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# COMPLEMENTARY DISTANCE SPECTRA AND COMPLEMENTARY DISTANCE ENERGY OF LINE GRAPHS OF REGULAR GRAPHS

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**Abstract.** The complementary distance (CD) matrix of a graph $G$ is defined as $CD(G) = [c_{ij}]$, where $c_{ij} = 1 + D - d_{ij}$ if $i \neq j$ and $c_{ij} = 0$, otherwise, where $D$ is the diameter of $G$ and $d_{ij}$ is the distance between the vertices $v_i$ and $v_j$ in $G$. The $CD$-energy of $G$ is defined as the sum of the absolute values of the eigenvalues of $CD$-matrix. Two graphs are said to be $CD$-equienergetic if they have same $CD$-energy. In this paper we show that the complement of the line graph of certain regular graphs has exactly one positive $CD$-eigenvalue. Further we obtain the $CD$-energy of line graphs of certain regular graphs and thus constructs pairs of $CD$-equienergetic graphs of same order and having different $CD$-eigenvalues.

*Key words and Phrases*: Complementary distance eigenvalues, adjacency eigenvalues, line graphs, complementary distance energy.

**Abstrak.** Matriks complementary distance (CD) dari sebuah graph $G$ didefinisikan sebagai $CD(G) = [c_{ij}]$, dimana $c_{ij} = 1 + D - d_{ij}$ jika $i \neq j$ dan $c_{ij} = 0$, atau yang lain, dimana $D$ adalah diameter $G$ dan $d_{ij}$ adalah jarak antara titik-titik $v_i$ dan $v_j$ di $G$. Energi-$CD$ dari $G$ didefinisikan sebagai jumlahan dari nilai mutlak nilai-nilai eigen matriks-$CD$. Dua graf disebut ekuienergetik-$CD$ jika mereka mempunyai energi-$CD$ yang sama. Dalam paper ini kami menunjukkan komplemen graf garis dari graf-graf regular tertentu mempunyai tepat satu nilai eigen-$CD$ positif. Lebih jauh, kami mendapatkan energi-$CD$ graf garis dari graf-graf regular tertentu dan selanjutnya mengkonstruksi pasangan graf-graf ekuienergetik-$CD$-equienergetic berorde sama dan mempunyai nilai-nilai eigen-$CD$ berbeda.

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*Kata kunci*: Nilai-nilai eigen complementary distance, Nilai-nilai eigen ketetanggaan, graf-graf garis, energi complementary distance.

## 1. Introduction

Let $G$ be a simple, undirected, connected graph with $n$ vertices and $m$ edges. Let the vertex set of $G$ be $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of a graph $G$ is the square matrix $A = A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$ and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ are the *adjacency eigenvalues* of $G$, and they are labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These form the *adjacency spectrum* of $G$ [4].

The *distance* between the vertices $v_i$ and $v_j$, denoted by $d_{ij}$, is the length of the shortest path joining $v_i$ and $v_j$. The *diameter* of a graph $G$, denoted by $diam(G)$, is the maximum distance between any pair of vertices of $G$ [3]. A graph $G$ is said to be *$r$-regular graph* if all of its vertices have same degree equal to $r$.

The *complementary distance* between the vertices $v_i$ and $v_j$, denoted by $c_{ij}$ is defined as $c_{ij} = 1 + D - d_{ij}$, where $D$ is the diameter of $G$ and $d_{ij}$ is the distance between $v_i$ and $v_j$ in $G$.

The *complementary distance matrix* or *$CD$-matrix* [7] of a graph $G$ is an $n \times n$ matrix $CD(G) = [c_{ij}]$, where

$$c_{ij} = \begin{cases} 1 + D - d_{ij}, & \text{if} \quad i \neq j \\ 0, & \text{if} \quad i = j. \end{cases}$$

The complementary distance matrix is an important source of structural descriptors in the quantitative structure property relationship (QSPR) model in chemistry [7, 8].

The eigenvalues of $CD(G)$ labeled as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are said to be the *complementary distance eigenvalues* or *$CD$-eigenvalues* of $G$ and their collection is called *$CD$-spectra* of $G$. Two non-isomorphic graphs are said to be *$CD$-cospectral* if they have same $CD$-spectra.

The *complementary distance energy* or *$CD$-energy* of a graph $G$ denoted by $CDE(G)$ is defined as

$$CDE(G) = \sum_{i=1}^{n} |\mu_i| \, . \tag{1}$$

The Eq. (1) is defined in full analogy with the *ordinary graph energy* $E(G)$, defined as [5]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ . \tag{2}$$

Two graphs $G_1$ and $G_2$ are said to be *equienergetic* if $E(G_1) = E(G_2)$. Results on non cospectral equienergetic graphs can be found in [1, 2, 12, 13, 17]. For more details about ordinary graph energy one can refer [9].

Two connected graphs $G_1$ and $G_2$ are said to be *complementary distance equienergetic* or *$CD$-equienergetic* if $CDE(G_1) = CDE(G_2)$. Trivially, the $CD$-cospectral graphs are $CD$-equienergetic. In this paper we obtain the $CD$-energy of line graphs of certain regular graphs and thus construct $CD$-equienergetic graphs having different $CD$-spectra.

We need following results.

**Theorem 1.1.** [4] *If $G$ is an $r$-regular graph, then its maximum adjacency eigenvalue is equal to $r$.*

The *line graph* of $G$, denoted by $L(G)$ is the graph whose vertices corresponds to the edges of $G$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in $G$ [6]. If $G$ is a regular graph of order $n$ and of degree $r$ then the line graph $L(G)$ is a regular graph of order $nr/2$ and of degree $2r-2$.

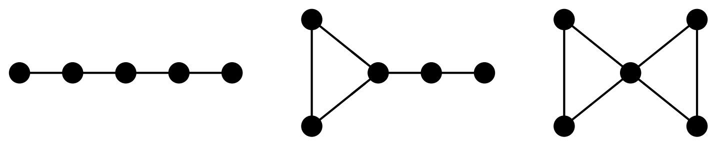


Figure 1: The forbidden induced subgraphs

**Theorem 1.2.** [10, 11] *For a connected graph $G$, $diam(L(G)) \leq 2$ if and only if none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$.*

**Theorem 1.3.** [15] *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph $G$ of order $n$ and of degree $r$, then the adjacency eigenvalues of $L(G)$ are*

$$\lambda_i + r - 2, \qquad i = 1, 2, \ldots, n, \qquad \textit{and}$$

$$-2, \qquad n(r-2)/2 \ \textit{times} \ .$$

**Theorem 1.4.** [14] *Let $G$ be an $r$-regular graph of order $n$. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of $G$, then the adjacency eigenvalues of $\overline{G}$, the complement of $G$, are $n-r-1$ and $-\lambda_i - 1$, $i = 2, 3, \ldots, n$.*

**Lemma 1.5.** [16] *If for any two adjacent vertices $u$ and $v$ of a graph $G$, there exists a third vertex $w$ which is not adjacent to either $u$ or $v$, then*
*(i) $\overline{G}$ is connected and*
*(ii) diam $(\overline{G}) = 2$.*

## 2. $CD$-eigenvalues

**Theorem 2.1.** *Let $G$ be an $r$-regular graph on $n$ vertices and $diam(G) = 2$. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of $G$, then $CD$-eigenvalues of $G$ are $n + r - 1$ and $\lambda_i - 1$, $i = 2, 3, \ldots, n$.*

*Proof.* Since $G$ is an $r$-regular graph, $\mathbf{1} = [1, 1, \ldots, 1]'$ is an eigenvector of $A = A(G)$ corresponding to the eigenvalue $r$. Set $\mathbf{z} = \frac{1}{\sqrt{n}}\mathbf{1}$ and let $P$ be an orthogonal matrix with its first column equal to $\mathbf{z}$ such that $P'AP = \text{diag}(r, \lambda_2, \ldots, \lambda_n)$. Since $diam(G) = 2$, the $CD$-matrix $CD(G)$ can be written as $CD(G) = J + A - I$, where $J$ is the matrix whose all entries are equal to 1 and $I$ is an identity matrix. Therefore

$$\begin{aligned} P'(CD)P &= P'(J + A - I)P \\ &= P'JP + P'AP - I \\ &= \text{diag}(n + r - 1, \lambda_2 - 1, \ldots, \lambda_n - 1), \end{aligned}$$

where we have used the fact that any column of $P$ other than the first column is orthogonal to the first column. Hence the eigenvalues of $CD(G)$ are $n + r - 1$ and $\lambda_i - 1$, $i = 2, 3, \ldots, n$. □

**Theorem 2.2.** *Let $G$ be an $r$-regular graph of order $n$. Let $L(G)$ be the line graph of $G$ such that for any two adjacent vertices $u$ and $v$ of $L(G)$, there exists a third vertex $w$ in $L(G)$ which is not adjacent to either $u$ or $v$. Then $\overline{L(G)}$, the complement of $L(G)$, has exactly one positive $CD$-eigenvalue, equal to $r(n - 2)$.*

*Proof.* Let the adjacency eigenvalues of $G$ be $r, \lambda_2, \ldots, \lambda_n$. From Theorem 1.3 , the adjacency eigenvalues of $L(G)$ are

$$\left.\begin{array}{rl} 2r - 2, & \text{and} \\ \lambda_i + r - 2, & i = 2, 3, \ldots, n, \quad \text{and} \\ -2, & n(r - 2)/2 \ \text{times.} \end{array}\right\} \tag{3}$$

From Theorem 1.4 and the Eq. (3), the adjacency eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{rl} (nr/2) - 2r + 1, & \text{and} \\ -\lambda_i - r + 1, & i = 2, 3, \ldots, n, \quad \text{and} \\ 1, & n(r - 2)/2 \ \text{times.} \end{array}\right\} \tag{4}$$

The graph $\overline{L(G)}$ is a regular graph of order $nr/2$ and of degree $(nr/2)-2r+1$. Since for any two adjacent vertice $u$ and $v$ of $L(G)$ there exists a third vertex $w$ which is not adjacent to either $u$ or $v$ in $L(G)$, by Lemma 1.5, $diam\left(\overline{L(G)}\right) = 2$. Therefore by Theorem 2.1 and Eq. (4), the $CD$-eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{rll} nr-2r, & \text{and} & \\ -\lambda_i - r, & i=2,3,\ldots,n, & \text{and} \\ 0, & n(r-2)/2 \ \text{ times.} & \end{array}\right\} \tag{5}$$

All adjacency eigenvalues of a regular graph of degree $r$ satisfy the condition $-r \leq \lambda_i \leq r$ [4]. Therefore $\lambda_i + r \geq 0$ , $i = 1,2,\ldots,n$ . The theorem follows from Eq. (5). □

## 3. $CD$-energy

**Theorem 3.1.** *Let $G$ be an $r$-regular graph of order $n$. Let $L(G)$ be the line graph of $G$ such that for any two adjacent vertices $u$ and $v$ of $L(G)$, there exists a third vertex $w$ in $L(G)$ which is not adjacent to either $u$ or $v$. Then $CDE\left(\overline{L(G)}\right) = 2r(n-2)$.*

*Proof.* Bearing in mind Theorem 2.2 and Eq. (5), the $CD$-energy of $\overline{L(G)}$ is computed as:

$$\begin{aligned} CDE\left(\overline{L(G)}\right) &= nr - 2r + \sum_{i=2}^{n}(\lambda_i + r) + |0| \times \frac{n(r-2)}{2} \\ &= 2r(n-2) \qquad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r. \end{aligned}$$

□

**Theorem 3.2.** *Let $G$ be a connected, $r$-regular graph with $n > 3$ vertices and let none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$.*
*(i) If the smallest adjacency eigenvalue of $G$ is greater than or equal to $3-r$, then $CDE(L(G)) = 3n(r-2)$.*
*(ii) If the second largest adjacency eigenvalue of $G$ is smaller than $3-r$, then $CDE(L(G)) = nr + 4r - 6$.*

*Proof.* Let $r, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the adjacency eigenvalues of a regular graph $G$ . Then from Theorem 1.3, the adjacency eigenvalues of $L(G)$ are

$$\left.\begin{array}{rll} 2r-2 & \text{and} & \\ \lambda_i + r - 2, & i=1,2,\ldots,n, & \text{and} \\ -2, & n(r-2)/2 \ \text{ times.} & \end{array}\right\} \tag{6}$$

The graph $G$ is regular of degree $r$ and has order $n$. Therefore $L(G)$ is a regular graph on $nr/2$ vertices and of degree $2r-2$. As none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$, from Theorem 1.2, $diam(L(G)) = 2$. Therefore from Theorem 2.1 and Eq. (6), the $CD$-eigenvalues of $L(G)$ are

$$\left.\begin{array}{rll} (nr+4r-6)/2, & \text{and} & \\ \lambda_i + r - 3, & i = 2,3,\ldots,n & \text{and} \\ -3, & n(r-2)/2 \ \text{ times.} & \end{array}\right\} \qquad (7)$$

Therefore

$$CDE(L(G)) = \left|\frac{nr+4r-6}{2}\right| + \sum_{i=2}^{n} |\lambda_i + r - 3| + |-3|\frac{n(r-2)}{2}. \qquad (8)$$

(i) By assumption, $\lambda_i + r - 3 \geq 0$, $i = 2,3,\ldots n$, then from Eq. (8)

$$\begin{aligned} CDE(L(G)) &= \frac{nr+4r-6}{2} + \sum_{i=2}^{n}(\lambda_i + r - 3) + \frac{3n(r-2)}{2} \\ &= \frac{nr+4r-6}{2} + \sum_{i=2}^{n}\lambda_i + (n-1)(r-3) + \frac{3n(r-2)}{2} \\ &= 3n(r-2) \qquad \text{since} \quad \sum_{i=2}^{n}\lambda_i = -r. \end{aligned}$$

(ii) By assumption, $\lambda_i + r - 3 < 0$, $i = 2,3,\ldots n$, then from Eq. (8)

$$\begin{aligned} CDE(L(G)) &= \frac{nr+4r-6}{2} - \sum_{i=2}^{n}(\lambda_i + r - 3) + \frac{3n(r-2)}{2} \\ &= \frac{nr+4r-6}{2} - \sum_{i=2}^{n}\lambda_i - (n-1)(r-3) + \frac{3n(r-2)}{2} \\ &= nr + 4r - 6 \qquad \text{since} \quad \sum_{i=2}^{n}\lambda_i = -r. \end{aligned}$$

□

**Corollary 3.3.** *Let $G$ be a connected, cubic graph with $n$ vertices and let none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$. Then $CDE(L(G)) = 3n + E(G)$.*

*Proof.* Substituting $r = 3$ in Eq. (8) we get

$$\begin{aligned} CD(L(G)) &= \left|\frac{3n+6}{2}\right| + \sum_{i=2}^{n} |\lambda_i| + |-3|\frac{n}{2} \\ &= \frac{3n+6}{2} + (E(G)-3) + \frac{3n}{2} \\ &= 3n + E(G). \end{aligned}$$

□

## 4. $CD$-equienergetic graphs

**Lemma 4.1.** *Let $G_1$ and $G_2$ be regular graphs of the same order and of the same degree. Then following holds:*
*(i) $L(G_1)$ and $L(G_2)$ are of the same order, same degree and have the same number of edges.*
*(ii) $\overline{L(G_1)}$ and $\overline{L(G_2)}$ are of the same order, same degree and have the same number of edges.*

*Proof.* Statement (i) follows from the fact that the line graph of a regular graph is a regular and that the number of edges of $G$ is equal to the number of vertices of $L(G)$. Statement (ii) follows from the fact that the complement of a regular graph is a regular and that the number of vertices of a graph and its complement is equal. □

**Lemma 4.2.** *Let $G_1$ and $G_2$ be regular graphs of the same order and of the same degree. Let for $i = 1, 2$, $L(G_i)$ be the line graph of $G_i$ such that for any two adjacent vertices $u_i$ and $v_i$ of $L(G_i)$, there exists a third vertex $w_i$ in $L(G_i)$ which is not adjacent to either $u_i$ or $v_i$. Then $\overline{L(G_1)}$ and $\overline{L(G_2)}$ are $CD$-cospectral if and only if $G_1$ and $G_2$ are cospectral.*

*Proof.* Follows from Eqs. (3), (4) and (5). □

**Lemma 4.3.** *Let $G_1$ and $G_2$ be connected, regular graphs of the same order $n > 3$ and of the same degree. Let none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 be an induced subgraph of $G_i$, $i = 1, 2$. Then $L(G_1)$ and $L(G_2)$ are $CD$-cospectral if and only if $G_1$ and $G_2$ are cospectral.*

*Proof.* Follows from Eqs. (6) and (7). □

**Theorem 4.4.** *Let $G_1$ and $G_2$ be regular, non $CD$-cospectral graphs of the same order and of the same degree. Let for $i = 1, 2$, $L(G_i)$ be the line graph of $G_i$ such that for any two adjacent vertices $u_i$ and $v_i$ of $L(G_i)$, there exists a third vertex $w_i$ in $L(G_i)$ which is not adjacent to either $u_i$ or $v_i$. Then $\overline{L(G_1)}$ and $\overline{L(G_2)}$ form a pair of non $CD$-cospectral, $CD$-equienergetic graphs of equal order and of equal number of edges.*

*Proof.* Follows from Lemma 4.1, Lemma 4.2 and Theorem 3.1. □

**Theorem 4.5.** *Let $G_1$ and $G_2$ be connected, regular, non $CD$-cospectral graphs of the same order $n > 3$ and of the same degree $r$. Let none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 be an induced subgraph of $G_i$ , $i = 1, 2$.*
*(i) If the smallest adjacency eigenvalue of $G_i$, $i = 1, 2$ is greater than or equal to $3 - r$, then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non $CD$-cospectral, $CD$-equienergetic graphs of equal order and of equal number of edges.*
*(ii) If the second largest adjacency eigenvalue of $G_i$, $i = 1, 2$ is smaller than $3 - r$, then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non $CD$-cospectral, $CD$-equienergetic graphs of equal order and of equal number of edges.*

*Proof.* Follows from Lemma 4.1, Lemma 4.3 and Theorem 3.2. □

**Theorem 4.6.** *Let $G_1$ and $G_2$ be connected, non $CD$-cospectral, cubic, equienergetic graphs of the same order. Let none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 be an induced subgraph of $G_i$ , $i = 1, 2$. Then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non $CD$-cospectral, $CD$-equienergetic graphs of equal order and of equal number of edges.*

*Proof.* Follows from Lemma 4.1, Lemma 4.3 and Corollary 3.3. □

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