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Construction Of An n-Norm On $\ell^p(\mathbb{R})$

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Abstract. In this article, we discuss an n-norm, with $n \geq 2$, which is defined through bounded linear functionals on p-summable sequence spaces. We also introduce a new norm induced by this n-norm, which will be examined for its equivalence to the usual norm. Next, we demonstrate the relationships between various mappings, including the n-norm, $(n\text{–}1)$-norm, $\cdots$, 2-norm, and the usual norm. Finally, our results show that the p-summable sequence spaces, equipped with these mappings are complete spaces.

Key words and Phrases: norm, n-norm, bounded linear functionals, ℓ^p-spaces.

1. INTRODUCTION

Let X be a real vector space and $\|\cdot\|$ be a real function on X satisfying the following three conditions:

(1) $\|x\| \geq 0$ for every $x \in X$; $\|x\| = 0$ if and only if $x = 0 \in X$;
(2) $\|\alpha x\| = |\alpha|\|x\|$ for every $x \in X$ and for every $\alpha \in \mathbb{R}$;
(3) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$.

The function $\|\cdot\|$ is called a norm on X and the pair $(X, \|\cdot\|)$ is called a normed space [1]. A norm can be viewed as a generalization of the concept of length within a vector space.

A set of p-summable sequences, with $1 \leq p \leq \infty$, is one of vector spaces in functional analysis. Here, the vector space is denoted by $\ell^p = \ell^p(\mathbb{R})$ which contains all sequences of real numbers $x = (x_j)$ for which $\sum\limits_{j=1}^{\infty} |x_j|^p < \infty$ (or $\sup\limits_{j\in\mathbb{N}} |x_j| < \infty$ while $p = \infty$). Usually, ℓ^p is equipped with the usual norm

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$$\|x\|_{\ell^p} := \begin{cases} \left(\sum\limits_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}} & , \quad \text{if } 1 \le p < \infty \\ \sup\limits_{j\in\mathbb{N}} |x_j| & , \quad \text{if } p = \infty \end{cases} \tag{1}$$

so that $\ell^p = (\ell^p, \|x\|_{\ell^p})$ becomes a normed space.

We also introduce an *n-norm.* Consider a vector space X with $dim(X) \ge n$. By [2], we give a mapping $\|\cdot, \dots, \cdot\|_{nX} : X \times \cdots \times X \longrightarrow \mathbb{R}$ such that it has four conditions:

(1) $\|v_1, \cdots, v_n\|_{nX} \ge 0$ holds for every $v_1, \dots, v_n \in X$; and $v_1, \dots, v_n$ are linearly dependent if and only if $\|v_1, \cdots, v_n\|_{nX} = 0$;
(2) $\|v_1, v_2, \cdots, v_n\|_{nX} = \|v_{i_1}, v_{i_2} \cdots, v_{i_n}\|_{nX}$ holds for every $v_1, v_2 \dots, v_n \in X$ and $(i_1, i_2 \cdots, i_n)$ is permutation of $(1, 2 \cdots, n)$;
(3) $\|\alpha v_1, \cdots, v_n\|_{nX} = |\alpha|\|v_1, \cdots, v_n\|_{nX}$ holds for every $v_1, \dots, v_n \in X$ and for every scalar $\alpha \in \mathbb{R}$;
(4) $\|v_1, \cdots, v_{n-1}, y + z\|_{nX} \le \|v_1, \cdots, v_{n-1}, y\|_{nX} + \|v_1, \cdots, v_{n-1}, z\|_{nX}$ holds for every $v_1, \dots, v_{n-1}, y, z \in X$.

We call that $\|\cdot, \cdots, \cdot\|_{nX}$ is an n-norm and the pair $(X, \|\cdot, \cdots, \cdot\|_{nX})$ is an n-normed space. On X, for simplification, we occasionally express the n-norm as

$$\|v_1, \cdots, v_n\|_{nX} = \|v_k|_{k=1}^n\|_{nX}, \tag{2}$$

for every $v_1, \cdots, v_n \in X$.

In functional analysis, the concept of n-normed spaces generalizes normed spaces. This idea was studied by various researchers who examined different properties of n-normed spaces. The notion of n-normed spaces was first introduced by Misiak [3], and subsequent contributions were made by Dutta [4], Gunawan and Mashadi [5], Gunawan [6], Kim and Cho [7], Lewandowska [8], and many others. In the specific case where $n = 2$, the notion of 2-normed spaces was introduced by Gähler in the mid-1960s (see [9]). This concept was further developed by Diminnie, Gähler, White [10, 11], among others.

We can add an extra property to the n-norm, as described in the following proposition.

Proposition 1.1. *On X, we have a property*

$$\left\| v_1, \cdots, v_{n-1}, \sum_{k=1}^{n} \alpha_k v_k \right\|_{nX} = |\alpha_n| \, \|v_1, \cdots, v_{n-1}, v_n\|_{nX}, \tag{3}$$

for every $v_1, \dots, v_n \in X$ and for every scalar $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$.

The proof of the above proposition is quite straightforward, relying only on the triangle inequality, the homogeneity property, and the concept of linearly dependent vectors. We leave the verification to the reader.

For $X = \ell^p$ where $1 \leq p < \infty$, we recall n-norm which is defined by Gunawan [2]. That is a mapping $\|\cdot, \cdots, \cdot\|_{n\ell^p} : \ell^p \times \cdots \times \ell^p \to \mathbb{R}$ which is stated by the following equation

$$\|x_k|_{k=1}^n\|_{n\ell^p} := \left(\frac{1}{n!} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left\| \begin{matrix} x_{1k_1} & x_{2k_1} & \cdots & x_{nk_1} \\ x_{1k_2} & x_{2k_2} & \cdots & x_{nk_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k_n} & x_{2k_n} & \cdots & x_{nk_n} \end{matrix} \right\|^p \right)^{\frac{1}{p}}, \tag{4}$$

for every $x_1, x_2, \cdots, x_n \in \ell^p$. For $p = \infty$, that is

$$\|x_k|_{k=1}^n\|_{n\ell^\infty} := \sup_{k_1, k_2, \cdots, k_n \in \mathbb{N}} \left\| \begin{matrix} x_{1k_1} & x_{2k_1} & \cdots & x_{nk_1} \\ x_{1k_2} & x_{2k_2} & \cdots & x_{nk_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k_n} & x_{2k_n} & \cdots & x_{nk_n} \end{matrix} \right\|,$$

for every $x_1, x_2, \cdots, x_n \in \ell^\infty$. Note that $\left\| \begin{matrix} x_{1k_1} & x_{2k_1} & \cdots & x_{nk_1} \\ x_{1k_2} & x_{2k_2} & \cdots & x_{nk_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k_n} & x_{2k_n} & \cdots & x_{nk_n} \end{matrix} \right\|$ means an absolute value of $det \begin{pmatrix} x_{1k_1} & x_{2k_1} & \cdots & x_{nk_1} \\ x_{1k_2} & x_{2k_2} & \cdots & x_{nk_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k_n} & x_{2k_n} & \cdots & x_{nk_n} \end{pmatrix}$. In this article, the n-norm defined by Gunawan (4) is referred to as the usual n-norm.

We now describe that a space X, equipped with a norm, is called a complete space (or Banach space) if every Cauchy sequence in X converges to a limit within the space. In the context of an n-normed space X, a sequence $(x(m))$ converges to x if and only if

$$\|v_1, \ldots, v_{(n-1)}, x(m) - x\|_{nX} \to 0$$

for all $v_1, \ldots, v_{(n-1)} \in X$ as $m \to \infty$. Likewise, the sequence $(x(m))$ is Cauchy if and only if

$$\|v_1, \ldots, v_{(n-1)}, x(m_1) - x(m_2)\|_{nX} \to 0$$

for all $v_1, \ldots, v_{(n-1)} \in X$ as $m_1, m_2 \to \infty$. An n-Banach space is an n-normed space where every Cauchy sequence converges. It was shown in [1] and [2] that both the usual norm and the usual n-norm result in complete spaces, meaning that both are a Banach spaces and an n-Banach spaces respectively. On $\ell^p(\mathbb{R})$, Konca *et al.* [12] conducted deeply investigation into the equivalence of two definitions of convergence (two definitions of Cauchy sequences).

In 2000, Gunawan introduced the norm induced by the usual n-norm and the first n standard basis vectors in $\ell^p(\mathbb{R})$ [2]. Later, in 2013, he and his students refined this definition by using any set of n linearly independent vectors in $\ell^p(\mathbb{R})$ [13]. This prompts the question: can we define an n-norm that is induced by a norm? For $p = 2$, defining a 2-norm on $\ell^2(\mathbb{R})$ is fairly straightforward, along with the corresponding 2-inner product. For $\ell^p(\mathbb{R})$ with $p \neq 2$, Konca *et al.* introduced a 2-inner product and derived a 2-norm (see [14]). They also investigated the completeness of these spaces. In 2016, Konca and Idris defined a 2-norm using the usual norm on $\ell^p(\mathbb{R})$ along with a set of bounded linear functionals [15], and later simplified this definition in [16]. They also explored the relationships between Gähler's 2-norm [9] and Gunawan's 2-norm (the usual 2-norm) [2]. In this work, we extend the approach of Konca and Idris to the case of $n > 2$. Additionally, we examine the relationship between their definition and the two different definitions of n-norms provided by Gähler and Gunawan. Furthermore, we investigate the significant role of bounded linear functionals on the space $F := \ell^q$ where $\frac{1}{q} = 1 - \frac{1}{p}$. Our findings confirm that all the fundamental properties of the n-norm on ℓ^p, constructed in this study are consistent with those discussed in the article "New 2-norms formed on ℓ^p by bounded linear functionals".

2. A SET OF BOUNDED LINEAR FUNCTIONALS

For further details on bounded linear functionals, we refer to [16]. Let X be a normed space and $f : X \to \mathbb{R}$ a bounded linear functional. The set of all bounded linear functionals on X, known as the dual space of X, is denoted by X'. Now, observe that for every $x \in X \setminus \{0\}$ and $f \in X'$, we have the inequality $|f(x)| \leq \|f\|_{X'} \|x\|_X$, or equivalently, $\frac{|f(x)|}{\|f\|_{X'}} \leq \|x\|_X$. Consequently, we obtain the following result:

$$\sup_{f \neq 0, f \in X'} \frac{|f(x)|}{\|f\|_{X'}} = \sup_{\|f\|_{X'} \leq 1} |f(x)| \leq \|x\|_X.$$

On $\ell^p(\mathbb{R})$, the usual norm $\|\cdot\|_{\ell^p}$ is defined as in (1). Let $w \in \ell^q$ with $\|w\|_{\ell^q} > 0$ and $\frac{1}{q} = 1 - \frac{1}{p}$. For every $x \in \ell^p$, we define the functional $f_w : \ell^p \to \mathbb{R}$ as follows:

$$f_w(x) := \sum_{k=1}^{\infty} w_k x_k \tag{5}$$

This functional is linear, and by applying Hölder's inequality, for every $x \in \ell^p$, we obtain

$$f_w(x) \leq |f_w(x)| \leq \|w\|_{\ell^q} \|x\|_{\ell^p}. \tag{6}$$

Thus, we conclude that f_w is a bounded linear functional. We then collect all such bounded linear functionals f_w into the set

$$F := \{f_w \mid w \in \ell^q\} = \ell^q.$$

Here, ℓ^q represents the dual space of ℓ^p, consisting of all bounded linear functionals on ℓ^p. As noted in [1], we have a norm $\|f_w\|_F = \|w\|_{\ell^q}$, and we say that $(F, \|\cdot\|_F)$ is a normed space.

Based on the previous description, we now present a definition of another norm on ℓ^p using sets of bounded linear functionals. Specifically, we define

$$\|x\|_{(f_w)\ell^p} := \sup_{\|f_w\|_F > 0} \frac{|f_w(x)|}{\|f_w\|_F} \tag{7}$$

for every $x \in \ell^p$. Readers can verify that this definition indeed defines a norm. Next, we explore its relationship with the usual norm. For $x \in \ell^p$, let $w_* \in \ell^q$ be defined as $w_* = (w_{*k})_{k\in\mathbb{N}}$, where $w_{*k} := |x_k|^{(p-1)}\,\mathrm{sign}(x_k)$. We have

$$|f_{w_*}(x)| = \sum_{k=1}^{\infty} |x_k|^{(p-1)} sign(x_k) x_k = \sum_{k=1}^{\infty} |x_k|^p = \|x\|_{\ell^p}^p.$$

Additionally, the norm of the functional is computed as

$$\|f_{w_*}\|_F = \left(\sum_{k=1}^{\infty} (|x_k|^{(p-1)})^q\right)^{\frac{1}{q}} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{q}} = \|x\|_{\ell^p}^{p-1}.$$

Thus, we obtain

$$\|x\|_{(f_w)\ell^p} \geq \frac{|f_{w_*}(x)|}{\|f_{w_*}\|_F} = \|x\|_{\ell^p}.$$

Since the inequality in equation (6) holds, we conclude that

$$\|x\|_{(f_w)\ell^p} = \|x\|_{\ell^p},$$

for every $x \in \ell^p$. This shows that the norms $\|\cdot\|_{(f_w)\ell^p}$ and $\|\cdot\|_{\ell^p}$ are equivalent.

3. MAIN RESULTS AND DISCUSSION

We begin by recalling Gähler's definition of the n-norm on ℓ^p, where $p \geq 1$. Let $F = \ell^q$ be the dual space consisting of all bounded linear functionals, where $\frac{1}{q} = 1 - \frac{1}{p}$. According to Gähler, the n-norm $\|\cdot, \cdots, \cdot\|_{n\odot\ell^p}^G : \ell^p \to \mathbb{R}$ is defined by

$$\|x_k|_{k=1}^n\|_{n\odot\ell^p}^G := \sup_{\substack{0<\|f_{w_j}\|_F\leq 1, f_{w_j}(x_i)\neq 0 \\ i,j\in\{1,2,\cdots,n\}}} \begin{vmatrix} f_{w_1}(x_1) & \cdots & f_{w_1}(x_n) \\ \vdots & \ddots & \vdots \\ f_{w_n}(x_1) & \cdots & f_{w_n}(x_n). \end{vmatrix} \tag{8}$$

for every $x_1, x_2, \cdots, x_n \in \ell^p$.

Lemma 3.1. *For $x_1, x_2, \cdots, x_n \in \ell^p$ and $w_1, w_2, \cdots, w_n \in \ell^q$ where $1 = \frac{1}{p} + \frac{1}{q}$, we have*

$$\begin{vmatrix} \sum\limits_{k=1}^{\infty} x_{1k} w_{1k} & \cdots & \sum\limits_{k=1}^{\infty} x_{nk} w_{1k} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=1}^{\infty} x_{1k} w_{nk} & \cdots & \sum\limits_{k=1}^{\infty} x_{nk} w_{nk} \end{vmatrix} = \sum_{k_n=1}^{\infty} \cdots \sum_{k_1=1}^{\infty} w_{nk_n} \cdots w_{1k_1} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{vmatrix}.$$

Proof. Take arbitrary $x_1, x_2, \cdots, x_n \in \ell^p$ and $w_1, w_2, \cdots, w_n \in \ell^q$ where $1 = \dfrac{1}{p} + \dfrac{1}{q}$.
Let

$$M_n := \begin{vmatrix} \sum\limits_{k=1}^{\infty} x_{1k} w_{1k} & \cdots & \sum\limits_{k=1}^{\infty} x_{nk} w_{1k} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=1}^{\infty} x_{1k} w_{nk} & \cdots & \sum\limits_{k=1}^{\infty} x_{nk} w_{nk} \end{vmatrix}.$$

Using the properties of determinant and applying Laplace expansion, we get

$$M_n = \begin{vmatrix} x_{11} w_{11} + \sum\limits_{k_1=2}^{\infty} x_{1k_1} w_{1k_1} & \cdots & x_{n1} w_{11} + \sum\limits_{k_1=2}^{\infty} x_{nk_1} w_{1k_1} \\ \sum\limits_{k_2=1}^{\infty} x_{1k_2} w_{2k_2} & \cdots & \sum\limits_{k_2=1}^{\infty} x_{nk_2} w_{2k_2} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix}$$

$$= \begin{vmatrix} x_{11} w_{11} & \cdots & x_{n1} w_{11} \\ \sum\limits_{k_2=1}^{\infty} x_{1k_2} w_{2k_2} & \cdots & \sum\limits_{k_2=1}^{\infty} x_{nk_2} w_{2k_2} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix} + \begin{vmatrix} \sum\limits_{k_1=2}^{\infty} x_{1k_1} w_{1k_1} & \cdots & \sum\limits_{k_1=2}^{\infty} x_{nk_1} w_{1k_1} \\ \sum\limits_{k_2=1}^{\infty} x_{1k_2} w_{2k_2} & \cdots & \sum\limits_{k_2=1}^{\infty} x_{nk_2} w_{2k_2} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix}.$$

Check that

$$\begin{vmatrix} x_{11} w_{11} & \cdots & x_{n1} w_{11} \\ \sum\limits_{k_2=1}^{\infty} x_{1k_2} w_{2k_2} & \cdots & \sum\limits_{k_2=1}^{\infty} x_{nk_2} w_{2k_2} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix} = w_{11} \begin{vmatrix} x_{11} & \cdots & x_{n1} \\ \sum\limits_{k_2=1}^{\infty} x_{1k_2} w_{2k_2} & \cdots & \sum\limits_{k_2=1}^{\infty} x_{nk_2} w_{2k_2} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix}.$$

So, we obtain

$$M_n = w_{11} \begin{vmatrix} x_{11} & \cdots & x_{n1} \\ \sum\limits_{k_2=1}^{\infty} x_{1k_2} w_{2k_2} & \cdots & \sum\limits_{k_2=1}^{\infty} x_{nk_2} w_{2k_2} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix} + \begin{vmatrix} \sum\limits_{k_1=2}^{\infty} x_{1k_1} w_{1k_1} & \cdots & \sum\limits_{k_1=2}^{\infty} x_{nk_1} w_{1k_1} \\ \sum\limits_{k_2=1}^{\infty} x_{1k_2} w_{2k_2} & \cdots & \sum\limits_{k_2=1}^{\infty} x_{nk_2} w_{2k_2} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix}.$$

Moving to w_{12}, we obtain

$$M_n = \sum_{k_1=1}^{2} w_{1k_1} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \sum\limits_{k_2=1}^{\infty} x_{1k_2} w_{2k_2} & \cdots & \sum\limits_{k_2=1}^{\infty} x_{nk_2} w_{2k_2} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix} + \begin{vmatrix} \sum\limits_{k_1=3}^{\infty} x_{1k_1} w_{1k_1} & \cdots & \sum\limits_{k_1=3}^{\infty} x_{nk_1} w_{1k_1} \\ \sum\limits_{k_2=1}^{\infty} x_{1k_2} w_{2k_2} & \cdots & \sum\limits_{k_2=1}^{\infty} x_{nk_2} w_{2k_2} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix}.$$

Next, moving to w_{1k_1} where k_1 goes to ∞

$$M_n = \sum_{k_1=1}^{\infty} w_{1k_1} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \sum\limits_{k_2=1}^{\infty} x_{1k_2} w_{2k_2} & \cdots & \sum\limits_{k_2=1}^{\infty} x_{nk_2} w_{2k_2} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix}.$$

We repeat the same process for $k_2, k_3, \cdots, k_n$ to get

$$\begin{aligned} M_n &= \sum_{k_1=1}^{\infty} w_{1k_1} \sum_{k_2=1}^{\infty} w_{2k_2} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ x_{1k_2} & \cdots & x_{nk_2} \\ \sum\limits_{k_3=1}^{\infty} x_{1k_3} w_{3k_3} & \cdots & \sum\limits_{k_3=1}^{\infty} x_{nk_3} w_{3k_3} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k_n=1}^{\infty} x_{1k_n} w_{nk_n} & \cdots & \sum\limits_{k_n=1}^{\infty} x_{nk_n} w_{nk_n} \end{vmatrix} \\ &\vdots \\ &= \sum_{k_1=1}^{\infty} w_{1k_1} \sum_{k_2=1}^{\infty} w_{2k_2} \cdots \sum_{k_n=1}^{\infty} w_{nk_n} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ x_{1k_2} & \cdots & x_{nk_2} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{vmatrix}. \end{aligned}$$

Applying the homogeneity property of summability,

$$M_n = \sum_{k_n=1}^{\infty} \cdots \sum_{k_1=1}^{\infty} w_{nk_n} \cdots w_{1k_1} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{vmatrix}$$

holds. The proof is thereby completed. □

By the above lemma, we have

$$\begin{vmatrix} f_{w_1}(x_1) & \cdots & f_{w_1}(x_n) \\ \vdots & \ddots & \vdots \\ f_{w_n}(x_1) & \cdots & f_{w_n}(x_n). \end{vmatrix} = \sum_{k_n=1}^{\infty}\cdots\sum_{k_1=1}^{\infty} w_{nk_n}\cdots w_{1k_1} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{vmatrix}$$

and we also use it for (8)

$$\|x_k|_{k=1}^n\|_{n\odot\ell^p}^G = \sup_{\substack{0<\|f_{w_j}\|_F\le 1 \\ f_{w_j}(x_i)\neq 0 \\ i,j\in\{1,2,\cdots,n\}}} \sum_{k_n=1}^{\infty}\cdots\sum_{k_1=1}^{\infty} w_{nk_n}\cdots w_{1k_1} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{vmatrix}.$$

In (8), there is a slight modification to Gähler's definition of the n-norm. Here, we add the condition $f_{w_j}(x_i) \neq 0$ for $i, j \in \{1, \cdots, n\}$ to obtain the supremum $\begin{vmatrix} f_{w_1}(x_1) & \cdots & f_{w_1}(x_n) \\ \vdots & \ddots & \vdots \\ f_{w_n}(x_1) & \cdots & f_{w_n}(x_n) \end{vmatrix}$. This condition is required since defining a new n-norm involves a division by f_w. Now, take arbitrary $x_1, x_2, \cdots, x_n \in \ell^p$ and let

$$A_{n1} := \frac{x_n}{f_w(x_n)} - \frac{x_1}{f_w(x_1)}.$$

Using (4), we get

$$\begin{aligned}
&\frac{\left\|x_1 f_w(x_n) - x_n f_w(x_1), \cdots, x_{(n-1)} f_w(x_n) - x_n f_w(x_{(n-1)})\right\|_{(n-1)\ell^p}}{|f_w(x_n)|^{(n-2)}} \qquad (9)\\
=&\left\| x_n f_w(x_1) - x_1 f_w(x_n), x_2 - x_n\frac{f_w(x_2)}{f_w(x_n)}, \cdots, x_{(n-1)} - x_n\frac{f_w(x_{(n-1)})}{f_w(x_n)} \right\|_{(n-1)\ell^p}\\
=&|f_w(x_1) f_w(x_n)|\\
&\times \left\| A_{n1}, x_2 - \frac{x_1 f_w(x_2)}{f_w(x_1)} - f_w(x_2) A_{n1}, \cdots, x_{(n-1)} - \frac{x_1 f_w(x_{(n-1)})}{f_w(x_1)} - f_w(x_{(n-1)}) A_{n1} \right\|_{(n-1)\ell^p}\\
=&|f_w(x_1) f_w(x_n)| \left\| A_{n1}, x_2 - x_1\frac{f_w(x_2)}{f_w(x_1)}, \cdots, x_{(n-1)} - x_1\frac{f_w(x_{(n-1)})}{f_w(x_1)} \right\|_{(n-1)\ell^p}\\
=&\frac{\|x_2 f_w(x_1) - x_1 f_w(x_2), \cdots, x_n f_w(x_1) - x_1 f_w(x_n)\|_{(n-1)\ell^p}}{|f_w(x_1)|^{(n-2)}}.
\end{aligned}$$

We proceed to define a mapping based on the usual $(n-1)$-norm on ℓ^p and the set of bounded linear functionals $F = \ell^q$. The definition is given as follows.

$$\begin{aligned}
&\|x_k|_{k=1}^n\|_{n\odot\ell^p}^I \qquad (10)\\
:=& \sup_{\substack{0<\|f_w\|_F\le 1 \\ f_w(x_i)\neq 0 \\ i\in\{1,2,\cdots,n\}}} \frac{\|x_2 f_w(x_1) - x_1 f_w(x_2), \cdots, x_n f_w(x_1) - x_1 f_w(x_n)\|_{(n-1)\ell^p}}{|f_w(x_1)|^{(n-2)}},
\end{aligned}$$

for every $x_1, x_2, \cdots, x_n \in \ell^p$.

Since $\|\cdot, \ldots, \cdot\|^I_{(n-1)\ell^p}$ is an $(n-1)$-norm, then we can use its properties to verify that (10) is an n-norm. Readers may check its properties. Now, we represent (9) in a form

$$\begin{aligned}\frac{\left\| x_k f_w(x_n) - x_n f_w(x_k)|_{k=1}^{(n-1)} \right\|_{(n-1)\ell^p}}{|f_w(x_n)|^{(n-2)}} &= |f_w(x_n)| \left\| x_k - x_n \frac{f_w(x_k)}{f_w(x_n)} |_{k=1}^{(n-1)} \right\|_{(n-1)\ell^p} \\ &= \left(\frac{1}{(n-1)!} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_{(n-1)}=1}^{\infty} |B|^p \right)^{\frac{1}{p}},\end{aligned}$$

where

$$B = f_w(x_n) \begin{vmatrix} x_{1k_1} - \frac{x_{nk_1} f_w(x_1)}{f_w(x_n)} & \cdots & x_{(n-1)k_1} - \frac{x_{nk_1} f_w(x_{(n-1)})}{f_w(x_n)} \\ \vdots & \ddots & \vdots \\ x_{1k_{(n-1)}} - \frac{x_{nk_{(n-1)}} f_w(x_1)}{f_w(x_n)} & \cdots & x_{(n-1)k_{(n-1)}} - \frac{x_{nk_{(n-1)}} f_w(x_{(n-1)})}{f_w(x_n)} \end{vmatrix}. \tag{11}$$

Using Laplace expansion, (11) becomes

$$\begin{aligned} B &= \begin{vmatrix} x_{1k_1} & \cdots & x_{(n-1)k_1} & x_{nk_1} \\ \vdots & \ddots & \vdots & \\ x_{1k_{(n-1)}} & \cdots & x_{(n-1)k_{(n-1)}} & x_{nk_{(n-1)}} \\ f_w(x_1) & \cdots & f_w(x_{(n-1)}) & f_w(x_n) \end{vmatrix} \\ &= \sum_{k_n=1}^{\infty} w_{k_n} \begin{vmatrix} x_{1k_1} & \cdots & x_{(n-1)k_1} & x_{nk_1} \\ \vdots & \ddots & \vdots & \\ x_{1k_{(n-1)}} & \cdots & x_{(n-1)k_{(n-1)}} & x_{nk_{(n-1)}} \\ x_{1k_n} & \cdots & x_{(n-1)k_n} & x_{nk_n} \end{vmatrix}. \end{aligned}$$

Consequently, (10) becomes

$$\begin{aligned} \|x_k|_{k=1}^n\|^I_{n\odot\ell^p} &= \sup_{\substack{0<\|f_w\|_F\le 1 \\ f_w(x_i)\neq 0 \\ i\in\{1,2,\cdots,n\}}} |f_w(x_n)| \left\| x_k - x_n \frac{f_w(x_k)}{f_w(x_n)} |_{k=1}^{(n-1)} \right\|_{(n-1)\ell^p} \\ &= \sup_{\substack{0<\|f_w\|_F\le 1 \\ f_w(x_i)\neq 0 \\ i\in\{1,2,\cdots,n\}}} \left(\frac{\sum_{k_1=1}^{\infty} \cdots \sum_{k_{(n-1)}=1}^{\infty} \left| \left(\sum_{k_n=1}^{\infty} w_{k_n} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{vmatrix} \right) \right|^p}{(n-1)!} \right)^{\frac{1}{p}}. \end{aligned}$$

3.1. **Equivalence of Three n-norms.**

Now, there are three n-norms on ℓ^p. We will investigate their relationships.

Proposition 3.2. *Let $p \geq 1$. For every $x_1, x_2, \cdots, x_n \in \ell^p$, we have*

$$C_1 \|x_1, x_2, \cdots, x_n\|^G_{n \odot \ell^p} \leq \|x_1, x_2, \cdots, x_n\|^I_{n \odot \ell^p} \leq C_2 \|x_1, x_2, \cdots, x_n\|_{n\ell^p},$$

where $C_1 = \left(\dfrac{1}{(n-1)!}\right)^{\frac{1}{p}}$ *and* $C_2 = \left(\dfrac{n!}{(n-1)!}\right)^{\frac{1}{p}}$.

Proof. For arbitrary $x_1, x_2, \ldots, x_n \in \ell^p$, recall (8)

$$\|x_k|_{k=1}^n\|^G_{n \odot \ell^p} = \sup_{\substack{0<\|f_{w_j}\|_F \leq 1 \\ f_{w_j}(x_i) \neq 0 \\ i,j \in \{1,2,\cdots,n\}}} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} w_{1k_1} \cdots w_{nk_n} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{vmatrix}.$$

Notice the following expression

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} w_{1k_1} \cdots w_{nk_n} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{vmatrix} = \sum_{k_1=1}^{\infty} \cdots \sum_{k_{(n-1)}=1}^{\infty} \Big(w_{1k_1} \cdots w_{(n-1)k_{(n-1)}} \Big) D$$

where $D = \sum\limits_{k_n=1}^{\infty} w_{nk_n} \begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{vmatrix}$. By applying Hölder's inequality, we obtain

$$\begin{aligned}
&\sum_{k_1=1}^{\infty} \cdots \sum_{k_{(n-1)}=1}^{\infty} \Big(w_{1k_1} \cdots w_{(n-1)k_{(n-1)}} \Big) D \\
&\qquad \leq \Big(\sum_{k_1=1}^{\infty} \cdots \sum_{k_{(n-1)}=1}^{\infty} \Big| w_{1k_1} \cdots w_{(n-1)k_{(n-1)}} \Big|^q \Big)^{\frac{1}{q}} \Big(\sum_{k_1=1}^{\infty} \cdots \sum_{k_{(n-1)}=1}^{\infty} \Big| D \Big|^p \Big)^{\frac{1}{p}} \\
&\qquad \leq \Big(\sum_{k_1=1}^{\infty} \cdots \sum_{k_{(n-1)}=1}^{\infty} \Big| D \Big|^p \Big)^{\frac{1}{p}}.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x_k|_{k=1}^n\|^G_{n \odot \ell^p} &= \sup_{\substack{0<\|f_{w_j}\|_F \leq 1 \\ f_{w_j}(x_i) \neq 0 \\ i,j \in \{1,2,\cdots,n\}}} \sum_{k_1=1}^{\infty} \cdots \sum_{k_{(n-1)}=1}^{\infty} \Big(w_{1k_1} \cdots w_{(n-1)k_{(n-1)}} \Big) D \\
&\leq \sup_{\substack{0<\|f_{w_j}\|_F \leq 1 \\ f_{w_j}(x_i) \neq 0 \\ i,j \in \{1,2,\cdots,n\}}} \Big(\sum_{k_1=1}^{\infty} \cdots \sum_{k_{(n-1)}=1}^{\infty} \Big| D \Big|^p \Big)^{\frac{1}{p}}.
\end{aligned}$$

Check that

$$\sup_{\substack{0<\left\|f_{w_j}\right\|_F\leq 1\\ f_{w_j}(x_i)\neq 0\\ i,j\in\{1,2,\cdots,n\}}}\Big(\sum_{k_1=1}^{\infty}\cdots\sum_{k_{(n-1)}=1}^{\infty}\Big|D\Big|^p\Big)^{\frac{1}{p}}$$

$$\leq ((n-1)!)^{\frac{1}{p}}\sup_{\substack{0<\left\|f_{w_j}\right\|_F\leq 1\\ f_{w_j}(x_i)\neq 0\\ i,j\in\{1,2,\cdots,n\}}}\left(\frac{\sum\limits_{k_1=1}^{\infty}\cdots\sum\limits_{k_{(n-1)}=1}^{\infty}\left|\left(\sum\limits_{k_n=1}^{\infty}w_{nk_n}\begin{vmatrix}x_{1k_1}&\cdots&x_{nk_1}\\ \vdots&\ddots&\vdots\\ x_{1k_n}&\cdots&x_{nk_n}\end{vmatrix}\right)\right|^p}{(n-1)!}\right)^{\frac{1}{p}}$$

$$\leq ((n-1)!)^{\frac{1}{p}}\|x_k|_{k=1}^n\|^I_{n\odot\ell^p}.$$

Next, we can express the inequality between the two definitions of the n-norms as below

$$\|x_k|_{k=1}^n\|^G_{n\odot\ell^p}\leq ((n-1)!)^{\frac{1}{p}}\|x_k|_{k=1}^n\|^I_{n\odot\ell^p}.$$

Meanwhile, we recall

$$\|x_k|_{k=1}^n\|^I_{n\odot\ell^p}=\sup_{\substack{0<\|f_{w_n}\|_F\leq 1\\ f_{w_n}(x_i)\neq 0\\ i\in\{1,2,\cdots,n\}}}\left(\frac{\sum\limits_{k_1=1}^{\infty}\cdots\sum\limits_{k_{(n-1)}=1}^{\infty}\left|\left(\sum\limits_{k_n=1}^{\infty}w_{nk_n}\begin{vmatrix}x_{1k_1}&\cdots&x_{nk_1}\\ \vdots&\ddots&\vdots\\ x_{1k_n}&\cdots&x_{nk_n}\end{vmatrix}\right)\right|^p}{(n-1)!}\right)^{\frac{1}{p}}$$

$$=\sup_{\substack{0<\|f_{w_n}\|_F\leq 1\\ f_{w_n}(x_i)\neq 0\\ i\in\{1,2,\cdots,n\}}}\left(\frac{\sum\limits_{k_1=1}^{\infty}\cdots\sum\limits_{k_{(n-1)}=1}^{\infty}|D|^p}{(n-1)!}\right)^{\frac{1}{p}},$$

where $D=\sum\limits_{k_n=1}^{\infty}w_{nk_n}\begin{vmatrix}x_{1k_1}&\cdots&x_{nk_1}\\ \vdots&\ddots&\vdots\\ x_{1k_n}&\cdots&x_{nk_n}\end{vmatrix}$. By Hölder's inequality, we get

$$|D|\leq\sum_{k_n=1}^{\infty}|w_{nk_n}|\left\|\begin{vmatrix}x_{1k_1}&\cdots&x_{nk_1}\\ \vdots&\ddots&\vdots\\ x_{1k_n}&\cdots&x_{nk_n}\end{vmatrix}\right\|$$

$$\leq \left(\sum_{k_n=1}^{\infty} |w_{nk_n}|^q \right)^{\frac{1}{q}} \left(\sum_{k_n=1}^{\infty} \left\| \begin{matrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{matrix} \right\|^p \right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{k_n=1}^{\infty} \left\| \begin{matrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{matrix} \right\|^p \right)^{\frac{1}{p}}.$$

Thus,

$$\|x_k|_{k=1}^n\|_{n\odot\ell^p}^I \leq \sup_{\substack{0<\|f_{w_n}\|_F\leq 1 \\ f_{w_n}(x_i)\neq 0 \\ i\in\{1,2,\cdots,n\}}} \left(\frac{\sum_{k_1=1}^{\infty}\cdots\sum_{k_{(n-1)}=1}^{\infty}\sum_{k_n=1}^{\infty} \left\| \begin{matrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n} \end{matrix} \right\|^p}{(n-1)!} \right)^{\frac{1}{p}}$$

and we obtain $\|x_k|_{k=1}^n\|_{n\odot\ell^p}^I \leq \left(\frac{n!}{(n-1)!}\right)^{\frac{1}{p}} \|x_k|_{k=1}^n\|_{n\ell^p}$.

Next, we combine the above inequalities. So

$$C_1\|x_1, x_2, \cdots, x_n\|_{n\odot\ell^p}^G \leq \|x_1, x_2, \cdots, x_n\|_{n\odot\ell^p}^I \leq C_2\|x_1, x_2, \cdots, x_n\|_{n\ell^p}$$

holds, where $C_1 = \left(\frac{1}{(n-1)!}\right)^{\frac{1}{p}}$ and $C_2 = \left(\frac{n!}{(n-1)!}\right)^{\frac{1}{p}}$. □

The above proposition makes it easier for us to check the equivalence among three n-norms. Next, we use the symbol $U \sim V$, which means that U and V are equivalent.

Theorem 3.3. *On ℓ^p, we have*

$$\|\cdot, \cdots, \cdot\|_{n\odot\ell^p}^G \sim \|\cdot, \cdots, \cdot\|_{n\odot\ell^p}^I \sim \|\cdot, \cdots, \cdot\|_{n\ell^p}.$$

Proof. We know that the equations (4) and (8) are equivalent (see [17]). By checking Proposition 3.2, we conclude that $\|\cdot, \cdots, \cdot\|_{n\odot\ell^p}^G$, $\|\cdot, \cdots, \cdot\|_{n\odot\ell^p}^I$, and $\|\cdot, \cdots, \cdot\|_{n\ell^p}$ are equivalent. □

Corollary 3.4. *Two spaces $(\ell^p, \|\cdot, \cdots, \cdot\|_{n\odot\ell^p}^G)$ and $(\ell^p, \|\cdot, \cdots, \cdot\|_{n\odot\ell^p}^I)$ are n-Banach spaces.*

Proof. As shown in [2, 18], $(\ell^p, \|\cdot, \ldots, \cdot\|_{n\ell^p})$ is an n-Banach space. This means that every Cauchy sequence with respect to $\|\cdot, \ldots, \cdot\|_{n\ell^p}$ in ℓ^p is convergent with respect to $\|\cdot, \ldots, \cdot\|_{n\ell^p}$. Theorem 3.3 demonstrates that

$$\|\cdot, \ldots, \cdot\|_{n\odot\ell^p}^G \sim \|\cdot, \ldots, \cdot\|_{n\ell^p}.$$

Let $(x(m))$ in ℓ^p be a Cauchy sequence with respect to $\|\cdot,\dots,\cdot\|^G_{n\odot\ell^p}$. So for every $v_1, v_2, \cdots, v_{(n-1)} \in \ell^p$, we have

$$\|v_1, \cdots, v_{(n-1)}, x(m_1) - x(m_2)\|_{n\ell^p} \sim \|v_1, \cdots, v_{(n-1)}, x(m_1) - x(m_2)\|^G_{n\odot\ell^p} \to 0,$$

as $m_1, m_2 \to \infty$. It means that this sequence becomes a Cauchy sequence with respect to $\|\cdot,\dots,\cdot\|_{n\ell^p}$. Since $(\ell^p, \|\cdot,\dots,\cdot\|_{n\ell^p})$ is an n-Banach space, then this sequence is also convergent with respect to $\|\cdot,\dots,\cdot\|_{n\ell^p}$. Furthermore, since

$$\|\cdot,\dots,\cdot\|^G_{n\odot\ell^p} \sim \|\cdot,\dots,\cdot\|_{n\ell^p},$$

then we can conclude that $(x(m))$ is also convergent with respect to $\|\cdot,\dots,\cdot\|^G_{n\odot\ell^p}$.

Next, in this part, we apply a similar argument to Cauchy sequences with respect to $\|\cdot,\dots,\cdot\|^I_{n\odot\ell^p}$. By Theorem 3.3, which states that

$$\|\cdot,\dots,\cdot\|^I_{n\odot\ell^p} \sim \|\cdot,\dots,\cdot\|_{n\ell^p},$$

we therefore conclude that the sequence is convergent with respect to $\|\cdot,\dots,\cdot\|^I_{n\odot\ell^p}$.

Hence, as a result, $(\ell^p, \|\cdot,\dots,\cdot\|^G_{n\odot\ell^p})$ and $(\ell^p, \|\cdot,\dots,\cdot\|^I_{n\odot\ell^p})$ are n-Banach spaces. $\square$

3.2. **Relationship Between Norms And n-norms.**

In the study of ℓ^p, researchers so far defined a norm induced by the n-norm. In 2000 [2], Gunawan related the usual type (norm and n-norm) and he refined it in 2013 (see [13]). Take fixed linearly independent vectors $a_1, a_2, \cdots, a_n \in \ell^p$ and we recall the definition of norm in [13], that is,

$$\|x\|^H_{\circ\ell^p} := \left(\sum_{\{k_1,k_2,\cdots,k_{(n-1)}\}\subset\{1,2,\cdots,n\}} \|a_{k_1}, a_{k_2}, \cdots, a_{k_{(n-1)}}, x\|^p_{n\ell^p} \right)^{\frac{1}{p}} \tag{12}$$

for every $x \in \ell^p$. His investigation showed that the norm $\|\cdot\|^H_{\circ\ell^p}$ induced by the usual n-norm is actually equivalent to the usual norm (see [13]). In this context, he could verify the convergence of a sequence $x(m)$ to x in $(\ell^p, \|\cdot,\cdots,\cdot\|_{n\ell^p})$ by selecting a fixed set of n linearly independent vectors $\{a_1, a_2, \dots, a_n\} \subset \ell^p$. So, for every subset $\{k_1, k_2, \dots, k_{(n-1)}\} \subset \{1, 2, \dots, n\}$, the condition

$$\|a_{k_1}, a_{k_2}, \dots, a_{k_{(n-1)}}, x(m) - x\|_{n\ell^p} \to 0$$

holds as $m \to \infty$. A similar approach can also be used to examine whether a sequence is Cauchy in $(\ell^p, \|\cdot,\cdots,\cdot\|_{n\ell^p})$.

Here, we will define two new norms induced by $\|\cdot,\cdots,\cdot\|^G_{n\odot\ell^p}$ and $\|\cdot,\cdots,\cdot\|^I_{n\odot\ell^p}$, these are

$$\|x\|^G_{\odot\ell^p} := \sum_{\{k_1,k_2,\cdots,k_{(n-1)}\}\subset\{1,2,\cdots,n\}} \|a_{k_1}, a_{k_2}, \cdots, a_{k_{(n-1)}}, x\|^G_{n\odot\ell^p} \tag{13}$$

$$\|x\|^I_{\odot\ell^p} := \sum_{\{k_1,k_2,\cdots,k_{(n-1)}\}\subset\{1,2,\cdots,n\}} \|a_{k_1}, a_{k_2}, \cdots, a_{k_{(n-1)}}, x\|^I_{n\odot\ell^p}, \tag{14}$$

for every $x \in \ell^p$. The relationship between the two norm definitions and the usual norm in ℓ^p will be examined. The following theorem explains it.

Theorem 3.5. *On ℓ^p with $p \geq 1$, we have*
$$\|\cdot\|^G_{\odot\ell^p} \sim \|\cdot\|^I_{\odot\ell^p} \sim \|\cdot\|_{\ell^p}.$$

Proof. Let $p \geq 1$. For $\alpha_1, \alpha_2, \dots, \alpha_n > 0$, we have the following:
$$\frac{(\alpha_1 + \cdots + \alpha_n)}{n^{1-\frac{1}{p}}} \leq (\alpha_1^p + \cdots + \alpha_n^p)^{\frac{1}{p}} \leq (\alpha_1 + \cdots + \alpha_n). \tag{15}$$
From Theorem 3.3, we know that
$$\|\cdot, \cdots, \cdot\|^G_{n\odot\ell^p} \sim \|\cdot, \cdots, \cdot\|_{n\ell^p}.$$
Take a fixed set of linearly independent vectors $\{a_1, a_2, \dots, a_n\} \subset \ell^p$, so for every $x \in \ell^p$, we have
$$\sum_{\{k_1,\dots,k_{(n-1)}\}\subset\{1,\dots,n\}} \|a_{k_1}, \dots, a_{k_{n-1}}, x\|^G_{n\odot\ell^p} \sim \sum_{\{k_1,\dots,k_{(n-1)}\}\subset\{1,\dots,n\}} \|a_{k_1}, \dots, a_{k_{n-1}}, x\|_{n\ell^p}$$
By the inequality in (15), we also have
$$\sum_{\{k_1,\dots,k_{(n-1)}\}\subset\{1,\dots,n\}} \|a_{k_1}, \dots, a_{k_{n-1}}, x\|_{n\ell^p} \sim \left(\sum_{\{k_1,\dots,k_{(n-1)}\}\subset\{1,\dots,n\}} \|a_{k_1}, \dots, a_{k_{n-1}}, x\|^p_{n\ell^p} \right)^{\frac{1}{p}}.$$
This shows that $\|\cdot\|^G_{\odot\ell^p} \sim \|\cdot\|^H_{\circ\ell^p}$.

We also have $\|\cdot, \dots, \cdot\|^I_{n\odot\ell^p} \sim \|\cdot, \dots, \cdot\|_{n\ell^p}$. By proceeding in a similar manner as above, we further obtain $\|\cdot\|^I_{\odot\ell^p} \sim \|\cdot\|^H_{\circ\ell^p}$. Since $\|\cdot\|_{\ell^p} \sim \|\cdot\|^H_{\circ\ell^p}$, then the norms $\|\cdot\|^G_{\odot\ell^p}$, $\|\cdot\|^I_{\odot\ell^p}$, and $\|\cdot\|_{\ell^p}$ are equivalent. □

We now have four definitions of norms based on the equations (1), (7), (13), and (14), all of which describe equivalent norms. As a result, the properties of the usual norm on ℓ^p are also valid for the space when equipped with any of the other three norms. As noted in [1], the ℓ^p with the usual norm is a Banach space, meaning that it is complete. Since the norms $\|\cdot\|_{\ell^p}$ and $\|\cdot\|_{(f_w)\ell^p}$ are equivalent, every Cauchy sequence in $(\ell^p, \|\cdot\|_{(f_w)\ell^p})$ is guaranteed to converge. Therefore, $(\ell^p, \|\cdot\|_{(f_w)\ell^p})$ is also a complete space. Based on the equivalence of norms and following a similar reasoning, it is clear that both $(\ell^p, \|\cdot\|^G_{\odot\ell^p})$ and $(\ell^p, \|\cdot\|^I_{\odot\ell^p})$ are complete spaces, or Banach spaces. The next theorem discusses the ℓ^p space equipped with the norm induced by an n-norm.

Corollary 3.6. *The spaces $(\ell^p, \|\cdot\|^G_{\odot\ell^p})$ and $(\ell^p, \|\cdot\|^I_{\odot\ell^p})$ are Banach spaces.*

Proof. Let $(x(m))$ be a sequence in ℓ^p that is Cauchy with respect to the norm $\|\cdot\|^G_{\odot\ell^p}$. According to Theorem 3.5, we have $\|\cdot\|^G_{\odot\ell^p} \sim \|\cdot\|_{\ell^p}$, meaning that $(x(m))$ is also a Cauchy sequence with respect to the norm $\|\cdot\|_{\ell^p}$. Since $(\ell^p, \|\cdot\|_{\ell^p})$ is a Banach space, the sequence $(x(m))$ converges with respect to the norm $\|\cdot\|_{\ell^p}$.

By Theorem 3.5 again, we conclude that $(x(m))$ also converges with respect to the norm $\|\cdot\|^G_{\odot\ell^p}$.

Similarly, using the fact that $\|\cdot\|^I_{\odot\ell^p} \sim \|\cdot\|_{\ell^p}$ as stated in Theorem 3.5, we can show that if $(x(m))$ is Cauchy with respect to the norm $\|\cdot\|^I_{\odot\ell^p}$, then it must also converge with respect to the norm $\|\cdot\|^I_{\odot\ell^p}$.

Therefore, both $(\ell^p, \|\cdot\|^G_{\odot\ell^p})$ and $(\ell^p, \|\cdot\|^I_{\odot\ell^p})$ are Banach spaces. $\square$

4. FURTHER RESULTS

From the above discussions and previous studies, it is clear that inducing a norm from the n-norm can be done effectively. Now, we also need to attempt to induce an n-norm from a norm on ℓ^p. Konca and Idris have defined the 2-norm from the norm (see [15, 16]). Furthermore, the above definition shows the induction of the n-norm from the $(n-1)$-norm. In the last section, we can define the n-norm recursively based on the norm.

Taking arbitrary $x_1, x_2, \cdots, x_n \in \ell^p$, we define 2-norm

$$\begin{aligned}
\|x_1, x_2\|^I_{2\bullet\ell^p} :=& \|x_1, x_2\|^I_{2\odot\ell^p} \\
&= \sup_{\substack{0<\|f_{w_2}\|_F\leq 1 \\ f_{w_2}(x_i)\neq 0 \\ i\in\{1,2\}}} |f_{w_2}(x_2)| \left\| x_1 - x_2 \frac{f_{w_2}(x_1)}{f_{w_2}(x_2)} \right\|_{\ell^p} \\
&= \sup_{\substack{0<\|f_{w_2}\|_F\leq 1 \\ f_{w_2}(x_i)\neq 0 \\ i\in\{1,2\}}} \left(\sum_{k_1=1}^{\infty} \left| \left(\sum_{k_2=1}^{\infty} w_{2k_2} \begin{vmatrix} x_{1k_1} & x_{2k_1} \\ x_{1k_2} & x_{2k_2} \end{vmatrix} \right) \right|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Next, we define 3-norm

$$\|x_1, x_2, x_3\|^I_{3\bullet\ell^p} := \sup_{\substack{0<\|f_{w_3}\|_F\leq 1 \\ f_{w_3}(x_i)\neq 0 \\ i\in\{1,2,3\}}} |f_{w_3}(x_3)| \left\| x_1 - x_3 \frac{f_{w_3}(x_1)}{f_{w_3}(x_3)}, x_2 - x_3 \frac{f_{w_3}(x_2)}{f_{w_3}(x_3)} \right\|^I_{2\bullet\ell^p}$$

Using Laplace expansion, verify that

$$\begin{aligned}
&|f_{w_3}(x_3)| \|x_1 - x_3 \frac{f_{w_3}(x_1)}{f_{w_3}(x_3)}, x_2 - x_3 \frac{f_{w_3}(x_2)}{f_{w_3}(x_3)}\|^I_{2\bullet\ell^p} \\
&= |f_{w_3}(x_3)|
\end{aligned}$$

$$\times \sup_{\substack{0<\left\|f_{w_2}\right\|_F\leq 1\\ ,f_{w_2}(x_i)\neq 0\\ i\in\{1,2\}}} \left(\sum_{k_1=1}^{\infty}\left|\left(\sum_{k_2=1}^{\infty} w_{2k_2}\begin{vmatrix} x_{1k_1}-x_{3k_1}\frac{f_{w_3}(x_1)}{f_{w_3}(x_3)} & x_{2k_1}-x_{3k_1}\frac{f_{w_3}(x_2)}{f_{w_3}(x_3)} \\ x_{1k_2}-x_{3k_2}\frac{f_{w_3}(x_1)}{f_{w_3}(x_3)} & x_{2k_2}-x_{3k_2}\frac{f_{w_3}(x_2)}{f_{w_3}(x_3)}\end{vmatrix}\right)\right|^p\right)^{\frac{1}{p}}$$

$$= \sup_{\substack{0<\left\|f_{w_2}\right\|_F\leq 1\\ ,f_{w_2}(x_i)\neq 0\\ i\in\{1,2\}}} \left(\sum_{k_1=1}^{\infty}\left|\left(\sum_{k_2=1}^{\infty} w_{2k_2}\begin{vmatrix} x_{1k_1} & x_{2k_1} & x_{3k_1} \\ x_{1k_2} & x_{2k_2} & x_{3k_2} \\ f_{w_3}(x_1) & f_{w_3}(x_2) & f_{w_3}(x_3)\end{vmatrix}\right)\right|^p\right)^{\frac{1}{p}}$$

$$= \sup_{\substack{0<\left\|f_{w_2}\right\|_F\leq 1\\ ,f_{w_2}(x_i)\neq 0\\ i\in\{1,2\}}} \left(\sum_{k_1=1}^{\infty}\left|\left(\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty} w_{2k_2}w_{3k_3}\begin{vmatrix} x_{1k_1} & x_{2k_1} & x_{3k_1} \\ x_{1k_2} & x_{2k_2} & x_{3k_2} \\ x_{1k_3} & x_{2k_3} & x_{3k_3}\end{vmatrix}\right)\right|^p\right)^{\frac{1}{p}}.$$

Now, we have

$$\|x_1,x_2,x_3\|^I_{3\bullet\ell^p} = \sup_{\substack{0<\left\|f_{w_j}\right\|_F\leq 1\\ ,f_{w_j}(x_i)\neq 0\\ i\in\{1,2,3\}\\ j\in\{2,3\}}} \left(\sum_{k_1=1}^{\infty}\left|\left(\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty} w_{2k_2}w_{3k_3}\begin{vmatrix} x_{1k_1} & x_{2k_1} & x_{3k_1} \\ x_{1k_2} & x_{2k_2} & x_{3k_2} \\ x_{1k_3} & x_{2k_3} & x_{3k_3}\end{vmatrix}\right)\right|^p\right)^{\frac{1}{p}}.$$

Repeat by defining the 4-norm up to the $(n-1)$-norm and the last, we get the definition of the n-norm

$$\|x_k|_{k=1}^n\|^I_{n\bullet\ell^p} := \sup_{\substack{0<\|f_{w_n}\|_F\leq 1\\ f_{w_n}(x_i)\neq 0\\ i\in\{1,2,\cdots,n\}}} |f_{w_n}(x_n)|\|x_k - x_n\frac{f_{w_n}(x_k)}{f_{w_n}(x_n)}|_{k=1}^{(n-1)}\|^I_{(n-1)\bullet\ell^p}$$

$$= \sup_{\substack{0<\left\|f_{w_j}\right\|_F\leq 1\\ f_{w_j}(x_i)\neq 0\\ i\in\{1,\cdots,n\}\\ j\in\{2,\cdots,n\}}} \left(\sum_{k_1=1}^{\infty}\left|\left(\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} w_{2k_2}\cdots w_{nk_n}\begin{vmatrix} x_{1k_1} & x_{2k_1} & \cdots & x_{nk_1} \\ x_{1k_2} & x_{2k_2} & \cdots & x_{nk_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k_n} & x_{2k_n} & \cdots & x_{nk_n}\end{vmatrix}\right)\right|^p\right)^{\frac{1}{p}}.$$

Interestingly, we also examine the relationship between $\|\cdot,\ldots,\cdot\|^I_{n\bullet\ell^p}$ and other n-norms mentioned above.

Proposition 4.1. *Let $p\geq 1$. We have an inequality*

$$\|x_k|_{k=1}^n\|^G_{n\odot\ell^p} \leq \|x_k|_{k=1}^n\|^I_{n\bullet\ell^p} \leq A_1\|x_k|_{k=1}^n\|^I_{n\odot\ell^p} \leq A_2\|x_k|_{k=1}^n\|_{n\ell^p},$$

for every $x_1,x_2,\cdots,x_n\in\ell^p$ with $A_1=(n-1)!)^{\frac{1}{p}}$ and $A_2=(n!)^{\frac{1}{p}}$.

Proof. Recall (8). For arbitrary $x_1, x_2, \cdots, x_n \in \ell^p$, we have

$$\begin{aligned}
&\|x_k|_{k=1}^n\|_{n\odot\ell^p}^G \\
&= \sup_{\substack{0<\|f_{w_j}\|_F\leq 1 \\ ,f_{w_j}(x_i)\neq 0 \\ i,j\in\{1,2,\cdots,n\}}} \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} w_{1k_1}w_{2k_2}\cdots w_{nk_n}\begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n}\end{vmatrix} \\
&= \sup_{\substack{0<\|f_{w_j}\|_F\leq 1 \\ ,f_{w_j}(x_i)\neq 0 \\ i,j\in\{1,2,\cdots,n\}}} \sum_{k_1=1}^{\infty} w_{1k_1}E,
\end{aligned}$$

where $E = \sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} w_{2k_2}\cdots w_{nk_n}\begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n}\end{vmatrix}$. By Hölder's inequality, we obtain

$$\begin{aligned}
&\|x_k|_{k=1}^n\|_{n\odot\ell^p}^G \\
&\leq \sup_{\substack{0<\|f_{w_j}\|_F\leq 1 \\ ,f_{w_j}(x_i)\neq 0 \\ i,j\in\{1,2,\cdots,n\}}} \left(\sum_{k_1=1}^{\infty}|E|^p\right)^{\frac{1}{p}}\left(\sum_{k_1=1}^{\infty}|w_{1k_1}|^q\right)^{\frac{1}{q}} \\
&\leq \sup_{\substack{0<\|f_{w_j}\|_F\leq 1 \\ f_{w_j}(x_i)\neq 0 \\ i\in\{1,\cdots,n\} \\ j\in\{2,\cdots,n\}}} \left(\sum_{k_1=1}^{\infty}\left|\left(\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} w_{2k_2}\cdots w_{nk_n}\begin{vmatrix} x_{1k_1} & x_{2k_1} & \cdots & x_{nk_1} \\ x_{1k_2} & x_{2k_2} & \cdots & x_{nk_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k_n} & x_{2k_n} & \cdots & x_{nk_n}\end{vmatrix}\right)\right|^p\right)^{\frac{1}{p}} \\
&= \|x_k|_{k=1}^n\|_{n\bullet\ell^p}^I.
\end{aligned}$$

Observe also that, by the triangle inequality and Hölder's inequality,

$$\begin{aligned}
|E| &\leq \sum_{k_2=1}^{\infty}\cdots\sum_{k_{(n-1)}=1}^{\infty}|w_{2k_2}|\cdots|w_{(n-1)k_{(n-1)}}||D| \\
&\leq \Big(\sum_{k_2=1}^{\infty}\cdots\sum_{k_{(n-1)}=1}^{\infty}\Big(|w_{2k_2}|\cdots|w_{(n-1)k_{(n-1)}}|\Big)^q\Big)^{\frac{1}{q}}\Big(\sum_{k_2=1}^{\infty}\cdots\sum_{k_{(n-1)}=1}^{\infty}|D|^p\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{k_2=1}^{\infty}\cdots\sum_{k_{(n-1)}=1}^{\infty}|D|^p\Big)^{\frac{1}{p}}
\end{aligned}$$

holds, where $D = \sum_{k_n=1}^{\infty} w_{nk_n} \begin{vmatrix} x_{1k_1} & x_{2k_1} & \cdots & x_{nk_1} \\ x_{1k_2} & x_{2k_2} & \cdots & x_{nk_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k_n} & x_{2k_n} & \cdots & x_{nk_n} \end{vmatrix}$. As a result,

$$\|x_k|_{k=1}^n\|_{n\bullet\ell^p}^I = \sup_{\substack{0<\|f_{w_j}\|_F\leq 1 \\ f_{w_j}(x_i)\neq 0 \\ i\in\{1,\cdots,n\} \\ j\in\{2,\cdots,n\}}} \left(\sum_{k_1=1}^{\infty} |E|^p\right)^{\frac{1}{p}} \leq \sup_{\substack{0<\|f_{w_j}\|_F\leq 1 \\ f_{w_j}(x_i)\neq 0 \\ i\in\{1,\cdots,n\} \\ j\in\{2,\cdots,n\}}} \left(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_{(n-1)}=1}^{\infty} |D|^p\right)^{\frac{1}{p}}.$$

Moreover, it can be observed that

$$\begin{aligned} &\sup_{\substack{0<\|f_{w_j}\|_F\leq 1 \\ f_{w_j}(x_i)\neq 0 \\ i\in\{1,\cdots,n\} \\ j\in\{2,\cdots,n\}}} \left(\sum_{k_1=1}^{\infty}\cdots\sum_{k_{(n-1)}=1}^{\infty} |D|^p\right)^{\frac{1}{p}} \\ &= \sup_{\substack{0<\|f_{w_j}\|_F\leq 1 \\ f_{w_j}(x_i)\neq 0 \\ i\in\{1,\cdots,n\} \\ j\in\{2,\cdots,n\}}} \left(\frac{\sum_{k_1=1}^{\infty}\cdots\sum_{k_{(n-1)}=1}^{\infty}\left|\left(\sum_{k_n=1}^{\infty} w_{nk_n}\begin{vmatrix} x_{1k_1} & \cdots & x_{nk_1} \\ \vdots & \ddots & \vdots \\ x_{1k_n} & \cdots & x_{nk_n}\end{vmatrix}\right)\right|^p}{(n-1)!}\right)^{\frac{1}{p}} \\ &\quad \times ((n-1)!)^{\frac{1}{p}} \\ &= ((n-1)!)^{\frac{1}{p}} \|x_k|_{k=1}^n\|_{n\odot\ell^p}^I \leq (n!)^{\frac{1}{p}} \|x_k|_{k=1}^n\|_{n\ell^p}. \end{aligned}$$

So, we have

$$\|x_k|_{k=1}^n\|_{n\bullet\ell^p}^I \leq ((n-1)!)^{\frac{1}{p}} \|x_k|_{k=1}^n\|_{n\odot\ell^p}^I \leq (n!)^{\frac{1}{p}} \|x_k|_{k=1}^n\|_{n\ell^p}.$$

Accordingly, the four definitions of n-norms give rise to an inequality

$$\|x_k|_{k=1}^n\|_{n\odot\ell^p}^G \leq \|x_k|_{k=1}^n\|_{n\bullet\ell^p}^I \leq ((n-1)!)^{\frac{1}{p}} \|x_k|_{k=1}^n\|_{n\odot\ell^p}^I \leq (n!)^{\frac{1}{p}} \|x_k|_{k=1}^n\|_{n\ell^p}.$$

The proof is complete. □

With the results from Theorem 3.3 and Proposition 4.1, on ℓ^p, we get

$$\|\cdot,\cdots,\cdot\|_{n\odot\ell^p}^G \sim \|\cdot,\cdots,\cdot\|_{n\bullet\ell^p}^I \sim \|\cdot,\cdots,\cdot\|_{n\odot\ell^p}^I \sim \|\cdot,\cdots,\cdot\|_{n\ell^p}. \tag{16}$$

Using fixed linear independent vectors $a_1, a_2, \cdots, a_n \in \ell^p$, we also define

$$\|x\|_{\bullet\ell^p}^I := \sum_{\{k_1,k_2,\cdots,k_{(n-1)}\}\subset\{1,2,\cdots,n\}} \|a_{k_1}, a_{k_2}, \cdots, a_{k_{(n-1)}}, x\|_{n\bullet\ell^p}^I. \tag{17}$$

for every $x \in \ell^p$. Since (16) holds, $\|\cdot\|^I_{\bullet\ell^p} \sim \|\cdot\|_{\ell^p}$.

Corollary 4.2. *The spaces* $(\ell^p, \|\cdot, \cdots, \cdot\|^I_{n\bullet\ell^p})$ *and* $(\ell^p, \|\cdot\|^I_{n\bullet\ell^p})$ *are an* n*-Banach space and a Banach space respectively.*

Proof. Use the method similar to the proof of Corollary 3.4 and the proof of Corollary 3.6. □

5. CONCLUDING REMARKS

We have defined several n-norms and norms on ℓ^p and demonstrated their interrelationships. This investigation could similarly be extended to $X = \mathbb{L}^p(\mathbb{R}^n)$, or more generally to X as a normed space with an appropriate set of bounded linear functionals. In such cases, new challenges may arise, potentially leading to different results.

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