem 2.1. *A complete graph K_n admits a graceful vit labeling if and only if* $n = 1, 2,$ *or* 4.

Proof. Golomb [4] established that a complete graph K_n is a graceful graph if and only if $n \leq 4$. Using this result, we determine which of these graphs also satisfy the conditions for being a graceful vit graph.

First, it is straightforward to verify that K_1 is a graceful vit graph since it trivially satisfies all labeling conditions. For K_2 and K_4, we provide examples of graceful vit labelings, as illustrated in Figure 2. Given an injective function $f : V(K_2) \to \{0, 1\}$ and $g : V(K_4) \to \{0, 1, \ldots, 6\}$ with $f(x_1) = 0$, $f(x_2) = 1$ and $g(v_1) = 0$, $g(v_2) = 1$, $g(v_3) = 4$, $g(v_4) = 6$, we obtain the induced edge labels

$$f^*(x_1x_2) = |1 - 0| = 1, \quad g^*(v_1v_2) = 1, \quad g^*(v_1v_3) = 4, \quad g^*(v_1v_4) = 6,$$
$$g^*(v_2v_3) = 3, \quad g^*(v_2v_4) = 5, \quad g^*(v_3v_4) = 2.$$

Since all edge labels in K_2 and all edge labels in K_4 are distinct, both f and g are graceful labelings. Furthermore, the graceful vit weights of the vertices are computed as follows:

$$w_f(x_1) = 0 + 1 = 1, \quad w_f(x_2) = 1 + 1 = 2, \quad w_g(v_1) = 0 + 1 + 4 + 6 = 11,$$
$$w_g(v_2) = 1 + 1 + 3 + 5 = 10, \quad w_g(v_3) = 13, \quad w_g(v_4) = 19.$$

Thus, all vertices of each graph have distinct weights, and therefore both f and g are graceful vit labelings. In both cases, the vertex weights derived from these labelings are distinct for all vertices, confirming that these graphs are graceful vit graphs.

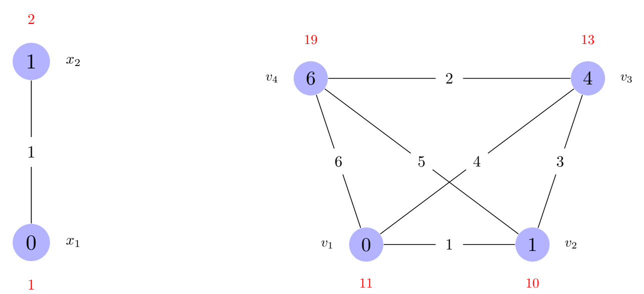

FIGURE 2. Graceful Vit Labelings of Complete Graphs K_2 and K_4

After that, we consider K_3, with vertices v_1, v_2, v_3 and considering graph isomorphism, K_3 has exactly two distinct graceful labelings. Table 1 displays these labelings along with their induced vertex weights.

Table 1. Graceful Labelings and Their Vertex Weights for K_3

$f(a_1), f(a_2), f(a_3)$	$w_{f^*}(a_1), w_{f^*}(a_2), w_{f^*}(a_3)$
$0, 3, 1$	$4, 8, 4$
$0, 3, 2$	$5, 7, 5$

In both cases, the vertex weights are not distinct, violating the conditions for graceful vit labeling. Therefore, K_3 is not a graceful vit graph. From this analysis, we conclude that the only complete graphs that admit graceful vit labelings are K_1, K_2, and K_4. □

Theorem 2.1 reveals that the graceful vit property is exclusive to complete graphs with a limited number of vertices—namely, $n = 1$, $n = 2$, and $n = 4$. It underscores the inherent restrictions on the applicability of graceful vit labelings in complete graphs, indicating that larger complete graphs fail to meet the stringent conditions required for this labeling. Building on the theorem, which characterizes the graceful vit property for complete graphs K_n, we now extend our discussion to the graphs formed after deleting one edge from a complete graph. Specifically, we examine the graph $K_n - e$, which is derived by eliminating one edge from the complete graph K_n. The next theorem investigates the graceful vitness of these modified graphs, establishing the conditions under which $K_n - e$ admits a graceful vit labeling.

Theorem 2.2. *The graph $K_n - e$ is a graceful vit graph if and only if $n = 3, 4, 5$.*

Proof. According to Beutner and Harborth [5], the graphs $K_n - e$ admits a graceful labeling if and only if $2 < n \leq 5$. This implies that the graph $K_n - e$ with $n > 5$ is not a graceful graph. Since every graceful vit graph is also a graceful graph, it follows that $K_n - e$ with $n > 5$ is not a graceful vit graph either. In the graphs $K_3 - e$, $K_4 - e$, and $K_5 - e$, graceful vit labelings are provided in Figure 3.

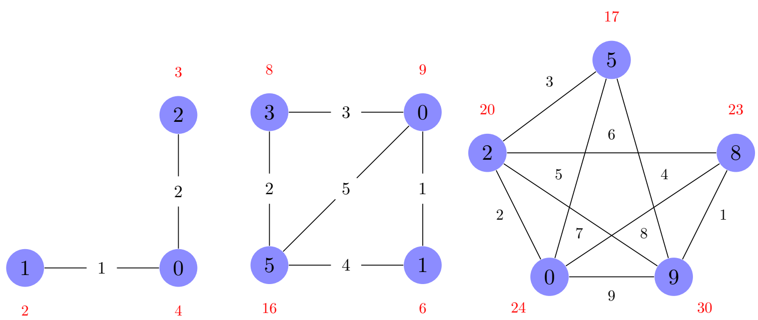

Figure 3. Graceful Vit Labeling of Graphs $K_3 - e, K_4 - e$, and $K_5 - e$

□

Theorem 2.2 highlights a key difference in the graceful vit labeling of complete graphs and their edge-deleted counterparts. While only K_1, K_2, and K_4 can be graceful vit graphs as complete graphs, the removal of a single edge from K_n opens up the possibility for $K_3 - e$, $K_4 - e$, and $K_5 - e$ to be graceful vit graphs. This suggests that removing an edge from a complete graph can relax some of the stringent conditions required for a graceful vit labeling, allowing for a broader range of graphs to satisfy the labeling criteria.

Thereafter, we focus on a particular class of graphs called star graphs. A star graph S_n is formed by one central vertex connected directly to n surrounding vertices. In this case, we investigate the graceful vit labeling property of star graphs. The next theorem provides a result regarding the graceful vit graph nature of star graphs for different values of n.

Theorem 2.3. *For any positive integer $n \geq 1$, the star graph S_n is a graceful vit graph if and only if $n \neq 3$.*

Proof. It is observed that the star graph S_1 is isomorphic to K_2. According to Theorem 2.1, the graph S_1 is a graceful vit graph. Subsequently, the star graph S_2 is isomorphic to the graph $K_3 - e$. According to Theorem 2.2, it follows that the graph S_2 is a graceful vit graph. Consider the star graph S_n with $n \geq 4$. Let

$$V(S_n) = \{v_i : 1 \leq i \leq n+1\} \text{ and } E(S_n) = \{v_1 v_i : 2 \leq i \leq n+1\}.$$

Let the vertex labeling $h : V(S_n) \to \{0, 1, \ldots, n\}$ be defined by assigning $h(v_i) = i - 1$ for each $i = 1, 2, \ldots, n+1$. Note that for $i = 2, 3, \ldots, n+1$,

$$h^*(v_1 v_i) = |h(v_1) - h(v_i)| = |0 - (i-1)| = i - 1.$$

For any $v_1 v_i, v_1 v_j \in E(G)$, where $v_1 v_i \neq v_1 v_j$, which means $i \neq j$, we have $h^*(v_1 v_i) = i - 1 \neq j - 1 = h^*(v_1 v_j)$. Thus, h and h^* are injective functions. Hence, h is a graceful labeling. Furthermore, compute the weight of each vertex in S_n under the labeling h and h^* as follows.

$$\begin{aligned} w_h(v_1) &= h(v_1) + \sum_{j=2}^{n+1} h^*(v_1 v_j) = 0 + \sum_{j=2}^{n+1} (j-1) = \frac{n(n+1)}{2}, \\ w_h(v_i) &= h(v_i) + h^*(v_1 v_i) = (i-1) + (i-1) = 2(i-1), \text{ for } i = 2, 3, \ldots, n+1. \end{aligned}$$

We know that $w_h(v_2) < w_h(v_3) < \cdots < w_h(v_{n+1})$. Since $n \geq 4$, we have $\frac{n+1}{2} > 2$ and

$$w_h(v_{n+1}) = 2(n - 1 + 1) = 2n < \frac{n(n+1)}{2} = w_h(v_1).$$

Consequently, we get $w_h(v_2) < w_h(v_3) < \cdots < w_h(v_{n+1}) < w_h(v_1)$. Thus, w_h is an injective function. Therefore, there exists a function h that is a graceful vit labeling, and hence the star graph S_n with $n \geq 4$ is a graceful vit graph. Now, let us consider the illustration of the star graph S_3.

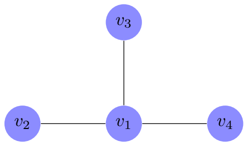

FIGURE 4. Star Graph S_3

Suppose there exists a graceful vit labeling $g : V(S_3) \to \{0, 1, 2, 3\}$. To get an edge weight of 3, we must have $g(v_1) = 0$ or $g(v_1) = 3$. If $g(v_1) = 0$, then without loss of generality, we can take $g(v_2) = 1$, $g(v_3) = 2$, and $g(v_4) = 3$. In this case, we have

$$\begin{aligned} w_g(v_1) &= g(v_1) + |g(v_1) - g(v_2)| + |g(v_1) - g(v_3)| + |g(v_1) - g(v_4)| \\ &= 0 + 1 + 2 + 3 = 6, \\ w_g(v_4) &= g(v_4) + |g(v_1) - g(v_4)| \\ &= 3 + 3 = 6. \end{aligned}$$

This brings about a contradiction, since g is supposed to be a graceful vit labeling. On the other hand, if $g(v_1) = 3$, then we have $g(v_1) > g(v_j)$ for $j = 2, 3, 4$. Note that

$$\begin{aligned} w_g(v_2) &= g(v_2) + |g(v_1) - g(v_2)| = g(v_2) + g(v_1) - g(v_2) = g(v_1), \\ w_g(v_3) &= g(v_3) + |g(v_1) - g(v_3)| = g(v_3) + g(v_1) - g(v_3) = g(v_1). \end{aligned}$$

These results contradict the fact that g is a graceful vit labeling. Hence, the assumption must be false. Therefore, the star graph S_3 is not a graceful vit graph. □

In this section, we examine the graceful vit graph property of path graphs. A path graph is a simple graph where each vertex is connected to at most two other vertices. The next lemma shows that the path graph P_4, which consists of four vertices, does not possess the graceful vit graph property.

Lemma 2.4. *The path graph P_4 is not graceful vit.*

Proof. Given the path P_4 with vertex set $V(P_4) = \{u_1, u_2, u_3, u_4\}$ and edge set $E(P_4) = \{u_1u_2, u_2u_3, u_3u_4\}$, it can be readily verified that considering isomorphism, there are just two distinct graceful labelings of the path P_4.

TABLE 2. Graceful Labelings and Their Vertex Weights for P_4

$h(u_1), h(u_2), h(u_3), h(u_4)$	$w_h(u_1), w_h(u_2), w_h(u_3), w_h(u_4)$
$0, 3, 1, 2$	$3, 8, 4, 3$
$3, 0, 2, 1$	$6, 5, 5, 2$

Since the resulting vertex weights are not unique, we conclude that the path P_4 is not a graceful vit graph. $\square$

In [3], it is demonstrated that a path P_k is graceful for all $k \geq 2$. However, according to Lemma 2.4, the path P_4 is not graceful vit. For $k = 2, 3, 5, 6, 7, 8, 9, 10, 11$, the corresponding graceful vit labels are as follows.

Table 3. Graceful Vit Labeling of Path P_n, for $2 \leq n \leq 11$

k	$h(u_1), h(u_2), \cdots, h(u_k)$
2	$1, 0$
3	$1, 0, 2$
5	$1, 2, 4, 0, 3$
6	$2, 3, 5, 0, 4, 1$
7	$3, 2, 4, 1, 5, 0, 6$
8	$4, 3, 5, 2, 6, 1, 7, 0$
9	$5, 4, 2, 7, 3, 6, 0, 8, 1$
10	$5, 6, 0, 9, 1, 8, 3, 7, 4, 2$
11	$4, 5, 3, 8, 2, 6, 9, 1, 10, 0, 7$

For larger values of k, a general formula for the number of graceful labelings of paths P_k remains elusive. However, Aldred et al. [10] showed that for sufficiently large k, the number of graceful labelings of paths P_k exceeds $\left(\frac{5}{3}\right)^k$. Given the results discussed above, we are led to propose the following conjecture.

Conjecture 1. *For every integer $k \geq 2$ and $k \neq 4$, the path P_k admits a graceful vit labeling.*

We now turn our attention to cycle graphs. Although cycle graphs are known to have interesting properties in the context of graceful labelings, it turns out that the cycle graphs C_3 and C_4 do not satisfy the conditions for being graceful vit graphs. This can be formally stated as follows.

Lemma 2.5. *The cycle graphs C_3 and C_4 are not graceful vit graphs.*

Proof. To investigate the graceful vit labeling of cycle graphs, we first consider the cycle graph C_3. Since $C_3 \cong K_3$, using Theorem 2.1, we know that C_3 is not graceful vit.For the cycle graph C_4, defined by the vertex set $\{u_1, u_2, u_3, u_4\}$ and the edge set $\{u_1u_2, u_2u_3, u_3u_4, u_1u_4\}$ There are only two possible graceful labelings of C_4, up to isomorphism.

Table 4. Graceful Labelings and Their Vertex Weights for C_4

$g(u_1), g(u_2), g(u_3), g(u_4)$	$w_g(u_1), w_g(u_2), w_g(u_3), w_g(u_4)$
$0, 3, 2, 4$	$7, 7, 5, 10$
$0, 2, 1, 4$	$6, 5, 5, 11$

We note that in both cases the induced vertex weights are not unique. Therefore, the cycle graph C_4 fails to be a graceful vit graph. Hence, we conclude that both C_3 and C_4 are not graceful vit graphs.

□

In [3], it is proved that a cycle C_k is graceful if and only if $k \equiv 3 \pmod 4$ or $k \equiv 0 \pmod 4$. Based on Lemma 2.5, the graphs C_3 and C_4 are not graceful vit graphs. For $k = 7$ and $k = 8$, the corresponding graceful vit labels are as follows.

TABLE 5. Graceful Vit Labeling of Cycles C_7 and C_8

k	$g(u_1), g(u_2), \ldots, g(u_k)$
7	$0, 7, 1, 6, 3, 2, 4$
8	$0, 8, 1, 7, 2, 3, 6, 4$

For larger k, a general formula could not be determined. Therefore, we present the following open question.

Open Problem 1. *Determine the properties of graceful vit labelings for cycle graphs C_k with $k \geq 11$ where $k \equiv 3 \pmod 4$ or $k \equiv 0 \pmod 4$.*

In this section, we analyze the structure of double star graphs and their graceful vit labeling properties. A double star graph $B_{n,m}$ consists of two central vertices connected by an edge, where one central vertex is adjacent to n leaf vertices and the other is adjacent to m leaf vertices. In Theorem 2.6, we provide conditions under which such graphs can be graceful vit graphs.

Theorem 2.6. *The double star graph $B_{n,m}$ is a graceful vit graph if and only if $n = 1$ and $m \geq 2$ or $m = 1$ and $n \geq 2$.*

Proof. Consider the double star graph $B_{n,m}$ with $n, m \geq 1$. For $(n, m) = (1, 1)$, the graph $B_{n,m} = B_{1,1}$ is isomorphic to the graph P_4. According to Lemma 2.4, the graph $B_{1,1}$ is not a graceful vit graph. Note that the graph $B_{n,m}$ is isomorphic to the graph $B_{m,n}$. Without loss of generality, consider the double star graph $B_{1,m}$ with $m \geq 2$. First, let

$$V(B_{1,m}) = \{x_a, x_b, a_1, b_1, b_2, \ldots, b_m\}$$

and

$$E(B_{1,m}) = \{x_a a_1, x_b b_j : 1 \leq j \leq m\} \cup \{x_a x_b\}.$$

Define the vertex labeling function $f : V(B_{1,m}) \to \{0, 1, \ldots, m + 2\}$ such that for every $i = 1, 2, \ldots, m$, we have

$$f(x_a) = 2, \quad f(x_b) = 0, \quad f(a_1) = 1, \quad f(b_i) = m + 3 - i.$$

Since $f(x_b) < f(a_1) < f(x_a) < f(b_m) < f(b_{m-1}) < \cdots < f(b_1)$, we get f injective. Furthermore, for $i = 1, 2, \ldots, m$,

$$f^*(x_b b_i) = |f(x_b) - f(b_i)| = |0 - (m + 3 - i)| = m + 3 - i$$

and

$$\begin{aligned}
f^*(x_a x_b) &= |f(x_a) - f(x_b)| = |2 - 0| = 2, \\
f^*(x_a a_1) &= |f(a_1) - f(x_a)| = |1 - 2| = 1.
\end{aligned}$$

Since $f^*(x_a a_1) < f^*(x_a x_b) < f^*(x_b b_m) < f^*(x_b b_{m-1}) < \cdots < f^*(x_b b_1)$, we get f^* injective and thus f is a graceful labeling. We now compute the weight of each vertex in $B_{1,m}$ under the labeling f and f^* as follows.

$$\begin{aligned}
w_f(x_b) &= f(x_b) + f^*(x_a x_b) + \sum_{i=1}^{m} f^*(x_b b_i) = 0 + 2 + \sum_{i=1}^{m} (m + 3 - i) \\
&= 0 + 2 + 3 + 4 + \cdots + (m + 2) \\
&= \frac{(m+4)(m+1)}{2} \\
&= \frac{m^2 + 5m + 4}{2}, \\
w_f(x_a) &= f(x_a) + f^*(x_a x_b) + f^*(x_a a_1) \\
&= 2 + 2 + 1 = 5, \\
w_f(a_1) &= f(a_1) + f^*(x_a a_1) \\
&= 1 + 1 = 2.
\end{aligned}$$

For $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
w_f(b_i) &= f(b_i) + f^*(x_b b_i) \\
&= (m + 3 - i) + (m + 3 - i) = 2(m + 3 - i) = 2m + 6 - 2i.
\end{aligned}$$

Consider that

$$6 = w_f(b_m) < w_f(b_{m-1}) < \cdots < w_f(b_1) = 2m + 4.$$

Since $m \geq 2$, $\frac{m^2+5m+4}{2} \geq 9$ and $\frac{m^2+m-4}{2} \geq 1 > 0$, we get

$$w_f(b_1) = \frac{4m+8}{2} < \frac{4m+8}{2} + \frac{m^2+m-4}{2} = \frac{m^2+5m+4}{2} = w_f(x_b).$$

Thus, $w_f(a_1) < w_f(x_a) < w_f(b_m) < w_f(b_{m-1}) < \cdots < w_f(b_1) < w_f(x_b)$. Hence, the weight function w_f is injective, demonstrating the existence of a function f that provides a graceful vit labeling. Consequently, the double star graph $B_{1,m}$ with $m \geq 2$ is a graceful vit graph.

Now, consider the double star graph $B_{n,m}$ where $n, m \geq 2$. Let

$$V(B_{n,m}) = \{x_a, x_b, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$$

and

$$E(B_{n,m}) = \{x_a a_i, x_b b_j : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\} \cup \{x_a x_b\}.$$

In this graph, x_a and x_b are the central vertices, while the vertices a_i and b_j (with $1 \leq i \leq n$ and $1 \leq j \leq m$) represent the leaves. Suppose there is a graceful vit labeling function $g : V(B_{n,m}) \to \{0, 1, 2, \ldots, n + m + 1\}$ on $B_{n,m}$. If a central vertex has a label greater than at least two labels on its leaves, for example, if $g(x_a) > g(a_1)$ and $g(x_a) > g(a_2)$, then we get the following weight calculations:

$$w_g(a_1) = g(x_a) - g(a_1) + g(a_1) = g(x_a)$$

and

$$w_g(a_2) = g(x_a) - g(a_2) + g(a_2) = g(x_a).$$

This will lead to a contradiction with the fact that g is a graceful vit labeling. On the other hand, since g is a graceful labeling, labels 0 and $n+m+1$ must be labels of two adjacent vertices. Examine two separate cases as follows.

Case 1. If there exists $i \in \{1, 2, \ldots, n\}$ such that $g(a_i) = 0$, then $g(x_a) = n+m+1$. This leads to $g(a_i) < g(x_a)$ for $i = 1, 2, \ldots, n$ resulting in a contradiction. Similarly, if there exists $j \in \{1, 2, \ldots, m\}$ such that $g(b_j) = 0$, then $g(x_b) = n+m+1$. This leads to $g(b_j) < g(x_b)$ for $j = 1, 2, \ldots, m$ resulting in a contradiction.

Case 2. If $g(x_a) = 0$, then we have $g(x_b) = n+m+1$ or $g(a_i) = n+m+1$ for some $i \in \{1, 2, \ldots, n\}$. If $g(x_b) = n+m+1$, then $g(b_j) < g(x_b)$ for $j = 1, 2, \ldots, m$, resulting in a contradiction. To simplify the argument, let us consider $g(a_1) = n+m+1$. Then, to get edge weight of $n+m$ it can only be formed from labels 0 and $n+m$ or from labels $n+m+1$ and 1. Since vertex a_1 has order 1, we get $g(a_2) = n+m$. Similarly, we get $g(a_3) = n+m-1, g(a_4) = n+m-2, \cdots, g(a_n) = m+2$. Then, to get an edge weight of $m+1$, we must have $g(x_b) = m+1$. This leads to $g(b_j) < g(x_b)$ for $j = 1, 2, \ldots, m$, resulting in a contradiction.
Similarly, if $g(x_b) = 0$, then we have $g(x_a) = n+m+1$ or $g(b_j) = n+m+1$ for some $j \in \{1, 2, \ldots, m\}$. If $g(x_a) = n+m+1$, then $g(a_i) < g(x_a)$ for $i = 1, 2, \ldots, n$, resulting in a contradiction. To simplify the argument, let us consider $g(b_1) = n+m+1$. Then, to get an edge weight of $n+m$ it can only be formed from labels 0 and $n+m$ or from labels 1 and $n+m+1$. Since vertex b_1 has order 1, we get $g(b_2) = n+m$. Similarly, we get $g(b_3) = n+m-1, \quad g(b_4) = n+m-2, \cdots, \quad g(b_m) = n+2$. Then, to get an edge weight of $n+1$, we must have $g(x_a) = n+1$. This leads to $g(a_i) < g(x_a)$ untuk $i = 1, 2, \ldots, n$, resulting in a contradiction.

Based on the analysis of Cases 1 and 2, we reach a contradiction, as both cases lead to inconsistencies with the assumption that g is a graceful vit labeling. Therefore, it follows that the double star graphs $B_{n,m}$ with $n, m \geq 2$ cannot be graceful vit graphs. $\square$

Let the generalized star graph $S_{2,n}$ be a graph that consists of one central vertex u connected to n vertices $x_1, x_2, \ldots, x_n$, and each x_i is connected to another vertex y_i. This creates a structure where u is the central vertex, and each x_i is connected to its corresponding leaf y_i. The edge set of this graph is:

$$E(S_{2,n}) = \{ux_i, x_iy_i : i = 1, 2, \ldots, n\}.$$

Now, we turn our attention to the graceful vit labeling of the generalized star graph $S_{2,n}$ for $n \leq 5$. The following theorem demonstrates that this graph is indeed a graceful vit graph for these values of n.

Theorem 2.7. *The generalized star graph $S_{2,n}$ is a graceful vit graph for every positive integer $n \leq 5$.*

Proof. Given the star graph $S_{2,n}$ with $n \leq 5$, first, let

$$V(S_{2,n}) = \{u, x_i, y_i : 1 \leq i \leq n\} \text{ and } E(S_{2,n}) = \{ux_i, x_iy_i : 1 \leq i \leq n\}.$$

For odd n, a labeling of points is defined as

$$f : V(S_{2,n}) \to \{0, 1, \ldots, 2n\}$$

such that $f(u) = 2n - 1$, $f(x_1) = 0$, and for every $i = 1, 2, \ldots, n$,

$$f(y_i) = 2n - 2(i - 1).$$

For every $i = 2, 3, \ldots, n$, $f(x_i) = 2i - 3$. Observe that $f(u)$ and each $f(x_i)$ (for $i = 2, 3, \ldots, n$) are assigned odd values, while $f(x_1)$ and $f(y_i)$ are even numbers for every $i = 1, 2, \ldots, n$. Since

$$f(x_2) < f(x_3) < \cdots < f(x_n) < f(u)$$

and

$$f(x_1) < f(y_n) < f(y_{n-1}) < \cdots < f(y_1),$$

we get f is an injective function. Subsequently, we observe the edge weights of the graph $S_{2,n}$,

$$f^*(ux_1) = 2n - 1, \quad f^*(x_1y_1) = 2n,$$

for $i = 2, 3, \ldots, n$,

$$\begin{aligned} f^*(ux_i) &= |2n - 1 - (2i - 3)| \\ &= |2n - 1 - 2i + 3| \\ &= 2n - 2i + 2, \quad \text{and} \\ f^*(x_iy_i) &= |2i - 3 - (2n - 2(i - 1))| \\ &= |2i - 3 - 2n + 2i - 2| \\ &= |4i - 2n - 5| \\ &= \begin{cases} 2n - 4i + 5, & \text{if } i = 2, 3, \ldots, \frac{n+1}{2}, \\ 4i - 2n - 5, & \text{if } i = \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n. \end{cases} \end{aligned}$$

Note that $f^*(x_1y_1) > f^*(ux_1) > f^*(ux_2) > f^*(ux_3) > \cdots > f^*(ux_n)$. On the other hand, for $i = 2, 3, \ldots, n$, $f^*(ux_1) > f^*(x_iy_i)$. Assuming that there exist $i \in \{2, 3, \ldots, n\}$ and $j \in \{2, 3, \ldots, \frac{n+1}{2}\}$ such that $f^*(ux_i) = f^*(x_jy_j)$, we get

$$2n - 2i + 2 = 2n - 4j + 5 \iff 4j - 2i = 3 \iff 2(2j - i) = 3,$$

which leads to a contradiction. Assume there exist i, j where $i \in \{2, 3, \ldots, n\}$ and $j \in \{\frac{n+3}{2}, \frac{n+5}{2}, \ldots, n\}$ such that $f^*(ux_i) = f^*(x_jy_j)$, we get

$$2n - 2i + 2 = 4j - 2n - 5 \iff 4j + 2i = 4n + 7 \iff 2(2j + i) = 2(2n + 3) + 1,$$

resulting in a contradiction. Assume there exist $i \in \{2,3,\ldots,\frac{n+1}{2}\}$ and $j \in \{\frac{n+3}{2},\frac{n+5}{2},\ldots,n\}$ such that $f^*(x_iy_i) = f^*(x_jy_j)$, we get

$$2n-4i+5 = 4j-2n-5 \iff 4j+4i = 4n+10 \iff 4(j+i) = 4(n+2)+2.$$

Contradiction. Thus, f^* is injective and it follows that f is a graceful labeling. In the following, we calculate the weight of each vertex in $S_{2,n}$ under the labeling f and f^* as follows. For $y_1, x_1 \in V(S_{2,n})$, we have

$$w_f(y_1) = 2n+2n = 4n \ , \ w_f(x_1) = 2n+2n-1 = 4n-1.$$

For $i = 2,3,\ldots,n$, we have

$$\begin{aligned} w_f(y_i) &= f(y_i) + f^*(x_iy_i) \\ &= \begin{cases} 2n-2i+2+2n-4i+5, & \text{if } i = 2,3,\ldots,\frac{n+1}{2}, \\ 2n-2i+2+4i-2n-5, & \text{if } i = \frac{n+3}{2},\frac{n+5}{2},\ldots,n \end{cases} \\ &= \begin{cases} 4n-6i+7, & \text{if } i = 2,3,\ldots,\frac{n+1}{2}, \\ 2i-3, & \text{if } i = \frac{n+3}{2},\frac{n+5}{2},\ldots,n, \end{cases} \end{aligned}$$

and

$$\begin{aligned} w_f(x_i) &= f(x_i) + f^*(x_iy_i) + f^*(ux_i) \\ &= \begin{cases} (2i-3)+(2n-2i+2)+(2n-4i+5), & \text{if } i = 2,3,\ldots,\frac{n+1}{2}, \\ (2i-3)+(2n-2i+2)+(4i-2n-5), & \text{if } i = \frac{n+3}{2},\frac{n+5}{2},\ldots,n \end{cases} \\ &= \begin{cases} 4n-4i+4, & \text{if } i = 2,3,\ldots,\frac{n+1}{2}, \\ 4i-6, & \text{if } i = \frac{n+3}{2},\frac{n+5}{2},\ldots,n. \end{cases} \end{aligned}$$

For $u \in V(S_{2,n})$, we have

$$\begin{aligned} w_f(u) &= f(u) + \sum_{i=1}^{n} f^*(ux_i) \\ &= 2n-1+2n-1+\sum_{i=2}^{n}(2n-2i+2) \\ &= 4n-2+\frac{(2n-2+2)(n-1)}{2} \\ &= 4n-2+n^2-n = n^2+3n-2. \end{aligned}$$

Note that the values of $w_f(x_1)$ and $w_f(y_i)$ for every $i = 2,\ldots,n$ are odd numbers. For $n \le 5$, we have

$$\begin{aligned} w_f(x_1) > w_f(y_2) > w_f(y_3) > \cdots > w_f(y_{\frac{n+1}{2}}) \\ > w_f(y_n) > w_f(y_{n-1}) > \cdots > w_f(y_{\frac{n+3}{2}}). \end{aligned}$$

On the other hand, $w_f(x_i) \equiv 0 \pmod 4$ for every $i = 2,3,\ldots,\frac{n+1}{2}$, while $w_f(x_i) \equiv 2 \pmod 4$ for every $i = \frac{n+3}{2},\frac{n+5}{2},\ldots,n$. The values $w_f(y_1)$ and $w_f(u)$ are even numbers. Observe that for $2 < n \leq 5$,

$$w_f(u) > w_f(y_1) > w_f(x_2) > w_f(x_3) > \cdots > w_f(x_{\frac{n+1}{2}})$$

and

$$w_f(u) > w_f(y_1) > w_f(x_n) > w_f(x_{n-1}) > w_f(x_{\frac{n+3}{2}}).$$

For $n = 1$, we have $w_f(y_1) > w_f(x_1) > w_f(u)$, hence w_f is an injective function.

For $n = 2$ and $n = 4$, the given labelings in Figures 5 are graceful vit labelings. Thus, the star graph $S_{2,n}$ with $n \leq 5$ is a graceful vit graph.

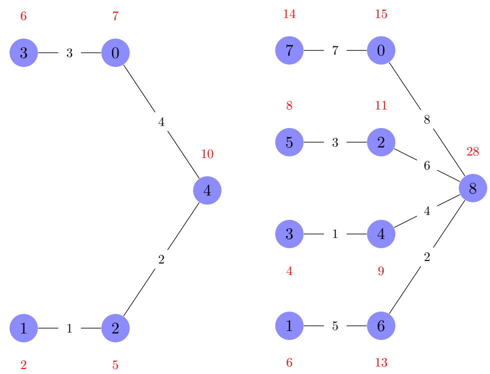

FIGURE 5. Graceful Vit Labeling on The Generalized Star Graph $S_{2,2}$ and $S_{2,4}$

□

For $n > 5$, we have not succeeded in identifying a general formula. Therefore, we present the following open problem.

Open Problem 2. *Find the properties of graceful vit labelings for generalized star graphs $S_{2,n}$ with $n > 5$.*

In addition to star graphs, double star graphs, and generalized star graphs, we are also exploring graceful vit labeling in another type of tree graph, specifically complete binary tree graphs.

Lemma 2.8. *The complete binary tree graph BTC_3 is not a graceful vit graph.*

Proof. The complete binary tree graph BTC_3 can be illustrated as shown in Figure 6.

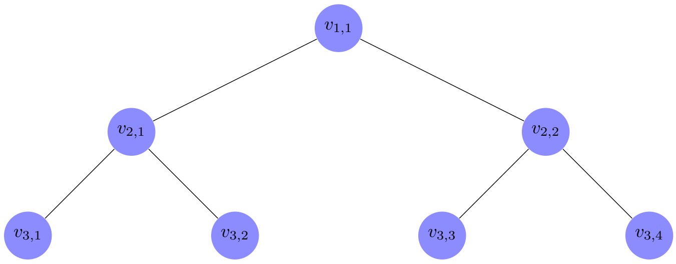

FIGURE 6. Complete Binary Tree Graph BTC_3

Let assume there exists a graceful vit labeling $g : V(BTC_3) \to \{0, 1, 2, \ldots, 7\}$. Suppose $g(v_{i,j}) = 0$ for some $i \in \{1, 2, 3\}$ and $j \in \{1, 2, \ldots, 2^{i-1}\}$, then the analysis can be divided into the following three cases.

Case 1. If $i = 1$, then $g(v_{2,1}) = 6$ or $g(v_{2,2}) = 6$. It is observed that if $g(v_{2,1}) = 6$, then $g(v_{3,1}) < g(v_{2,1})$ and $g(v_{3,2}) < g(v_{2,1})$, resulting in vertex weights

$$w_g(v_{3,1}) = g(v_{2,1}) - g(v_{3,1}) + g(v_{3,1}) = g(v_{2,1}),$$

$$w_g(v_{3,2}) = g(v_{2,1}) - g(v_{3,2}) + g(v_{3,2}) = g(v_{2,1}).$$

This is impossible since g is a graceful vit labeling. Analogously, for the case $g(v_{2,2}) = 6$, it follows that $w_g(v_{3,3}) = w_g(v_{3,4})$, leading to a contradiction with g being a graceful vit labeling.

Case 2. If $i = 3$, then $g(v_{2,1}) = 6$ or $g(v_{2,2}) = 6$. Analogously, this results in a contradiction as in Case 1.

Case 3. If $i = 2$, then $g(v_{1,1}) = 6$ or $g(v_{3,2j-1}) = 6$ or $g(v_{3,2j}) = 6$. If $g(v_{1,1}) = 6$, then $g(v_{2,1}) < g(v_{1,1})$ and $g(v_{2,2}) < g(v_{1,1})$, resulting in vertex weights

$$w_g(v_{2,1}) = g(v_{1,1}) - g(v_{2,1}) + g(v_{2,1}) = g(v_{1,1}),$$

$$w_g(v_{2,2}) = g(v_{1,1}) - g(v_{2,2}) + g(v_{2,2}) = g(v_{1,1}).$$

This causes a logical inconsistency since g is a graceful vit labeling. Furthermore, if $g(v_{3,2j-1}) = 6$, then $g(v_{3,2j}) = 5$ or $g(v_{1,1}) = 5$. If $g(v_{1,1}) = 5$, then $g(v_{2,1}) < g(v_{1,1})$ and $g(v_{2,2}) < g(v_{1,1})$. In this case, a contradiction is obtained using a similar approach as in the case $g(v_{1,1}) = 6$. On the other hand, if $g(v_{3,2j}) = 5$, then $g(v_{1,1}) = 4$. This implies that $g(v_{2,1}) < g(v_{1,1})$ and $g(v_{2,2}) < g(v_{1,1})$. Hence, by applying a similar argument as in the previous cases, we obtain a contradiction.

Based on Cases 1, 2, and 3, a contradiction is always obtained with the fact that g is a graceful vit labeling. Thus, the assumption must be false. Therefore, the graph BTC_3 is not a graceful vit graph. □

After that, we examine the graceful vit labeling on wheel graphs. A wheel graph W_n consists of a cycle of n vertices with an additional central vertex connected to all vertices in the cycle. In this section, we aim to demonstrate that W_n

can be labeled gracefully as a vit graph for every odd $n \geq 3$. The following theorem formalizes this result.

Theorem 2.9. *The wheel graph W_n is a graceful vit graph for every odd number $n \geq 3$.*

Proof. Given the wheel graph W_n with n being an odd number and $n \geq 5$. First, let

$$V(W_n) = \{v_i : 0 \leq i \leq n\}$$

and

$$E(W_n) = \{v_{i-1}v_i : 1 \leq i \leq n-1\} \cup \{v_0 v_{n-1}\} \cup \{v_i v_n : 0 \leq i \leq n-1\}.$$

Moreover, a labeling of the vertices $f : V(W_n) \to \{0, 1, \ldots, 2n\}$ can be defined such that for every vertex in the graph W_n,

$$f(v_0) = 0,\ f(v_1) = 2,\ f(v_n) = 2n,$$

and

$$f(v_i) = \begin{cases} n+i, & \text{if } i = 2, 4, \ldots, n-1, \\ n+1-i, & \text{if } i = 3, 5, \ldots, n-2. \end{cases}$$

Since $f(v_0) < f(v_1) < f(v_{n-2}) < f(v_{n-4}) < \cdots < f(v_3) < f(v_2) < f(v_4) < \cdots < f(v_{n-1}) < f(v_n)$, we get f is an injective function. Furthermore, the edge weights of the edges of W_n can be determined as follows. For each edge in the graph W_n, we obtain,

$$\begin{aligned} f^*(v_{i-1}v_i) &= |f(v_{i-1}) - f(v_i)| \\ &= \begin{cases} |n+1-(i-1)-(n+i)|, \text{ if } i = 4, 6, \ldots, n-1, \\ |n+i-1-(n+1-i)|, \text{ if } i = 3, 5, \ldots, n-2 \end{cases} \\ &= \begin{cases} |2-2i|, \text{ if } i = 4, 6, \ldots, n-1, \\ |2i-2|, \text{ if } i = 3, 5, \ldots, n-2 \end{cases} \\ &= 2i-2, \text{ for } i = 3, 4, \ldots, n-1, \\ f^*(v_0v_1) &= |f(v_0) - f(v_1)| = |0-2| = 2, \\ f^*(v_1v_2) &= |f(v_1) - f(v_2)| = |2-(n+2)| = n, \\ f^*(v_0v_{n-1}) &= |f(v_0) - f(v_{n-1})| = |0-(n+n-1)| = 2n-1, \end{aligned}$$

$$\begin{aligned} f^*(v_iv_n) &= |f(v_i) - f(v_n)| \\ &= \begin{cases} |(n+i)-2n|, \text{ if } i = 2, 4, \ldots, n-1, \\ |(n+1-i)-2n|, \text{ if } i = 3, 5, \ldots, n-2 \end{cases} \\ &= \begin{cases} 2n-n-i, \text{ if } i = 2, 4, \ldots, n-1, \\ 2n-n-1+i, \text{ if } i = 3, 5, \ldots, n-2 \end{cases} \\ &= \begin{cases} n-i, \text{ if } i = 2, 4, \ldots, n-1, \\ n-1+i, \text{ if } i = 3, 5, \ldots, n-2, \end{cases} \end{aligned}$$

$$\begin{aligned} f^*(v_0v_n) &= |f(v_0) - f(v_n)| = |0 - 2n| = 2n, \\ f^*(v_1v_n) &= |f(v_1) - f(v_n)| = |2 - 2n| = 2n - 2. \end{aligned}$$

Observe that the values of $f^*(v_1v_2)$, $f^*(v_0v_{n-1})$, and $f^*(v_iv_n)$ for $2 \le i \le n-1$ are all odd integers. On the other hand, $f^*(v_0v_1)$, $f^*(v_0v_n)$, $f^*(v_1v_n)$, and $f^*(v_{i-1}v_i)$ for $i = 3, 4, \ldots, n-1$ are even numbers. Therefore,

$$\begin{aligned} f^*(v_{n-1}v_n) &< f^*(v_{n-3}v_n) < \cdots < f^*(v_2v_n) < f^*(v_1v_2) \\ &< f^*(v_3v_n) < f^*(v_5v_n) < \cdots < f^*(v_{n-2}v_n) < f^*(v_0v_{n-1}) \text{ and} \\ f^*(v_0v_1) &< f^*(v_3v_4) < f^*(v_4v_5) < \cdots < f^*(v_{n-2}v_{n-1}) \\ &< f^*(v_1v_n) < f^*(v_0v_n), \end{aligned}$$

Since f^* is injective, we get f is a graceful labeling. Furthermore, the weight of each vertex in W_n under the labeling f and f^* can be computed as follows. For $v_0, v_1, v_2, v_{n-1}, v_n \in V(W_n)$, we have

$$\begin{aligned} w_f(v_0) &= 0 + 2 + (2n - 1) + 2n = 4n + 1, \\ w_f(v_1) &= 2 + 2 + n + 2n - 2 = 3n + 2, \\ w_f(v_2) &= (n + 2) + n + (6 - 2) + (n - 2) = 3n + 4, \end{aligned}$$

$$\begin{aligned} w_f(v_{n-1}) &= (2n - 1) + 2(n - 1) - 2 + 1 + (2n - 1) \\ &= 2n - 1 + 2n - 4 + 1 + 2n - 1 = 6n - 5, \\ w_f(v_n) &= 0 + \sum_{2 \le i \le n-1,\ i \text{ even}} (n - i) + \sum_{3 \le i \le n-2,\ i \text{ odd}} (n - 1 + i) + 2n + 2n - 2 \\ &= \frac{(n-1)(n-1)}{4} + \frac{(3n-1)(n-3)}{4} + 2n + 2n - 2 \\ &= \frac{(n^2 - 2n + 1) + (3n^2 - 10n + 3) + 16n - 8}{4} \\ &= \frac{4n^2 - 12n + 4 + 16n - 8}{4} \\ &= \frac{4n^2 + 4n - 4}{4} = n^2 + n - 1. \end{aligned}$$

For $i = 3, 4, \ldots, n-2$, we have

$$\begin{aligned} w_f(v_i) &= f(v_i) + f^*(v_{i-1}v_i) + f^*(v_iv_{i+1}) + f^*(v_iv_n) \\ &= \begin{cases} (n+i) + 2i - 2 + 2(i+1) - 2 + (n-i), \text{ if } i = 4, 6, \ldots, n-3, \\ (n+1-i) + 2i - 2 + 2(i+1) - 2 + n - 1 + i, \text{ if } i = 3, 5, \ldots, n-2 \end{cases} \\ &= \begin{cases} 2n + 2i - 2 + 2(i+1) - 2, \text{ if } i = 4, 6, \ldots, n-1, \\ 2n + 2i - 2 + 2(i+1) - 2, \text{ if } i = 3, 5, \ldots, n-2 \end{cases} \\ &= \begin{cases} 2n + 4i - 2, \text{ if } i = 4, 6, \ldots, n-1, \\ 2n + 4i - 2, \text{ if } i = 3, 5, \ldots, n-2 \end{cases} \\ &= 2n + 4i - 2, \text{ for } i = 3, 4, \ldots, n-2. \end{aligned}$$

Note that $w_f(v_1) < w_f(v_2)$ and

$$w_f(v_3) < w_f(v_4) < \cdots < w_f(v_{n-2}) < w_f(v_{n-1}).$$

For $n \geq 5$, $w_f(v_n) > w_f(v_i)$ for every $i = 0, 1, 2, \ldots, n-1$. Assuming $w_f(v_0) = w_f(v_1)$, we have

$$4n + 1 = 3n + 2 \iff n = 1.$$

Moreover, it can be shown that the vertex weights in the wheel graph W_n are all unique. Assuming $w_f(v_0) = w_f(v_2)$, we get

$$4n + 1 = 3n + 4 \iff n = 3.$$

For $w_f(v_0) = w_f(v_{n-1})$, we have

$$4n + 1 = 6n - 5 \iff n = 3.$$

If there exists $i \in \{3, 4, \ldots, n-2\}$ such that $w_f(v_0) = w_f(v_i)$, then

$$4n + 1 = 2n + 4i - 2 \iff 4i = 2n + 3,$$

creating a contradiction since $2n + 3$ is odd. Assuming $w_f(v_1) = w_f(v_{n-1})$, we get

$$3n + 2 = 6n - 5 \iff n = 3.$$

If there exists $i \in \{3, 4, \ldots, n-2\}$ such that $w_f(v_1) = w_f(v_i)$, we have

$$3n + 2 = 2n + 4i - 2 \iff 4i = n + 4,$$

yielding an inconsistency because $n + 4$ is odd. Assuming $w_f(v_2) = w_f(v_{n-1})$, we get

$$3n + 4 = 6n - 5 \iff n = 3.$$

If there exists $i \in \{3, 4, \ldots, n-2\}$ such that $w_f(v_2) = w_f(v_i)$, then

$$3n + 4 = 2n + 4i - 2 \iff 4i = n + 6,$$

yielding an inconsistency since $n + 6$ is odd.

For $n = 3$, the labeling given in Figure 7 is a graceful vit labeling.

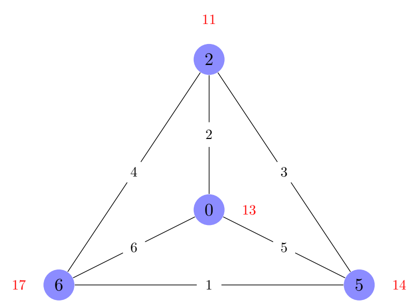

FIGURE 7. Graceful Vit Labeling on Wheel Graph W_3

Thus, it is shown that the function w_f is injective, meaning that f is a graceful vit labeling. In conclusion, the wheel graph W_n with an odd number $n \geq 3$ is a graceful vit graph. □

3. CONCLUDING REMARKS

This paper presents a novel concept of graceful labeling, which we call "graceful vit labeling." In this approach, for every couple of vertices that are not the same, the vertex weight values must be different. The weight assigned to a vertex is calculated by adding its graceful label to the total weights of all edges connected to it in a given graceful labeling. For a graph to qualify as graceful vit, it is required that the graph is graceful in the first place. Through our analysis, we identified several connected graphs—including $C_3, C_4, K_3, P_4, S_3, BTC_3,$ and $B_{n,m}$ (where $n, m \geq 2$)—that are graceful but not graceful vit. This observation raises an interesting question about the generality of this property, and we propose the following open problem:

Open Problem 3. *Can we describe the entire class of graphs that are graceful yet not graceful vit?*

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