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An SQP Regularization with Double Conjugate Gradient Implementation for Solving Nonlinear Complementarity Problems

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Abstract. Building upon the works proposed in [1] and [2], we introduce an advanced version of regularized proximal point methods to solve nonlinear complementarity problems (NCP). Our contribution is characterized by two key innovations. Firstly, we introduce an innovative square root quadratic term as part of the regularized subproblem framework, replacing the commonly used logarithmic quadratic term. Secondly, we implement the conjugate gradient algorithm in two stages: the intermediate step and the correction step. This dual approach employs two optimal descent directions with two step lengths to achieve multiplicative progress in each iteration, significantly accelerating convergence. We establish the global convergence of our innovative algorithm, under the condition that F exhibits monotonicity. Initial numerical experiments are presented to confirm the algorithm's practical effectiveness.

Key words and Phrases: Nonlinear complementarity problems, monotone operator, proximal point method, logarithmic quadratic term, square root quadratic term, conjugate gradient algorithm.

1. Introduction

NCP seeks to identify a vector $x \in R^n$ satisfying

$$x \geq 0, \quad F(x) \geq 0 \quad \text{and} \qquad x^T F(x) = 0, \tag{1}$$

F represents a nonlinear function from $\mathbb{R}^n$ onto itself. This study considers $F(x)$ to be continuous and monotone with respect to $\mathbb{R}^n_+$. Additionally, it is assumed that the solution set for (1), represented by Ω^*, is not empty.

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Richard W. Cottle's introduction of NCP marked in his Ph.D. thesis during the early 1960s. Since then, complementarity problems have captured the interest of researchers, leading to numerous publications that lay down the essential theoretical foundations of this field (see [3, 4]). A standard approach to addressing the NCP involves identifying a vector $x^* \in R^n_+$ such that $0 \in O(x^*)$, where the operator $O(x) = F(x) + C_{R^n_+}(x)$. Here, $C_{R^n_+}(.)$ denotes the normal cone to the nonnegative orthant.

A widely adopted strategy for tackling the NCP is the PPA method. This method starts with an arbitrary $x^0 \in R^n_+$ and $\beta_k \geq \beta > 0$, generates x^{k+1} for solving :

$$\text{(PPA)} \qquad 0 \in \beta_k O(x) + \nabla_x q\left(x, x^k\right). \tag{2}$$

with

$$q\left(x, x^k\right) = \frac{1}{2}\left\|x - x^k\right\|^2 \tag{3}$$

Recently, numerous studies have focused on developing innovative interior point methods to address NCP. These methods share a common characteristic which enforce the new iterates $\{x^{k+1}\}$ to stay in the interior of R^n_{++}. Auslender, *et al.* [5] have proposed a new type of proximal interior algorithms via replacing the quadratic function (3) by $dist_\phi(x, x^k)$ which could be defined as

$$dist_\phi(x, y) = \sum_{j=1}^{n} y_j^2 \phi\left(y_j^{-1} x_j\right).$$

Let $\nu > \mu > 0$ be two predetermined constants, let

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu\varphi(t) & \text{if} \quad t > 0 \\ +\infty & \text{otherwise} \end{cases}$$

where $\varphi(t)$ is a φ-divergence function that respects these necessary aspects:

1) The function φ is assumed to be twice continuously differentiable within the interior of $\mathbb{R}^n$.
2) The function φ exhibits strict convexity throughout its defined domain
3) $\lim\limits_{x \to 0^+} \frac{d\varphi(x)}{dx} = -\infty$.
4) $\varphi(1) = \frac{d\varphi(1)}{dx} = 0$ and $\frac{d^2\varphi(1)}{dx^2} > 0$.
5) There exists $\nu \in \left(\frac{1}{2}\frac{d^2\varphi(1)}{dx^2}, \frac{d^2\varphi(1)}{dx^2}\right)$ such that

$$\left(1 - \frac{1}{t}\right)\left(\frac{d^2\varphi(1)}{dx^2} + \nu(t-1)\right) \leq \frac{d\varphi(t)}{dx} \leq \frac{d^2\varphi(1)}{dx^2}(t-1) \quad \forall t > 0.$$

In [6], Auslender, *et al.* A specialized logarithmic-quadratic proximal (LQP) algorithm has been employed by leveraging by using $\varphi_1(t) = t - \log(t) - 1$ in the definition of $\phi(t)$ (with $\nu = 2, \mu = 1$).

Later on, Noor and Bnouhachem [7] and [8], have proposed a new modified LQP method by using $\varphi_2(t) = t\log(t) - t + 1$ as a φ-divergence function (with $\nu = 1, \mu \in (0,1)$)

Let $\nu = \dfrac{1}{2}$ and $\mu \in \left(0, \dfrac{1}{2}\right)$, in our contribution, we used the φ function $\varphi_3(t) = (\sqrt{t} - 1)^2$ as proposed in [9] and [2], we get

$$\phi(t) = \begin{cases} \dfrac{1}{4}(t-1)^2 + \mu(\sqrt{t}-1)^2 & \text{if} \quad t > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Assume $x^k \in R^n_+$, $\beta_k \geq \beta > 0$, the updated iteration x^{k+1} of the problem (2) becomes the unique solution of the following set-valued equation:

$$\text{(SRQP)} \qquad 0 \in \beta_k O(x) + \nabla_x dist_\phi\left(x, x^k\right), \tag{4}$$

where

$$dist_\phi\left(x, x^k\right) = \begin{cases} \dfrac{1}{4}\|x - x^k\|^2 + \mu \sum\limits_{j=1}^{n} \left(x_j^k x_j - 2\left(x_j^k\right)^2 \sqrt{\dfrac{x_j}{x_j^k}} + \left(x_j^k\right)^2 \right) & \text{if} \quad x \in R^n_{++}, \\ +\infty & \text{otherwise.} \end{cases} \tag{5}$$

It is evident to see that

$$\begin{aligned} \nabla_x dist_\phi(x, x^k) &= \frac{1}{2}\left(x - x^k\right) + \mu \sum_{j=1}^{n} \left(x_j^k - \frac{\left(x_j^k\right)^2}{\sqrt{x_j^k}} \frac{1}{\sqrt{x_j}} \right) \\ &= \frac{1}{2}(x - x^k) + \mu\left(x^k - X_k\left(\sqrt{x}\right)^{-1}\right). \end{aligned} \tag{6}$$

$X_k = diag\left(\sqrt{x_1^k}^3, ..., \sqrt{x_n^k}^3\right)$ and $\sqrt{x} = \left(\sqrt{x_1}, ..., \sqrt{x_n}\right)$.

Now, the problem (4) is equivalent to :

$$\beta_k F(x) + \frac{1}{2}\left(x - x^k\right) + \mu\left(x^k - X_k\left(\sqrt{x}\right)^{-1}\right) = 0. \tag{7}$$

Solving the subproblem (7) exactly presents significant challenges in practice, often excluding practical applications. To mitigate this issue, it is advisable to pursue approximate solutions $\tilde{x}^k$ instead of exact ones. For this reason, we introduce ξ^k such that :

$$0 \approx \beta_k F(x) + \frac{1}{2}\left(x - x^k\right) + \mu\left(x^k - X_k\left(\sqrt{x}\right)^{-1}\right) = \xi^k \tag{8}$$

and $\xi^k := \beta_k\left(F\left(\tilde{x}^k\right) - F\left(x^k\right)\right)$ satisfies

$$\|\xi^k\| \leq \eta\|x^k - \tilde{x}^k\|, \qquad 0 < \mu, \eta < \frac{1}{2}. \tag{9}$$

In this paper, we proposed a prediction-correction method to solve (7) approximately. Numerical results are provided to substantiate the efficacy of the proposed method.

2. Preliminaries

Key properties are essential for our subsequent analysis.
First, we denote $Pr_{R^n_+}(.)$ as : $Pr_{R^n_+}(z) = \min\left\{\|z - x\| \| \quad x \in R^n_+\right\}$.
A fundamental characteristic of this projection mapping is :

$$\left(y - Pr_{R^n_+}(y)\right)^T \left(Pr_{R^n_+}(y) - x\right) \geq 0, \quad \forall y \in R^n, \quad \forall x \in R^n_+. \tag{10}$$

From (10), one can readily confirm that :

$$\|Pr_{R^n_+}(v) - u\|^2 \leq \|v - u\|^2 - \|v - Pr_{R^n_+}(v)\|^2, \qquad \forall v \in R^n, u \in R^n_+. \tag{11}$$

Definition 2.1. *The operator $F : R^n \to R^n$ is said to be monotone, if*

$$\forall u, v \in R^n, \qquad (v - u)^T(F(v) - F(u)) \geq 0.$$

3. The proposed method and convergence results

At the kth iteration, Using a three-step SRQP approach, compute the exact solution for the system of equations specified below:

$$\beta_k F(x) + \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}) = 0. \tag{12}$$

We now introduce an SRQP approach for solving problem (1). For given $x^1 > 0$, $D_0 = 0$ and $\tilde{D}_0 = 0$, the suggested approach comprises three steps.
Step 1 : Find $\tilde{x}^k$ of (12), such that

$$0 \approx \beta_k F(x) + \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}) = \xi^k \tag{13}$$

and $\xi^k := \beta_k(F(\tilde{x}^k) - F(x^k))$ satisfies

$$\|\xi^k\| \leq \eta\|x^k - \tilde{x}^k\|, \qquad 0 < \mu, \eta < \frac{1}{2}. \tag{14}$$

Step 2: For $\alpha_k > 0$. Compute

$$d(x^k) = \frac{1}{2}(x^k - \tilde{x}^k) + \frac{1}{1+\mu}\xi^k, \tag{15}$$

and

$$D_k = d(x^k) + \theta_k D_{k-1}, \tag{16}$$

where

$$\theta_k = \max\left(0, \frac{-d(x^k)^T D_{k-1}}{\|D_{k-1}\|^2}\right) \tag{17}$$

$\bar{x}^k(\alpha_k)$ is defined by

$$\bar{x}^k(\alpha_k) = P_{R^n_+}\left[x^k - \alpha_k D_k\right], \tag{18}$$

where

$$\alpha_k = \frac{\psi(x^k)}{\|D_k\|^2} \quad \textit{and} \quad \psi(x^k) = \frac{1}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}\left(x^k - \tilde{x}^k\right)^T \xi^k. \tag{19}$$

Step 3: For $0 < \rho < 1$ Compute

$$g(x^k) = x^k - \bar{x}^k, \tag{20}$$

and

$$\tilde{D}_k = g(x^k) + \lambda_k \tilde{D}_{k-1}, \tag{21}$$

where

$$\tilde{\theta}_k = \max\left(0, \frac{-g(x^k)^T \tilde{D}_{k-1}}{\|\tilde{D}_{k-1}\|^2}\right) \tag{22}$$

The new updated $x^{k+1}(\delta_k)$ is

$$x^{k+1}(\delta_k) = \rho x^k + (1-\rho) P_{R^n_+}\left[x^k - \delta_k \tilde{D}_k\right], \tag{23}$$

where

$$\delta_k = \frac{\tilde{\psi}(x^k)}{\|\tilde{D}_k\|^2} \quad \text{and} \quad \tilde{\psi}(x^k) = \frac{\|x^k - \bar{x}^k\|^2 + \alpha_k \psi(x^k)}{2}. \tag{24}$$

Remark 3.1. *(14) Leads to the conclusion that*

$$|(x^k - \tilde{x}^k)^T \xi^k| \leq \eta \|x^k - \tilde{x}^k\|^2, \qquad \eta < \frac{1}{2}. \tag{25}$$

Remark 3.2. *Consider the case where $\xi^k = \beta_k(F(\tilde{x}^k) - F(x^k))$. If F is Lipschitz continuous within R^n_+, with $L > 0$, i.e.,*

$$\|F(x^k) - F(\tilde{x}^k)\| \leq L\|x^k - \tilde{x}^k\|.$$

If β_k satisfying $0 < \beta_k \leq \frac{\eta}{L}$, then the above inequalities (14) are satisfied.

This Lemma is essential in analyzing convergence and plays a pivotal role in this regard.

Lemma 3.3. *if we let $x > 0$ and $q \in \mathbb{R}^n$, Let x be the positive solution of the following equation :*

$$q + \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}) = 0, \tag{26}$$

and $X_k = diag(\sqrt{x_1^k}^3, ..., \sqrt{x_n^k}^3)$ where $\sqrt{x} = (\sqrt{x_1}, ..., \sqrt{x_n})$,
then $\forall y \geq 0$ we have

$$(x - y)^T(-q) \geq \frac{1+\mu}{4}\left(\|x - y\|^2 - \|x^k - y\|^2\right) + \frac{1-\mu}{4}\|x^k - x\|^2. \tag{27}$$

Proof. [2] □

Lemma 3.4. [10] *Using the definition of $d(x^k)$, $g(x^k)$, S_k and D_k, then*

$$\|D_k\| \leq \|d(x^k)\|. \tag{28}$$

Proof. [2] □

Lemma 3.5. [10] *For any $k \geq 1$, we have*

$$D_{k-1}^T \left(x^k - x^*\right) \geq 0$$

Proof. [2] □

Theorem 3.6. [11] *Let x^* represent any solution of (1). For given $x^k \in R^n_{++}$ and $\beta_k > 0$, let $\tilde{x}^k$ and ξ^k satisfy the condition (14), then it holds*

$$\left(x^k - x^*\right)^T D_k \geq \psi(x^k) \geq \frac{1-2\eta}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2 \geq 0. \tag{29}$$

Proof. [2] □

To guarantee that $\bar{x}^k(\alpha_k)$ moves closer to the solution set compared to x^k, we introduce the following definition:

$$\Theta(\alpha_k) = \|x^k - x^*\|^2 - \|\bar{x}^k(\alpha_k) - x^*\|^2, \tag{30}$$

Theorem 3.7. *Let $\Theta(\alpha_k)$, D_k and $\psi(x^k)$ be defined by (30), (21) and (24) respectively, then $\forall x^* \in \Omega^*$ and $\alpha_k > 0$, we have*

$$\Theta(\alpha_k) \geq \Phi(\alpha_k), \tag{31}$$

where

$$\Phi(\alpha_k) = 2\alpha_k\psi(x^k) - \alpha^2\|D_k\|^2 \tag{32}$$

Proof.

$$\begin{aligned} \|\bar{x}^k(\alpha_k) - x^*\|^2 &= \|P_{R^n_+}[x^k - \alpha_k D_k] - x^*\|^2 \\ &\leq \|x^k - x^* - \alpha_k D_k\|^2 \\ &\leq \|x^k - x^*\|^2 - 2\alpha_k\psi(x^k) + \alpha_k^2\|D_k\|^2. \end{aligned} \tag{33}$$

Using the definition of $\Theta(\alpha_k)$ and $\Phi(\alpha_k)$, then (31) is proved. □

The function $\Phi(\alpha)$ evaluates the progress achieved during the kth iteration. A logical choice is to select a step length α_k that maximizes this progress. It is important to note that $\Phi(\alpha_k)$ represents a quadratic function of α, achieving its maximum at

$$\alpha_k^* = \frac{\psi(x^k)}{\|D_k\|^2} \tag{34}$$

and

$$\Phi(\alpha_k^*) = \alpha_k^*\psi(x^k). \tag{35}$$

In the following theorem, we demonstrate that both α_k^* and $\Phi(\alpha_k^*)$ maintain bounds strictly greater than zero. This result plays a pivotal role in establishing the proof of global convergence.

Theorem 3.8. *Given $x^k \in \mathbb{R}^n_+$ and $\beta_k > 0$, let $\tilde{x}^k$ and ξ^k satisfy the condition (14). Under these assumptions, we arrive at the following results :*

$$\alpha_k^* \geq \frac{1-2\eta}{4(1+\mu)} \tag{36}$$

and

$$\Phi(\alpha_k^*) \geq \frac{(1-2\eta)^2}{8(1+\mu)^2}\|x^k - \tilde{x}^k\|^2. \tag{37}$$

Proof. If $(x^k - \tilde{x}^k)^T\xi^k \leq 0$, since $\mu > 0$ it follows from (14), (20), (21) and (28) that

$$\begin{aligned}
\|D_k\|^2 &\leq \|d(x^k)\|^2 \\
&\leq \frac{1}{4}\|x^k - \tilde{x}^k\|^2 + \frac{1}{(1+\mu)^2}\|\xi^k\|^2 \\
&\leq \|x^k - \tilde{x}^k\|^2 + \|\xi^k\|^2 \\
&\leq 2\|x^k - \tilde{x}^k\|^2,
\end{aligned} \tag{38}$$

from (29) and (38), we obtain

$$\alpha_k^* = \frac{\psi(x^k)}{\|D_k\|^2} \geq \frac{1-2\eta}{4(1+\mu)}.$$

Otherwise, if $(x^k - \tilde{x}^k)^T\xi^k \geq 0$, it follows that

$$\begin{aligned}
\psi(x^k) &= \frac{1}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k \\
&\geq \frac{1}{1+\mu}\{\frac{1}{4}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k\frac{1}{4}\|x^k - \tilde{x}^k\|^2\} \\
&\geq \frac{1}{1+\mu}\{\frac{1}{16}\|x^k - \tilde{x}^k\|^2 + \frac{1}{4(1+\mu)}(x^k - \tilde{x}^k)^T\xi^k + \frac{1}{4(1+\mu)^2}\|\xi^k\|^2\} \\
&= \frac{1}{4(1+\mu)}\|d(x^k)\|^2 \\
&\geq \frac{1}{4(1+\mu)}\|D_k\|^2
\end{aligned}$$

and thus

$$\alpha_k^* \geq \frac{1}{4(1+\mu)} \geq \frac{1-2\eta}{4(1+\mu)}.$$

□

To ensure that $x^{k+1}(\delta_k)$ is closer to the solution set than x^k. For this purpose, we define

$$\tilde{\Theta}(\delta_k) = \|x^k - x^*\|^2 - \|x^{k+1}(\delta_k) - x^*\|^2, \tag{39}$$

Theorem 3.9. *Let $x^* \in \Omega^*$, then we have*

$$\tilde{\Theta}(\delta_k) \geq \tilde{\Phi}(\delta_k), \tag{40}$$

where

$$\tilde{\Phi}(\delta_k) = (1-\rho)(\delta_k \left\{\|g(x^k)\|^2 + \|x^k - x^*\|^2 - \|\bar{x}^k - x^*\|^2\right\} - \delta^2 \|\tilde{D}_k\|^2) \tag{41}$$

Proof. [1].□

Remark 3.10. *By using Theorem 3 and Theorem 1 in the reference* [1], *we get*

$$\delta_k \geq \frac{1}{2},$$

and

$$\tilde{\Theta}(\delta_k) \geq \frac{(1-\eta)^2}{(1+\mu)^2} \|x^k - \tilde{x}^k\|^2 \tag{42}$$

From the computational point of view, a relaxation factor $\gamma \in [1, 2)$ is preferable in the new iteration. It follows from (39) and (42) that there is a constant $c > 0$ such that

$$\|x^{k+1}(\gamma\delta_k) - x^*\|^2 \leq \|x^k - x^*\|^2 - c\|x^k - \tilde{x}^k\|^2 \qquad \forall x^* \in \Omega^{*\prime}$$

The following result can be proved by similar arguments as those in [1]. Hence the proof will be omitted.

Theorem 3.11. [1, 7] *If $\inf\limits_{k=0}^{\infty} \beta_k = \beta > 0$, then the sequence $\{x^k\}$ generated by the proposed method converges to some x^∞ which is a solution of the NCP.*

4. Preliminary Computational Results

In numerical experiments, determining the value of the approximate solution $\tilde{x}^k$ is essential. In the specific scenario where

$$\xi^k = \beta_k(F(\tilde{x}^k) - F(x^k)),$$

equation (13) can be rewritten as an equivalent system of nonlinear equations

$$\beta_k F(x^k) + \frac{1}{2}(\tilde{x}^k - x^k) + \mu(x^k - X_k(\sqrt{\tilde{x}^k})^{-1}) = 0, \tag{43}$$

hence

$$\frac{1}{2}\tilde{x}_j^k - \mu \frac{\left(\sqrt{x_j^k}\right)^3}{\sqrt{\tilde{x}_j^k}} + \left(\beta_k F_j(x^k) - \frac{1}{2}x_j^k + \mu x_j^k\right) = 0, \qquad j = 1, ..., n.$$

Then

$$\frac{1}{2}\tilde{x}_j^k - \mu \frac{\left(\sqrt{x_j^k}\right)^3}{\sqrt{\tilde{x}_j^k}} + \left(\beta_k F_j(x^k) - \frac{1}{2}x_j^k + \mu \frac{\left(\sqrt{x_j^k}\right)^3}{\sqrt{x_j^k}}\right) = 0, \qquad j = 1, ..., n.$$

The iterative procedure of the Newton method for addressing the specified problem can be outlined as follows:

$$\tilde{x_j}^k := x_j{}^k - \frac{2\beta_k}{1+\mu} F_j\left(x^k\right)$$

The solution satisfies $\tilde{x}^k > 0$. To prevent non-positive values of $\tilde{x_j}^k$ during the iteration process, we implement the following approach

$$\tilde{x_j}^k := \max\left\{ x_j{}^k - \frac{2\beta_k}{1+\mu} F_j\left(x^k\right), 0 \right\}, \qquad j = 1, ..., n.$$

To test the suggested algorithm, we consider the NCP :

$$x \geq 0, \qquad F(x) \geq 0, \qquad x^T F(x) = 0, \tag{44}$$

where

$$F(x) = D(x) + Mx + q,$$

with $D(x)$ representing the nonlinear component and $Mx + q$ denoting the linear component of $F(x)$.

The linear component of the test problems is constructed in a manner similar to the approach outlined by Harker and Pang [4]. Specifically, the matrix M is formed as $M = A^T A + B$, with A being an $n \times n$ matrix whose entries are randomly selected within the range $(-5, +5)$. Additionally, B is a skew-symmetric matrix generated under the same conditions. The vector q is drawn from a uniform distribution within the interval $(-500, 500)$. Regarding $D(x)$, which represents the nonlinear part of $F(x)$, its components are defined as $D_j(x) = d_j * \arctan(x_j)$, where d_j is a random variable within the range $(0, 1)$. Problems of a similar nature have been previously explored in [12] and [13].

The iterations begin with $x^1 = (1, \ldots, 1)^T$ and are terminated once the condition

$$\|\min(x^k, F(x^k))\|_\infty \leq 10^{-7},$$

is satisfied. All codes were implemented in Matlab, and the proposed method is compared with those presented in [14]. The test results for problem (44) are summarized in Tables 4.1 and 4.2. Here, k represents the number of iterations, and l refers to the count of mapping calculations for F.

Comparison to the method in [14] using only the first and second step of the proposed method:

Table 4.1 Numerical results for problem (44)
with $q \in (-500,\ 500)$

n	Algorithm in [14]			Suggested approach		
	k	l	CPU time in seconds	k	l	CPU time in seconds
200	371	792	0.105	243	496	0.035
300	410	875	0.087	269	548	0.048
400	417	886	0.125	290	595	0.081
500	455	952	0.197	318	646	0.142
700	441	922	0.841	294	598	0.475
800	376	800	0.942	264	544	0.645
1000	426	895	1.687	287	586	1.107

Comparison to the method in [1] using the three-steps proposed method:

Table 4.2 Numerical results for problem (44)
with $q \in (-500,\ 500)$

n	Algorithm in [1]			Suggested approach		
	k	l	CPU time in seconds	k	l	CPU time in seconds
200	264	572	0.065	135	278	0.007
300	259	561	0.07	144	297	0.011
400	333	720	0.13	162	333	0.015
500	336	726	0.18	180	367	0.021
700	279	605	0.31	165	339	0.03
1000	295	638	1.17	168	345	0.15

5. Concluding remarks

Tables 4.1 and 4.2 illustrate the enhanced efficiency of the proposed method. The numerical findings reveal that this method substantially decreases the iteration count and computational effort needed to compute the function F. This paper introduces a novel category of proximal algorithms aimed at addressing nonlinear complementarity problems, utilizing a novel SRQP term and a two-stage conjugate gradient algorithm. This approach integrates two refined descent directions with two optimal step sizes to achieve multiplicative improvements in each iteration, considerably speeding up the convergence process.

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REFERENCES

[1] A. Bnouhachem, A. Ou-yassine, M. A. Noor, and G. Lakhnati, “Modified lqp method with a new search direction for nonlinear complementarity problems,” *Applied Mathematics and Information Sciences*, vol. 10, no. 4, pp. 1375–1383, 2016. http://dx.doi.org/10.18576/amis/100416.

[2] A. Bnouhachem and A. Ou-yassine, “Square quadratic proximal method for nonlinear complementarity problems,” *Communications of the Korean Mathematical Society*, vol. 34, no. 2, pp. 671–684, 2019. https://doi.org/10.4134/CKMS.c180109.

[3] M. C. Ferris and J. S. Pang, “Engineering and economic applications of complementarity problems,” *SIAM Review*, vol. 39, pp. 669–713, 1997. https://doi.org/10.1137/S0036144595285963.

[4] P. T. Harker and J. S. Pang, “A damped-newton method for the linear complementarity problem,” *Lectures in Applied Mathematics*, vol. 26, pp. 265–284, 1990. https://www.researchgate.net/publication/263734739.

[5] A. Auslender, M. Teboulle, and S. Ben-Tiba, “Interior proximal and multiplier methods based on second order homogenous kernels,” *Mathematics of Operations Research*, vol. 24, pp. 646–668, 1999. https://doi.org/10.1287/moor.24.3.645.

[6] A. Auslender, M. Teboulle, and S. Ben-Tiba, “A logarithmic-quadratic proximal method for variational inequalities,” *Computational Optimization and Applications*, vol. 12, pp. 31–40, 1999. https://doi.org/10.1023/A:1008607511915.

[7] A. Noor and A. Bnouhachem, “Modified proximal-point method for nonlinear complementarity problems,” *Journal of Computational and Applied Mathematics*, vol. 197, no. 2, pp. 395–405, 2006. https://doi.org/10.1016/j.cam.2005.11.021.

[8] A. Ou-yassine, A. Bnouhachem, and F. Benssi, “Lqp method with a new optimal step size rule for nonlinear complementarity problems,” *Journal of Inequalities and Applications*, vol. 2015, no. 216, 2015. https://doi.org/10.1186/s13660-015-0736-4.

[9] A. Ou-yassine and S. Krit, “An introduction of a new square quadratic proximal point scalarization method for multiobjective programming,” *IAENG International Journal of Applied Mathematics*, vol. 51, no. 2, 2024.

[10] D. Han, “A new class of projection and contraction methods for solving variational inequality problems,” *Computers and Mathematics with Applications*, vol. 51, pp. 125–140, 2006. https://doi.org/10.1016/j.camwa.2005.11.029.

[11] X. M. Yuan, “A new criterion for the inexact logarithmic-quadratic proximal method and its derived hybrid methods,” *Journal of Global Optimization*, vol. 40, pp. 529–543, 2008. https://doi.org/10.1007/s10898-006-9123-z.

[12] P. Marcotte and J. P. Dussault, “A note on a globally convergent newton method for solving variational inequalities,” *Operations Research Letters*, vol. 6, pp. 35–42, 1987. https://doi.org/10.1016/0167-6377(87)90007-1.

[13] K. Taji, M. Fukushima, and T. Ibaraki, “A globally convergent newton method for solving strongly monotone variational inequalities,” *Mathematical Programming*, vol. 58, pp. 369–383, 1993. https://doi.org/10.1007/BF01581276.

[14] X. M. Yuan, “The prediction-correction approach to nonlinear complementarity problems,” *European Journal of Operational Research*, vol. 176, pp. 1357–1370, 2007. https://doi.org/10.1016/j.ejor.2005.11.006.