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$\mathcal{W}_1$-Curvature Tensor within the framework of Lorentzian α-Sasakian Manifold

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Abstract. The objective of this paper is to investigate the curvature properties of Lorentzian α-Sasakian manifolds under specific geometric conditions. In particular, we examine these manifolds when they satisfy the following conditions: ζ-$\mathcal{W}_1$-flatness, φ-$\mathcal{W}_1$-semi-symmetry, and the vanishing of certain curvature operators, specifically $\mathcal{W}_1 \cdot \mathcal{Q} = 0$ and $\mathcal{W}_1 \cdot R = 0$. Through our analysis, we derive several interesting results regarding the geometric structure and behavior of these manifolds under the given conditions.

Key words and Phrases: Almost contact manifolds, trans-Sasakian manifolds, Lorentzian α-Sasakian manifolds, φ-symmetric, φ-semisymmetric.

1. INTRODUCTION

S. Tanno classified linked almost contact metric manifolds with the largest dimension automorphism groups in [1]. For such a manifold, the sectional curvature of plane sections containing the vector field ζ is constant, denoted by c. He identified three distinct classes for these manifolds:

(i) Homogeneous normal contact Riemannian manifolds when $c > 0$
(ii) Global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature when $c = 0$,

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(iii) Warped product spaces when $c < 0$.

It is well established that manifolds in class (i) are characterized by their ability to admit a Sasakian structure. In the Gray-Hervella classification of almost Hermitian manifolds [2], the class W_4 is closely associated with locally conformal Kähler manifolds [3]. When the product manifold $M \times R$ belongs to the W_4 class, the almost contact metric structure on the manifold M is referred to as a trans-Sasakian structure [4, 5]. The class $C_6 \oplus C_5$ [6] corresponds to trans-Sasakian structures of type (α, β). Moreover, the local nature of the two subclasses of trans-Sasakian structures, namely C_5 and C_6, is well defined in [6].

It is noted that trans-Sasakian structures of types $(0, 0)$, $(0, \beta)$, and $(\alpha, 0)$ correspond to cosymplectic [5], β-Kenmotsu [7], and α-Sasakian [7] structures, respectively. As shown in [8], trans-Sasakian structures are generalized quasi-Sasakian structures [7]. Consequently, trans-Sasakian structures encompass a wide range of generalized quasi-Sasakian structures. Following this, Yildiz and Murathan introduced Lorentzian α-Sasakian manifolds in [9].

An almost contact metric structure $(\varphi, \zeta, \eta, g)$ on a manifold M is referred to as a trans-Sasakian structure [4] if the product manifold $(M \times \mathbb{R}, \mathcal{J}, \mathcal{G})$ belongs to the class W_4 [2]. Here, $\mathcal{J}$ denotes the almost complex structure on $M \times \mathbb{R}$, defined as follows

$$\mathcal{J}(\mathcal{F}, f\frac{d}{dt}) = (\varphi\mathcal{F} - f\zeta, \eta(\mathcal{F})\frac{d}{dt}),$$

for all vector field $\mathcal{F}$ on M and any smooth function f on $M \times \mathbb{R}$, with $\mathcal{G}$ being the product metric on $M \times \mathbb{R}$, this can be described by the following condition [10]

$$(\nabla_{\mathcal{F}}\varphi)\mathcal{G} = \alpha(g(\mathcal{F}, \mathcal{G})\zeta - \eta(\mathcal{G})\mathcal{F}) + \beta(g(\varphi\mathcal{F}, \mathcal{G})\zeta - \eta(\mathcal{G})\varphi\mathcal{F}), \tag{1}$$

for some smooth functions α and β on M, the trans-Sasakian structure is said to be of type (α, β).

From equation (1), it follows that

$$\nabla_{\mathcal{F}}\zeta = -\alpha\varphi\mathcal{F} + \beta(\mathcal{F} - \eta(\mathcal{F})\zeta), \tag{2}$$

$$(\nabla_{\mathcal{F}}\eta)(\mathcal{G}) = -\alpha g(\varphi\mathcal{F}, \mathcal{G}) + \beta g(\varphi\mathcal{F}, \varphi\mathcal{G}). \tag{3}$$

More generally, the concept of an α-Sasakian structure [7] can be defined as follows

$$(\nabla_{\mathcal{F}}\varphi)\mathcal{G} = \alpha(g(\mathcal{F}, \mathcal{G})\zeta - \eta(\mathcal{G})\mathcal{F}), \tag{4}$$

where α is a non-zero constant. From this condition, it can be easily deduced that

$$\nabla_{\mathcal{F}}\zeta = -\alpha\varphi\mathcal{F}, \tag{5}$$

$$(\nabla_{\mathcal{F}}\eta)(\mathcal{G}) = -\alpha g(\varphi\mathcal{F}, \mathcal{G}). \tag{6}$$

Thus, $\beta = 0$, meaning a trans-Sasakian structure of type (α, β) with α as a non-zero constant is always an α-Sasakian structure [7, 11]. When $\alpha = 1$, an α-Sasakian manifold becomes a Sasakian manifold. Marrero [12] explored the relationship between trans-Sasakian, α-Sasakian, and β-Kenmotsu structures.

$\mathcal{W}_1$-curvature tensor [13] is defined as

$$\mathcal{W}_1(\mathcal{F},\mathcal{G})\mathcal{H} = R(\mathcal{F},\mathcal{G})\mathcal{H} + \frac{1}{n-1}[S(\mathcal{G},\mathcal{H})\mathcal{F} - S(\mathcal{F},\mathcal{H})\mathcal{G}], \tag{7}$$

where R and S represent the curvature tensor and Ricci tensor of the manifold, respectively.

K. Kenmotsu [14] introduced a new class of almost contact Riemannian manifolds, known as Kenmotsu manifolds. He investigated the fundamental properties of their local structure. Kenmotsu manifolds are locally isometric to warped product spaces with a one-dimensional base and a Kähler fiber. Kenmotsu showed that if a Kenmotsu manifold satisfies the condition $R(\mathcal{F},\mathcal{G})\mathcal{H} = 0$, then the manifold has a constant negative curvature of -1, where R is the Riemannian curvature tensor and $R(\mathcal{F},\mathcal{G})\mathcal{H}$ represents the derivative of the tensor algebra at each point of the tangent space. It is well known that odd-dimensional spheres admit Sasakian structures, whereas odd-dimensional hyperbolic spaces do not; instead, they admit Kenmotsu structures. Kenmotsu manifolds are normal almost contact Riemannian manifolds. Various geometric properties of Kenmotsu manifolds, as well as several curvature structures including the Bochner curvature tensor [15], M-Projective curvature tensor [16] etc., have been the subject of extensive investigation in recent years [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. Building upon these contributions, the present study is devoted to a detailed examination of the $\mathcal{W}_1$-curvature tensor in the setting of Lorentzian α-Sasakian manifolds, following the foundational framework outlined in [9].

2. PRELIMINARIES

A $(2n+1)$-dimensional smooth manifold M is termed a Lorentzian α-Sasakian manifold if it possesses a $(1,1)$-tensor field φ, a vector field ζ, a 1-form η, and a Lorentzian metric g that satisfy the following conditions [9]:

$$\varphi^2 = I + \eta \otimes \zeta, \tag{8}$$

$$\eta(\zeta) = -1, \varphi(\zeta) = 0, \eta \circ \varphi = 0, \tag{9}$$

$$g(\mathcal{F},\zeta) = \eta(\mathcal{F}), \tag{10}$$

$$g(\varphi\mathcal{F},\varphi\mathcal{G}) = g(\mathcal{F},\mathcal{G}) + \eta(\mathcal{F})\eta(\mathcal{G}), \tag{11}$$

$$(\nabla_{\mathcal{F}}\varphi)\mathcal{G} = \alpha\{g(\mathcal{F},\mathcal{G})\zeta + \eta(\mathcal{G})\mathcal{F}\}, \tag{12}$$

for all $\mathcal{F},\mathcal{G} \in TM$. Also a Lorentzian α-Sasakian manifold M satisfies the following [9]

$$\nabla_{\mathcal{F}}\zeta = \alpha\varphi\mathcal{F}, \tag{13}$$

$$(\nabla_{\mathcal{F}}\eta)(\mathcal{G}) = \alpha g(\mathcal{F},\varphi\mathcal{G}), \tag{14}$$

where ∇ denotes the covariant differentiation operator associated with the Lorentzian metric g, and α is a constant.

Additionally, a Lorentzian α-Sasakian manifold M satisfies the following relations [9]:

$$\eta(R(\mathcal{F},\mathcal{G})\mathcal{H}) = \alpha^2\{g(\mathcal{G},\mathcal{H})\eta(\mathcal{F}) - g(\mathcal{F},\mathcal{H})\eta(\mathcal{G})\}, \quad (15)$$

$$R(\mathcal{F},\mathcal{G})\zeta = \alpha^2\{\eta(\mathcal{G})\mathcal{F} - \eta(\mathcal{F})\mathcal{G}\}, \quad (16)$$

$$R(\zeta,\mathcal{F})\mathcal{G} = \alpha^2\{g(\mathcal{F},\mathcal{G})\zeta - \eta(\mathcal{G})\mathcal{F}\}, \quad (17)$$

$$R(\zeta,\mathcal{F})\zeta = \alpha^2\{\eta(\mathcal{F})\zeta + \mathcal{F}\}, \quad (18)$$

$$S(\mathcal{F},\zeta) = 2n\alpha^2\eta(\mathcal{F}), \quad (19)$$

$$\mathcal{Q}\zeta = 2n\alpha^2\zeta, \quad (20)$$

$$S(\zeta,\zeta) = -2n\alpha^2, \quad (21)$$

for any vector fields $\mathcal{F},\mathcal{G},\mathcal{H}$, where S is the Ricci curvature and $\mathcal{Q}$ is the Ricci operator, defined by the relation $S(\mathcal{F},\mathcal{G}) = g(\mathcal{Q}\mathcal{F},\mathcal{G})$.

Definition 2.1. *A Lorentzian α-Sasakian manifold M is referred to as η-Einstein if its Ricci tensor S takes the following form:*

$$S(\mathcal{F},\mathcal{G}) = \lambda_1 g(\mathcal{F},\mathcal{G}) + \lambda_2\eta(\mathcal{F})\eta(\mathcal{G}), \quad (22)$$

for any vector fields $\mathcal{F},\mathcal{G}$, where λ_1 and λ_2 are functions on M. If $\lambda_1 = 0$, then M is classified as a special η-Einstein manifold.

3. $\mathcal{W}_1$-FLAT LORENTZIAN α-SASAKIAN MANIFOLD

In this section, we study $\mathcal{W}_1$-flat in Lorentzian α-Sasakian manifold.

Definition 3.1. *A Lorentzian α-Sasakian manifold is said to be $\mathcal{W}_1$-flat if*

$$\mathcal{W}_1(\mathcal{F},\mathcal{G})\mathcal{H} = 0, \quad (23)$$

for any vector fields $\mathcal{F},\mathcal{G}$ and $\mathcal{H}$ on M.

$\mathcal{W}_1$-curvature tensor [13, 32] is defined as

$$\mathcal{W}_1(\mathcal{F},\mathcal{G})\mathcal{H} = R(\mathcal{F},\mathcal{G})\mathcal{H} + \frac{1}{(n-1)}[S(\mathcal{G},\mathcal{H})\mathcal{F} - S(\mathcal{F},\mathcal{H})\mathcal{G}], \quad (24)$$

Using Equation (23) in (24), we have

$$R(\mathcal{F},\mathcal{G})\mathcal{H} + \frac{1}{(n-1)}[S(\mathcal{G},\mathcal{H})\mathcal{F} - S(\mathcal{F},\mathcal{H})\mathcal{G}] = 0. \quad (25)$$

Replacing $\mathcal{F}$ by ζ in (25) we get

$$R(\zeta,\mathcal{G})\mathcal{H} + \frac{1}{(n-1)}[S(\mathcal{G},\mathcal{H})\zeta - S(\zeta,\mathcal{H})\mathcal{G}] = 0. \quad (26)$$

Using Equations (17) and (19) in (26), we have

$$\alpha^2\{g(\mathcal{G},\mathcal{H})\zeta - \eta(\mathcal{H})\mathcal{G}\} + \frac{1}{(n-1)}[S(\mathcal{G},\mathcal{H})\zeta - 2n\alpha^2\eta(\mathcal{H})\mathcal{G}] = 0,$$

$$\alpha^2 g(\mathcal{G},\mathcal{H})\zeta - \alpha^2\eta(\mathcal{H})\mathcal{G} + \frac{1}{(n-1)}S(\mathcal{G},\mathcal{H})\zeta - \frac{2n}{n-1}\alpha^2\eta(\mathcal{H})\mathcal{G} = 0,$$

$$\frac{1}{(n-1)}S(\mathcal{G},\mathcal{H})\zeta = \frac{2n}{n-1}\alpha^2\eta(\mathcal{H})\mathcal{G} + \alpha^2\eta(\mathcal{H})\mathcal{G} - \alpha^2 g(\mathcal{G},\mathcal{H})\zeta,$$

$$S(\mathcal{G},\mathcal{H})\zeta = -(n-1)\alpha^2 g(\mathcal{G},\mathcal{H})\zeta + (3n-1)\alpha^2\eta(\mathcal{H})\mathcal{G}.$$

Taking inner product with ζ, we get

$$S(\mathcal{G},\mathcal{H}) = -(n-1)\alpha^2 g(\mathcal{G},\mathcal{H}) + (1-3n)\alpha^2\eta(\mathcal{G})\eta(\mathcal{H}). \tag{27}$$

Hence from the above discussion, we state the following theorem.

Theorem 3.1. *If a Lorentzian α-Sasakian manifold satisfying $\mathcal{W}_1$-flat condition then the manifold is an η-Einstein manifold.*

4. $\zeta - \mathcal{W}_1$-FLAT LORENTZIAN α-SASAKIAN MANIFOLD

In this section, we study $\zeta - \mathcal{W}_1$-flat in Lorentzian α-Sasakian manifold.

Definition 4.1. *A Lorentzian α-Sasakian manifold is said to be $\zeta - \mathcal{W}_1$-flat if* [33]

$$\mathcal{W}_1(\mathcal{F},\mathcal{G})\zeta = 0, \tag{28}$$

for any vector fields $\mathcal{F},\mathcal{G}$ on M.

$\mathcal{W}_1$-curvature tensor [13] is defined as

$$\mathcal{W}_1(\mathcal{F},\mathcal{G})\mathcal{H} = R(\mathcal{F},\mathcal{G})\mathcal{H} + \frac{1}{(n-1)}[S(\mathcal{G},\mathcal{H})\mathcal{F} - S(\mathcal{F},\mathcal{H})\mathcal{G}], \tag{29}$$

Replacing $\mathcal{H}$ by ζ in (29), we get

$$\mathcal{W}_1(\mathcal{F},\mathcal{G})\zeta = R(\mathcal{F},\mathcal{G})\zeta + \frac{1}{(n-1)}[S(\mathcal{G},\zeta)\mathcal{F} - S(\mathcal{F},\zeta)\mathcal{G}]. \tag{30}$$

By using (28) in (30), we get

$$R(\mathcal{F},\mathcal{G})\zeta + \frac{1}{(n-1)}[S(\mathcal{G},\zeta)\mathcal{F} - S(\mathcal{F},\zeta)\mathcal{G}] = 0. \tag{31}$$

By virtue of (16), (19) in (31) and on simplification, we obtain

$$\alpha^2\{\eta(\mathcal{G})\mathcal{F} - \eta(\mathcal{F})\mathcal{G}\} + \frac{1}{(n-1)}[2n\alpha^2\eta(\mathcal{G})\mathcal{F} - 2n\alpha^2\eta(\mathcal{F})\mathcal{G}] = 0, \tag{32}$$

$$\{\eta(\mathcal{G})\mathcal{F} - \eta(\mathcal{F})\mathcal{G}\} + \frac{2n}{n-1}\{\eta(\mathcal{G})\mathcal{F} - \eta(\mathcal{F})\mathcal{G}\} = 0,$$

$$\frac{(3n-1)}{(n-1)}\{\eta(\mathcal{G})\mathcal{F} - \eta(\mathcal{F})\mathcal{G}\} = 0,$$

$$\{\eta(\mathcal{G})\mathcal{F} - \eta(\mathcal{F})\mathcal{G}\} = 0. \tag{33}$$

Putting $\mathcal{F} = \zeta$ and $\mathcal{G} = \mathcal{Q}\mathcal{G}$ in (33), we get

$$\{\eta(\mathcal{Q}\mathcal{G})\zeta - \eta(\zeta)\mathcal{Q}\mathcal{G}\} = 0, \tag{34}$$

$$\mathcal{Q}\mathcal{G} = -2n\alpha^2\eta(\mathcal{G})\zeta,$$

$$S(\mathcal{G},\mathcal{H}) = -2n\alpha^2\eta(\mathcal{G})\eta(\mathcal{H}). \tag{35}$$

Hence from above discussion, we state the following theorem:

Theorem 4.1. *If a Lorentzian α-Sasakian manifold satisfying $\zeta-\mathcal{W}_1$-flat condition then the Lorentzian α-Sasakian manifold is a special type of η-Einstein manifold.*

5. $\varphi-\mathcal{W}_1$-SEMI-SYMMETRIC CONDITION IN LORENTZIAN α-SASAKIAN MANIFOLD

In this section, we study $\varphi-\mathcal{W}_1$-semi-symmetric condition in Lorentzian α-Sasakian manifold.

Definition 5.1. *A Lorentzian α-Sasakian manifold is said to be $\varphi-\mathcal{W}_1$- semi-symmetric if* [34]

$$\mathcal{W}_1(\mathcal{F},\mathcal{G})\cdot\varphi = 0, \tag{36}$$

for any vector fields $\mathcal{F},\mathcal{G}$ on M.

Now, (36) turns into

$$(\mathcal{W}_1(\mathcal{F},\mathcal{G})\cdot\varphi)\mathcal{H} = \mathcal{W}_1(\mathcal{F},\mathcal{G})\varphi\mathcal{H} - \varphi\mathcal{W}_1(\mathcal{F},\mathcal{G})\mathcal{H} = 0. \tag{37}$$

From Equation (24), we get

$$\mathcal{W}_1(\mathcal{F},\mathcal{G})\mathcal{H} = R(\mathcal{F},\mathcal{G})\mathcal{H} + \frac{1}{(n-1)}[S(\mathcal{G},\mathcal{H})\mathcal{F} - S(\mathcal{F},\mathcal{H})\mathcal{G}]. \tag{38}$$

Replace $\mathcal{H}$ by $\varphi\mathcal{H}$ in (38), we obtain

$$\mathcal{W}_1(\mathcal{F},\mathcal{G})\varphi\mathcal{H} = R(\mathcal{F},\mathcal{G})\varphi\mathcal{H} + \frac{1}{(n-1)}[S(\mathcal{G},\varphi\mathcal{H})\mathcal{F} - S(\mathcal{F},\varphi\mathcal{H})\mathcal{G}]. \tag{39}$$

Making use of (38) and (39) in (37) and on simplification, we get

$$R(\mathcal{F},\mathcal{G})\varphi\mathcal{H} + \frac{1}{(n-1)}[S(\mathcal{G},\varphi\mathcal{H})\mathcal{F} - S(\mathcal{F},\varphi\mathcal{H})\mathcal{G}] - \varphi(R(\mathcal{F},\mathcal{G})\mathcal{H} + \frac{1}{(n-1)}[S(\mathcal{G},\mathcal{H})\mathcal{F} - S(\mathcal{F},\mathcal{H})\mathcal{G}]) = 0. \tag{40}$$

Putting $\mathcal{F}=\zeta$ in (40) and by virtue of (17),(19) and on simplification, we obtain

$$\alpha^2 g(\mathcal{G},\varphi\mathcal{H})\zeta + \frac{1}{(n-1)}S(\mathcal{G},\varphi\mathcal{H})\zeta = 0. \tag{41}$$

Replace $\varphi\mathcal{H}$ by $\mathcal{H}$ in (41) and on simplification, we get

$$S(\mathcal{G},\mathcal{H})\zeta = -(n-1)\alpha^2 g(\mathcal{G},\mathcal{H})\zeta. \tag{42}$$

By taking inner product with ζ in (42), we get

$$S(\mathcal{G},\mathcal{H}) = -(n-1)\alpha^2 g(\mathcal{G},\mathcal{H}). \tag{43}$$

Hence, we state the following theorem.

Theorem 5.1. *If a Lorentzian α-Sasakian manifold satisfying $\varphi - \mathcal{W}_1-$semi-symmetric condition then the manifold is an Einstein Manifold.*

6. LORENTZIAN $\alpha-$SASAKIAN MANIFOLD SATISFYING $\mathcal{W}_1 \cdot \mathcal{Q} = 0$

In this section, we study the Lorentzian α-Sasakian manifold satisfying $\mathcal{W}_1 \cdot \mathcal{Q} = 0$. Then, we have

$$\mathcal{W}_1(\mathcal{F}, \mathcal{G})\mathcal{Q}\mathcal{H} - \mathcal{Q}(\mathcal{W}_1(\mathcal{F}, \mathcal{G})\mathcal{H}) = 0. \tag{44}$$

Putting $\mathcal{G} = \zeta$ in (44), we obtain

$$\mathcal{W}_1(\mathcal{F}, \zeta)\mathcal{Q}\mathcal{H} - \mathcal{Q}(\mathcal{W}_1(\mathcal{F}, \zeta)\mathcal{H}) = 0. \tag{45}$$

By virtue of (24) in (45), we get

$$\begin{aligned} R(\mathcal{F}, \zeta)\mathcal{Q}\mathcal{H} + \frac{1}{(n-1)}[S(\zeta, \mathcal{Q}\mathcal{H})\mathcal{F} - S(\mathcal{F}, \mathcal{Q}\mathcal{H})\zeta] - \mathcal{Q}\{R(\mathcal{F}, \zeta)\mathcal{H} \\ + \frac{1}{(n-1)}[S(\zeta, \mathcal{H})\mathcal{F} - S(\mathcal{F}, \mathcal{H})\zeta]\} = 0. \end{aligned} \tag{46}$$

By using (17), (19) in (46), we obtain

$$\begin{aligned} &-\alpha^2[g(\mathcal{F}, \mathcal{Q}\mathcal{H})\zeta - \eta(\mathcal{Q}\mathcal{H})\mathcal{F}] + \frac{1}{(n-1)}[S(\zeta, \mathcal{Q}\mathcal{H})\mathcal{F} - S(\mathcal{F}, \mathcal{Q}\mathcal{H})\zeta] \\ &\quad - \mathcal{Q}\{-(\alpha^2[g(\mathcal{F}, \mathcal{H})\zeta - \eta(\mathcal{H})\mathcal{F}] + \frac{1}{(n-1)}[S(\zeta, \mathcal{H})\mathcal{F} - S(\mathcal{F}, \mathcal{H})\zeta]\} = 0, \end{aligned} \tag{47}$$

$$\begin{aligned} &-\alpha^2[S(\mathcal{F}, \mathcal{H})\zeta - \eta(\mathcal{Q}\mathcal{H})\mathcal{F}] + \frac{1}{(n-1)}[2n\alpha^2\eta(\mathcal{Q}\mathcal{H})\mathcal{F} - S(\mathcal{F}, \mathcal{Q}\mathcal{H})\zeta] \\ &\quad - \mathcal{Q}\{-\alpha^2 g(\mathcal{F}, \mathcal{H})\zeta + \alpha^2\eta(\mathcal{H})\mathcal{F} + \frac{1}{(n-1)}[2n\alpha^2\eta(\mathcal{H})\mathcal{F} - S(\mathcal{F}, \mathcal{H})\zeta]\} = 0. \end{aligned} \tag{48}$$

Using (20) and simplify (48), we have

$$S(\mathcal{F}, \mathcal{H})\zeta = 2n\alpha^2 g(\mathcal{F}, \mathcal{H})\zeta. \tag{49}$$

Taking inner product with ζ in (49), we have

$$S(\mathcal{F}, \mathcal{H}) = 2n\alpha^2 g(\mathcal{F}, \mathcal{H}). \tag{50}$$

Hence, we state the following theorem.

Theorem 6.1. *A Lorentzian α-Sasakian manifold satisfying $\mathcal{W}_1 \cdot \mathcal{Q} = 0$, then the manifold is an Einstein manifold.*

7. LORENTZIAN α-SASAKIAN MANIFOLD SATISFYING $\mathcal{W}_1 \cdot R = 0$.

In this section, we study the Lorentzian α-Sasakian manifold satisfying $\mathcal{W}_1 \cdot R = 0$. Then, we have

$$\mathcal{W}_1(\zeta,\mathcal{U})R(\mathcal{F},\mathcal{G})\mathcal{H} - R(\mathcal{W}_1(\zeta,\mathcal{U})\mathcal{F},\mathcal{G})\mathcal{H} - R(\mathcal{F},\mathcal{W}_1(\zeta,\mathcal{U})\mathcal{G})\mathcal{H} - R(\mathcal{F},\mathcal{G})\mathcal{W}_1(\zeta,\mathcal{U})\mathcal{H} = 0. \quad (51)$$

Putting $\mathcal{H} = \zeta$ in (51), we have

$$\mathcal{W}_1(\zeta,\mathcal{U})R(\mathcal{F},\mathcal{G})\zeta - R(\mathcal{W}_1(\zeta,\mathcal{U})\mathcal{F},\mathcal{G})\zeta - R(\mathcal{F},\mathcal{W}_1(\zeta,\mathcal{U})\mathcal{G})\zeta - R(\mathcal{F},\mathcal{G})\mathcal{W}_1(\zeta,\mathcal{U})\zeta = 0. \quad (52)$$

By using (16) in (52) and on simplification, we get

$$\alpha^2\eta(\mathcal{W}_1(\zeta,\mathcal{U})\mathcal{F})\mathcal{G} - \alpha^2\eta(\mathcal{W}_1(\zeta,\mathcal{U})\mathcal{G})\mathcal{F} - R(\mathcal{F},\mathcal{G})\mathcal{W}_1(\zeta,\mathcal{U})\zeta = 0. \quad (53)$$

By using (24) in (53), we get

$$\alpha^2\eta[R(\zeta,\mathcal{U})\mathcal{F} + \frac{1}{(n-1)}\{S(\mathcal{U},\mathcal{F})\zeta - S(\zeta,\mathcal{F}\}\mathcal{U}]\mathcal{G} - \alpha^2\eta[R(\zeta,\mathcal{U})\mathcal{G} + \frac{1}{(n-1)}\{S(\mathcal{U},\mathcal{G})\zeta - S(\zeta,\mathcal{G}\}\mathcal{U}]\mathcal{F} - R(\mathcal{F},\mathcal{G})[R(\zeta,\mathcal{U})\zeta + \frac{1}{(n-1)}\{S(\mathcal{U},\zeta)\zeta - S(\zeta,\zeta)\mathcal{U}\}] = 0. \quad (54)$$

By using (17), (19), (21) in (54) and on simplification, we get

$$\alpha^2\{g(\mathcal{U},\mathcal{F})\mathcal{G} - g(\mathcal{U},\mathcal{G})\mathcal{F}\} + \frac{2n}{(n-1)}\alpha^2\eta(\mathcal{U})\eta(\mathcal{F})\mathcal{G} - \frac{2n}{(n-1)}\alpha^2\eta(\mathcal{U})\eta(\mathcal{G})\mathcal{F} + \frac{1}{(n-1)}\{S(\mathcal{U},\mathcal{G})\mathcal{F} - S(\mathcal{U},\mathcal{F})\mathcal{G}\} + R(\mathcal{F},\mathcal{G})\mathcal{U} = 0. \quad (55)$$

Putting $\mathcal{G} = \zeta$ in (55), we get

$$\alpha^2\{g(\mathcal{U},\mathcal{F})\zeta - g(\mathcal{U},\zeta)\mathcal{F}\} + \frac{2n}{(n-1)}\alpha^2\eta(\mathcal{U})\{\eta(\mathcal{F})\zeta - \eta(\zeta)\mathcal{F}\} + \frac{1}{(n-1)}\{S(\mathcal{U},\zeta)\mathcal{F} - S(\mathcal{U},\mathcal{F})\zeta\} + R(\mathcal{F},\zeta)\mathcal{U} = 0. \quad (56)$$

By using (9), (20), (22) in (56) and on simplification, we get

$$S(\mathcal{F},\mathcal{U})\zeta = 2n\alpha^2\eta(\mathcal{U})\eta(\mathcal{F})\zeta. \quad (57)$$

By taking inner product with ζ in (57), we have

$$S(\mathcal{F},\mathcal{U}) = 2n\alpha^2\eta(\mathcal{U})\eta(\mathcal{F}). \quad (58)$$

Hence, we state the following theorem.

Theorem 7.1. *If a Lorentzian α-Sasakian manifold satisfying $\mathcal{W}_1 \cdot R = 0$, then the manifold is a special type of η-Einstein manifold.*

8. EXAMPLE OF A FIVE-DIMENSIONAL LORENTZIAN α-SASAKIAN MANIFOLD

Consider a five dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5 : v \neq 0\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$\varrho_1 = \varrho^v \frac{\partial}{\partial x}, \quad \varrho_2 = \varrho^v \frac{\partial}{\partial y}, \quad \varrho_3 = \varrho^v \frac{\partial}{\partial z}, \quad \varrho_4 = \varrho^v \frac{\partial}{\partial u}, \quad \varrho_5 = \alpha \frac{\partial}{\partial v},$$

which are linearly independent at each point of the manifold M.
Let the Lorentzian metric g defined by

$$g(\varrho_i, \varrho_j) = \begin{cases} 1, & if \quad i = j \quad and \quad i, j \in \{1, 2, 3, 4\}, \\ -1, & if \quad i = j = 5, \\ 0, & otherwise. \end{cases}$$

and given by

$$g = \frac{1}{\varrho^{2v}} \left[dx \otimes dx + dy \otimes dy + dz \otimes dz + du \otimes du - dv \otimes dv \right].$$

Let η be a 1-form which satisfies the relation

$$\eta(\varrho_5) = -1.$$

Let φ be a $(1, 1)$-tensor field defined by $\varphi(\varrho_1) = -\varrho_2, \quad \varphi(\varrho_3) = -\varrho_4, \quad \varphi(\varrho_5) = 0.$
Then, we have

$$\varphi^2(\mathcal{F}) = \mathcal{F} + \eta(\mathcal{F})\varrho_5,$$
$$g(\varphi\mathcal{F}, \varphi\mathcal{G}) = g(\mathcal{F}, \mathcal{G}) + \eta(\mathcal{F})\eta(\mathcal{G}),$$

for any $\mathcal{F}, \mathcal{G} \in \chi(M^5)$. Thus for $\varrho_5 = \zeta$, $M^5(\varphi, \zeta, \eta, g)$ defines an almost contact metric structure on M. Now, we have

$$[\varrho_1, \varrho_2] = 0, \quad [\varrho_1, \varrho_3] = 0, \quad [\varrho_1, \varrho_4] = 0, \quad [\varrho_1, \varrho_5] = -\alpha\varrho_1, \quad [\varrho_2, \varrho_3] = 0,$$
$$[\varrho_2, \varrho_4] = 0, \quad [\varrho_2, \varrho_5] = -\alpha\varrho_2, \quad [\varrho_3, \varrho_4] = 0, \quad [\varrho_3, \varrho_5] = -\alpha\varrho_3, \quad [\varrho_4, \varrho_5] = -\alpha\varrho_4.$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula

$$2g(\nabla_{\mathcal{F}}\mathcal{G}, \mathcal{H}) = \mathcal{F}g(\mathcal{G}, \mathcal{H}) + \mathcal{G}g(\mathcal{H}, \mathcal{F}) - \mathcal{H}g(\mathcal{F}, \mathcal{G}) - g(\mathcal{F}, [\mathcal{G}, \mathcal{H}])$$
$$- g(\mathcal{G}, [\mathcal{F}, \mathcal{H}]) + g(\mathcal{H}, [\mathcal{F}, \mathcal{G}]).$$

Taking $\varrho_5 = \zeta$ and using the Koszul's formula, we get the following

$$\nabla_{\varrho_1}\varrho_1 = \alpha\varrho_1, \quad \nabla_{\varrho_2}\varrho_1 = 0, \quad \nabla_{\varrho_3}\varrho_1 = 0, \quad \nabla_{\varrho_4}\varrho_1 = 0, \quad \nabla_{\varrho_5}\varrho_1 = 0,$$
$$\nabla_{\varrho_1}\varrho_2 = 0, \quad \nabla_{\varrho_2}\varrho_2 = \alpha\varrho_2, \quad \nabla_{\varrho_3}\varrho_2 = 0, \quad \nabla_{\varrho_4}\varrho_2 = 0, \quad \nabla_{\varrho_5}\varrho_2 = 0,$$
$$\nabla_{\varrho_1}\varrho_3 = 0, \quad \nabla_{\varrho_2}\varrho_3 = 0, \quad \nabla_{\varrho_3}\varrho_3 = \alpha\varrho_3, \quad \nabla_{\varrho_4}\varrho_3 = 0, \quad \nabla_{\varrho_5}\varrho_3 = 0,$$
$$\nabla_{\varrho_1}\varrho_4 = 0, \quad \nabla_{\varrho_2}\varrho_4 = 0, \quad \nabla_{\varrho_3}\varrho_4 = 0, \quad \nabla_{\varrho_4}\varrho_4 = \alpha\varrho_4, \quad \nabla_{\varrho_5}\varrho_4 = 0,$$
$$\nabla_{\varrho_1}\varrho_5 = -\alpha\varrho_1, \quad \nabla_{\varrho_2}\varrho_5 = -\alpha\varrho_2, \quad \nabla_{\varrho_3}\varrho_5 = -\alpha\varrho_3, \quad \nabla_{\varrho_4}\varrho_5 = -\alpha\varrho_4, \quad \nabla_{\varrho_5}\varrho_5 = 0.$$

Consequently, it is clear that M^5 satisfies the condition. These results shows that the manifold satisfies

$$\nabla_{\mathcal{F}}\zeta = -\alpha\varphi\mathcal{F},$$

for $\zeta = \varrho_5$. Hence the manifold under consideration is a Lorentzian α-Sasakian manifold of dimension five.
This manifold allows for the verification of the results stated in this article.
The components of the curvature tensor with respect to the Levi-Civita connection ∇ are as follows

$$\begin{gathered}
R(\varrho_1,\varrho_2)\varrho_1 = -\alpha^2\varrho_2, \quad R(\varrho_1,\varrho_2)\varrho_2 = \alpha^2\varrho_1, \quad R(\varrho_1,\varrho_3)\varrho_1 = -\alpha^2\varrho_3, \\
R(\varrho_1,\varrho_3)\varrho_3 = \alpha^2\varrho_1, \quad R(\varrho_1,\varrho_4)\varrho_1 = -\alpha^2\varrho_4, \quad R(\varrho_1,\varrho_4)\varrho_4 = \alpha^2\varrho_1, \\
R(\varrho_1,\varrho_5)\varrho_1 = -\alpha^2\varrho_5, \quad R(\varrho_1,\varrho_5)\varrho_5 = -\alpha^2\varrho_1, \quad R(\varrho_2,\varrho_3)\varrho_2 = -\alpha^2\varrho_3, \\
R(\varrho_2,\varrho_3)\varrho_3 = \alpha^2\varrho_2, \quad R(\varrho_2,\varrho_4)\varrho_2 = -\alpha^2\varrho_4, \quad R(\varrho_2,\varrho_4)\varrho_4 = \alpha^2\varrho_2, \\
R(\varrho_2,\varrho_5)\varrho_2 = -\alpha^2\varrho_5, \quad R(\varrho_2,\varrho_5)\varrho_5 = -\alpha^2\varrho_2, \quad R(\varrho_3,\varrho_4)\varrho_3 = -\alpha^2\varrho_4, \\
R(\varrho_3,\varrho_4)\varrho_4 = \alpha^2\varrho_3, \quad R(\varrho_3,\varrho_5)\varrho_3 = -\alpha^2\varrho_5, \quad R(\varrho_3,\varrho_5)\varrho_5 = -\alpha^2\varrho_3, \\
R(\varrho_4,\varrho_5)\varrho_4 = -\alpha^2\varrho_5, \quad R(\varrho_4,\varrho_5)\varrho_5 = -\alpha^2\varrho_4.
\end{gathered}$$

From the previous results, we can obtain the Ricci tensor S of the Levi-Civita connection ∇, which is as follows: $S(\varrho_1,\varrho_1) = 4\alpha^2$, $S(\varrho_2,\varrho_2) = 4\alpha^2$, $S(\varrho_3,\varrho_3) = 4\alpha^2$, $S(\varrho_4,\varrho_4) = 4\alpha^2, S(\varrho_5,\varrho_5) = -4\alpha^2$. The scalar curvature r with respect to the Levi-Civita connection ∇ is given by $r = \sum_{i=1}^{5} S(\varrho_i,\varrho_i) = 20\alpha^2$.

9. CONCLUDING REMARKS

In this paper, we introduce the concepts of $\mathcal{W}_1$-flat and $\zeta - \mathcal{W}_1$-flat Lorentzian α-Sasakian manifolds, which were identified as special types of η-Einstein manifolds. We also explored the $\varphi - \mathcal{W}_1$-semi-symmetric condition in Lorentzian α-Sasakian manifolds, discovering that it results in an Einstein manifold. Furthermore, we examined Lorentzian α-Sasakian manifolds that satisfy the condition $\mathcal{W}_1 \cdot \mathcal{Q} = 0$, establishing that they are also Einstein manifolds. Finally, we discuss Lorentzian α-Sasakian manifolds that meet the condition $\mathcal{W}_1 \cdot R = 0$, concluding that they are η-Einstein manifolds.

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