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Atom of the Lattice of All $N-$Radicals

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Abstract. In this paper, we study atomic elements in the lattices of radical classes. A minimal element of the lattice of all N-radicals (respectively, supernilpotent radicals and special radicals) is called an N-atom (respectively, a supernilpotent atom and a special atom). We construct a class of N-atoms generated by prime essential rings and investigate their structural properties. We show that these N-atoms are closely related to supernilpotent atoms and special atoms, and we clarify the precise relationships among these three types of atoms. In particular, we prove that for a prime essential ring R, the associated radicals $\widetilde{l}_R$ and $\overline{l}_R$ generate, respectively, a special atom and a supernilpotent atom. This result provides a positive answer to an open question concerning which prime essential rings give rise to atomic elements in the lattices of all $N-$ radical, special radicals and supernilpotent radicals.

Key words and Phrases: $N-$atom, $N-$radical, Prime Essential Ring, Special atom, Supernilpotent atom.

1. INTRODUCTION

Radical theory has long played a central role in the structural investigation of associative rings, particularly in understanding how classes of rings can be characterized through hereditary, strong, and special radical properties. Foundational studies by Divinsky, Krempa, and Sulinski [1] demonstrated the importance of strong radical properties in associative rings, while the systematic exposition of radical constructions and lattice-theoretic approaches was later consolidated in [2].

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Within this framework, supernilpotent radicals and their associated lattice structures have attracted considerable attention because of their close relationship with hereditary and special radicals.

One of the fundamental directions in radical theory concerns the study of atoms in lattices of radicals. The investigation of atoms of supernilpotent radicals was initiated by France-Jackson in [3], where minimal supernilpotent radicals properly containing the prime radical were characterized through the existence of $*$-rings. Further developments by Puczylowski and Roszkowska [4] clarified several structural properties of atoms in lattices of radicals, while Korolczuk [5] investigated the lattice structure of special radicals through the introduction of $*$-rings. These results established a strong connection between prime rings, special classes, and the existence of minimal radical classes.

The relationship between special radicals and prime essential rings has also become an important topic in radical theory. Gardner and Stewart [6] introduced prime essential rings and demonstrated several of their structural properties. Later, France-Jackson and Groenewald [7] showed that certain rings can generate both supernilpotent atoms and special atoms. This direction was further strengthened by Wahyuni, Wijayanti, and France-Jackson [8], who constructed a prime essential ring generating a special atom. These developments indicate that prime essential rings provide an effective mechanism for constructing minimal radicals and studying their lattice-theoretic behavior.

Parallel to these developments, the study of normal radicals and N-radicals has evolved significantly through the works of Sands [9, 10], Jaegermann [11], and Jaegermann and Sands [12]. In particular, the characterization of supernilpotent radicals that are simultaneously hereditary and strong revealed deep relationships among normal radicals, N-radicals, and A-radicals. Moreover, Beidar and Salavova [13] investigated the lattice structure of N-radicals and established several important properties of the lattice $\mathbb{L}_N$. Despite these substantial developments, the relationship between prime essential rings, supernilpotent atoms, and N-atoms remains only partially understood.

N-radicals form an important class of radicals and have been investigated by many prominent authors. In 1983, Beidar and Salavova discussed the lattice of $N-$radicals in [13]. Moreover, some further properties of normal radicals are explained in [11]. In fact, a supernilpotent radical that is normal forms an $N-$radical. The relationship between normal radicals and $N-$radicals are explained in [9] and [10]. On the other hand, Jaegerman and Sands also gave additional information about relationship between normal radicals, N-radicals, and $A-$radicals in [12].

It is well known [2] that many important radicals, such as the prime radical β, the locally nilpotent radical $\mathcal{L}$, and the Jacobson radical $\mathcal{J}$, are N-radicals. In contrast, the Brown–McCoy radical $\mathcal{G}$ is not an N-radical. Moreover, the famous Köthe problem, which asks whether the nilradical $\mathcal{N}$ is left strong, is equivalent to the question of whether $\mathcal{N}$ is an N-radical. Furthermore, N-radicals are linked to some other long standing open problems in radical theory, as shown in [14] and [15].

So, there is a good reason for studying N-radicals. A theoretical implementation of the Jacobson radical $\mathcal{J}$ as $N-$radical can be seen in [16]. While [8] established that certain prime essential rings generate special atoms, our results explained in this paper extend this by: (1) demonstrating that some prime essential rings can generate N-atoms, supernilpotent atoms, and special atoms—and (2) revisiting the Questions 1 and 2 in [8] through the lens of N-radicals. Moreover, as one of the famous radical class, the theoretical implementation of Jacobson radical $\mathcal{J}$ can be seen in [17].

Recent studies indicate that problems related to special radicals and supernilpotent radicals naturally extend to broader algebraic settings. For example, Prasetyo et al. [18] generalized several radical-theoretic constructions into module theory through weakly special classes of modules. Moreover, Yuwaningsih et al. [19] extended several classical concepts of module homomorphisms into the framework of (R, S)-modules, showing that radical-theoretic and module-theoretic structures continue to develop in more generalized algebraic settings. In addition, Prasetyo et al. [20] constructed supernilpotent radical classes using topological methods related to Tychonoff spaces. Connections between algebraic and topological structures also appear in the work of Burgess and Raphael [21]. Furthermore, recent investigations on β-classes in [15] and the prime radical β in [22] revisited several open problems related to special radicals and hereditary constructions. These developments suggest that the study of N-radicals remains highly relevant, both structurally and in terms of possible generalizations.

Motivated by the above developments, this paper investigates N-atoms as minimal elements of the lattice $\mathbb{L}_N$ and studies their relationship with supernilpotent atoms in the lattice $\mathbb{K}$. In particular, we revisit Questions 1 and 2 posed in [8] from the perspective of N-radicals and establish conditions under which certain prime essential rings generate N-atoms, supernilpotent atoms, and special atoms simultaneously. Hence, the present work extends several earlier results concerning special atoms and provides additional insight into the structure of lattices of N-radicals.

2. PRELIMINARY

In this paper, all rings are associative, and all classes of rings are closed under isomorphisms and contain the one-element ring $R = \{0\}$. The fundamental definitions and properties of radicals can be found in [2]. We will use the following definitions and notations. If R is a ring, then $I \lhd R$ (respectively, $I <_l R$, $I <_r R$) means that I is an ideal (respectively, a left ideal, a right ideal) of R. If A is a subring of R, we use A^* to denote the ideal of R generated by A. R^0 denotes the ring defined on the same underlying additive group as that of a ring R with zero multiplication [2].

A subring A of a ring R is said to be accessible (respectively, left accessible) if there exist subrings $A = A_0 \subseteq A_1 \subseteq ... \subseteq A_n = R$ of R such that A_i is an ideal (respectively, left ideal) of A_{i+1}, for every $i = 1, 2, ..., n-1$ [2].

A class μ of rings is said to be hereditary (respectively, left hereditary, right hereditary) if for every ring $R \in \mu$, we have $I \in \mu$ for every ideal (respectively, left ideal, right ideal) I of R. Let α be a class of rings. The class α is called a radical class if α satisfies the following conditions [2]:

(i) For any ring $A \in \alpha$, then $A/I \in \alpha$ for every $I \lhd A$.
(ii) For every ring A, $\alpha(A) \in \alpha$ where $\alpha(A) = \Sigma(I \lhd A | I \in \alpha)$.
(iii) $\alpha(A/\alpha(A)) = 0$ for every ring A.

For any radical α and any ring R, $\alpha(R)$ denotes the α-radical of R, that is, the largest ideal of R that belongs to α. A hereditary class μ of prime rings is called a special class if μ is closed under essential extensions [2].

The smallest radical containing a given class μ of rings will be denoted by $l_{(\mu)}$. In a special case, the smallest radical $l_{\mathcal{Z}}$ containing the class $\mathcal{Z} = \{A : A^2 = 0\}$ of all zero rings is precisely the prime radical and it will be denoted by β [2].

Some results related to problems in ring radical theory have also been generalized in module theory, for example [18]. In addition, [20] explores the construction of a supernilpotent radical class using topological concept, namely a Thyconoff space. The definition of a Thyconoff space will be given later in the Results and Discussion Section.

A left hereditary and a left strong radical α that contains the prime radical β is called an N-radical. It is well known [11] that a supernilpotent radical α is an N-radical if and only if α is right hereditary and right strong. Moreover, the collection $\mathbb{L}_N$ of all N-radicals forms a sublattice of the lattice $\mathbb{K}$ of all supernilpotent radicals.

We now present the following useful previous results from [23] and [4]:

Theorem 2.1. *[23] If $\mu \subseteq \{R : R^0 \in \mu\}$ is a hereditary (respectively, left hereditary) class of rings, then the radical $l_{s(\mu)}$ is hereditary (respectively, left hereditary).*

Theorem 2.2. *[4] Let μ be a homomorphically closed class of rings and let R be a semiprime ring such that $l_{s(\mu)}(R) \neq 0$. Then*

(1) R contains a left accessible subring S such that $0 \neq S/\beta(S) \in \mu$.
(2) If the class μ is hereditary, then R contains a left ideal S such that $0 \neq S/\beta(S) \in \mu$.

We recall the following useful observation.

Definition 2.3. *[2] A hereditary radical $\alpha \supseteq \beta$ is called a supernilpotent radical. The class $\mathcal{U}(\mu) = \{A | A/I \notin \mu$ for every nonzero homomorphic image A/I of $A\}$ of rings of a given special class μ forms a radical class and it is called a special radical. A radical α is called left strong (respectively, right strong), if for every ring R and every left ideal (respectively, right ideal) $L \in \alpha$ of R, we have $L^* \in \alpha$, where L^* is the ideal of R generated by L.*

It is well known [1] that for any homomorphically closed class μ of rings, there exists the smallest strong radical $l_{s(\mu)}$ containing μ.

Lemma 2.4. *Let $\alpha \supsetneqq \beta$ be a supernilpotent radical (respectively, an N-radical). Then α is a supernilpotent atom (respectively, an N-atom) if and only if for every ring $A \in \alpha$ and every supernilpotent radical (respectively N-radical) γ, we have either $\gamma(A) = A$ or $\gamma(A) = \beta(A)$.*

Proof. $\Rightarrow$: Assume that $\alpha \supsetneqq \beta$ is a supernilpotent atom and let $A \in \alpha$ and $\gamma \in \mathbb{K}$. If $\alpha \subseteq \gamma$, then $A \in \gamma$ and then $\gamma(A) = A$. Otherwise, we have $\beta \subseteq \alpha \wedge \gamma \subsetneqq \alpha$, which implies that $\beta = \alpha \wedge \gamma$ since $\alpha \wedge \gamma \in \mathbb{K}$ and α is a supernilpotent atom. But, being a supernilpotent atom, α is hereditary. So, since $\gamma(A) \lhd A \in \alpha$, it follows that $\gamma(A) \in \alpha$ which implies that $\gamma(A) \in \alpha \wedge \gamma = \beta$. This means that $\gamma(A) \subseteq \beta(A)$ and, since $\beta(A) \in \beta$ and $\beta \subseteq \gamma$, it follows that $\beta(A) \in \gamma$. Hence, $\beta(A) \subseteq \gamma(A)$. So, $\gamma(A) = \beta(A)$.

$\Leftarrow$: Assume that $\alpha \supsetneqq \beta$ is a supernilpotent radical such that for every ring $A \in \alpha$ and every supernilpotent radical γ, we have either $\gamma(A) = A$ or $\gamma(A) = \beta(A)$. Suppose that there exists a supernilpotent radical γ such that $\beta \subsetneq \gamma \subsetneq \alpha$. Then, there exist rings $B \in \gamma$ and $C \in \alpha$ such that $\beta(B) \neq B$ and $\gamma(C) \neq C$. Then the ring $D := B \oplus C \in \alpha$ and $\gamma(D) = B \oplus \gamma(C)$ which implies that $\gamma(D) \neq \beta(D)$ and $\gamma(D) \neq D$ and we have a contradiction. Hence, α is a supernilpotent atom.

The same proof applies to N-radicals by replacing $\mathbb{K}$ with $\mathbb{L}_N$ throughout. $\square$

3. RESULTS AND DISCUSSION

Let α be a supernilpotent atom (respectively, an N-atom). Then $\beta \subsetneqq \alpha$. Consequently, α contains the smallest supernilpotent radical $\overline{l}_A$ (respectively, the smallest N-radical $\widetilde{l}_A$) containing A, for every semiprime ring $A \in \alpha$. Therefore, in studying supernilpotent atoms (respectively, N-atoms), it is important to know what rings the radicals $\overline{l}_A$ (respectively, $\widetilde{l}_A$) consist of. To describe such rings, we will need the following well known fact which follows from the lower radical construction:

Proposition 3.1. *[2] If μ is a homomorphically closed class of rings, then $l_{(\mu)} = \{R :$ Every nonzero homomorphic image of R has a nonzero accessible subring in $\mu\}$.*

We therefore have

Proposition 3.2. *[2] Let $\overline{l}_A$ be the supernilpotent radical generated by $\mu := \{A\}$. Then $\overline{l}_A = l_{(H(IA)\cup\beta)}$, where $H(IA) = \{R : R \simeq S/K,$ for some accessible subring S of $A\}$. Thus we have*

$$\overline{l}_A = l_{(H(IA)\cup\beta)}$$

Consider a class μ of rings. The construction of the smallest special radical $\widehat{l}_\mu$ containing μ has been already explained in [8]. Now, we will give an explanation of the construction of the smallest supernilpotent radical $\bar{l}_A$ containing a given ring A. It follows from Proposition 3.2 that the semisimple class $\mathcal{S}(\bar{l}_A)$ of the smallest hereditary and homomorphically closed class $\mathcal{S}(\bar{l}_A)$ of rings containing A is described below

$$\mathcal{S}(\bar{l}_A = l_{(H(IA)\cup\beta)}).$$

In fact, the class $\mathcal{S}(\bar{l}_A)$ is essentially closed, whence the class

$$\mathcal{S}(\bar{l}_A = l_{(H(IA)\cup\beta))} \cap \pi,$$

is essentially closed, where π is the class of all prime rings. It is clear that the class π of all prime rings is a special class since it is essentially closed and hereditary for nonzero ideals. We therefore have that the smallest special radical $\bar{l}_A$ containing A is equal to

$$\mathcal{U}(\mathcal{S}(\bar{l}_A = l_{(H(IA)\cup\beta))} \cap \pi)).$$

It was proved in [3] that, if A is a nonzero $*$-ring, that is, a semiprime ring A such that $A/I \in \beta$ for every nonzero ideal I of A [3], then $\bar{l}_A$ is a supernilpotent atom and

$$\bar{l}_A = l_{(\beta\cup IA)} = l_{(\beta\cup\{R:R\simeq S \text{ for some accessible subring } S \text{ of } A\})}.$$

Moreover, it was shown in [7] that every nonzero $**$-ring A also generates a supernilpotent atom $\bar{l}_A$, where a prime ring A is called $**$-ring if the smallest special class containing A is closed under semiprime homomorphic images of A.

Our next result shows how to build the smallest N-radical $\widetilde{l}_R$ containing a nonzero semiprime ring R. This construction is similar to the construction of the smallest supernilpotent radical $\bar{l}_R$ containing a nonzero semiprime ring R that was given above and it follows directly from Theorem 2.2 and Theorem 2.1 as well as from the very well known fact that the lower radical generated by a left hereditary and homomorphically closed class of rings is left hereditary.

Corollary 3.3. *Let R be a nonzero semiprime ring and let $H(I_l(R))$ denote the smallest left hereditary and homomorphically closed class of rings containing R. Then $H(I_l(R)) = \{A : A \simeq S/K$ for some left accessible subring S of $R\}$ and $\widetilde{l}_R = l_{s(H(I_l(R))\cup\beta)}$.*

Suppose R is a nonzero semiprime ring. Then the following characterization of supernilpotent atoms(respectively, N-atoms of the form $\widetilde{l}_R$) follows directly from the definition of a supernilpotent atom (respectively, an N-atom):

Lemma 3.4. *Let R be a nonzero semiprime ring. The following conditions are equivalent:*

(i) $\overline{l}_R$ *is a supernilpotent atom (respectively,* $\widetilde{l}_R$ *is an* N*-atom).*

(ii) $\overline{l}_R = \overline{l}_{\overline{R}}$ *(respectively,* $\widetilde{l}_R = \widetilde{l}_{\overline{R}}$*), for every nonzero semiprime homomorphic image* $\overline{R}$ *of* R*.*

(iii) $\overline{l}_R = \overline{l}_S$ *(respectively,* $\widetilde{l}_R = \widetilde{l}_S$*), for every nonzero semiprime ring* $S \in \overline{l}_R$ *(respectively,* $S \in \widetilde{l}_R$*).*

(iv) $\overline{l}_R = \overline{l}_{\overline{R}}$ *(respectively,* $\widetilde{l}_R = \widetilde{l}_{\overline{R}}$*), for every nonzero prime homomorphic image* $\overline{R}$ *of* R*.*

(v) $\overline{l}_R = \overline{l}_S$ *(respectively,* $\widetilde{l}_R = \widetilde{l}_S$*), for every nonzero prime ring* $S \in \overline{l}_R$ *(respectively,* $S \in \widetilde{l}_R$*).*

Our main result describes the relationship between supernilpotent atoms and N-atoms. It also shows that N-atoms exist and how to construct some N-atoms.

Theorem 3.5. *If R is a semiprime ring such that* $\overline{l}_R$ *is a supernilpotent atom, then* $\widetilde{l}_R$ *is an* N*-atom.*

Proof. We will prove this theorem into three claims as follows.

(1) Claim I: Assume that R is a semiprime ring such that $\overline{l}_R$ is a supernilpotent atom and let A be any nonzero semiprime ring such that $A \in \widetilde{l}_R$. Then $\widetilde{l}_A \subseteq \widetilde{l}_R$. We are going to prove that A contains a left ideal S such that $0 \neq S/\beta(S) \in H(I_l(R)) \cup \beta$. In view of Lemma 3.4 (iii), it is sufficient to show that $\widetilde{l}_R \subseteq \widetilde{l}_A$. Now, $\widetilde{l}_R = l_{s(H(I_l(R))\cup\beta)}$ and the class $H(I_l(R)) \cup \beta$ is homomorphically closed and left hereditary (hence hereditary). So, since $0 \neq A \in \widetilde{l}_R$, it follows from Theorem 2.2 (2) that A contains a left ideal S such that $0 \neq S/\beta(S) \in H(I_l(R)) \cup \beta$, that is, $S/\beta(S) \simeq J/I$, for some $0 \neq J = J_1 < J_2 < ... < J_m = R$.

(2) Claim II: We will show that $\overline{S}/\beta\left(\overline{S}\right) \simeq T/\beta(T)$, where $\bar{S}$ is a homomorphic image of S and $T \lhd S$. Observe that the ring A contains a left ideal P such that $0 \neq P/\beta(P) \simeq M/I$, for some nonzero left ideal M of R. Indeed, if $J^m = J$, then $J \subseteq J_m J_{m-1} ... J_1 \subseteq J$ which implies that $J = J_m J_{m-1} ... J_1$ is a left ideal of R. Then $M := J$ and $P := S$ fit the bill. If $J^m \neq J$, then consider $K := J_m J_{m-1} ... J_1$. Clearly, $K < R$ and $K \lhd J$ which implies that $((K + I)/I) \lhd J/I$. Moreover, $(K + I)/I \neq 0$ since, otherwise, $K + I = I$ which gives $J^m \lhd K \lhd I$. But this implies that $J/I \simeq (J/K)/(I/K)$ is a nonzero nilpotent ring isomorphic to a nonzero semiprime ring $S/\beta(S)$ which cannot be. Now, since $S/\beta(S) \simeq J/I$ and $(K + I) < R$, it follows that

$$0 \neq (K + I)/I \simeq (T + \beta(S))/(\beta(S)) \simeq T/(T \cap (\beta(S))),$$

for some $T \lhd S$ such that $T < A$. But, $\beta(T) = T \cap (\beta(S))$ since β is hereditary. Hence $0 \neq (K + I)/I \simeq T/\beta(T)$. Then take $P := T$ and $M := K + I$.

Since M/I is a nonzero semiprime ring, it follows from Lemma 2.2 of [14] that there exists a semiprime homomorphic image $\overline{R}$ of R and a left ideal $\overline{S}$ of $\overline{R}$ such that $M/I \simeq \overline{S}/\beta\left(\overline{S}\right)$ so that $\overline{S}/\beta\left(\overline{S}\right) \simeq T/\beta\left(T\right)$.

(3) Claim III: We will show that $\widetilde{l}_R$ is an N-atom by proving $\widetilde{l}_R$ is an atom of the lattice of all $N-$radicals. Since $\overline{S}/\beta\left(\overline{S}\right) \in \widetilde{l}_{\overline{S}/\beta\left(\overline{S}\right)}$, $\beta\left(\overline{S}\right) \in \beta \subseteq \widetilde{l}_{\overline{S}/\beta\left(\overline{S}\right)}$ and radicals are closed under extensions, it follows that $\overline{S} \in \widetilde{l}_{\overline{S}/\beta\left(\overline{S}\right)}$ which implies that $\overline{S} \subseteq \widetilde{l}_{\overline{S}/\beta\left(\overline{S}\right)}\left(\overline{R}\right)$ because $\overline{S} < \overline{R}$ and $\widetilde{l}_{\overline{S}/\beta\left(\overline{S}\right)}$ is left strong. So, $\widetilde{l}_{\overline{S}/\beta\left(\overline{S}\right)}\left(\overline{R}\right) \neq 0$ because $0 \neq \overline{S}$. On the other hand, the semiprime ring $\overline{R} \in \overline{l}_R$ so, since $\overline{l}_R$ is a supernilpotent atom and $\widetilde{l}_{\overline{S}/\beta\left(\overline{S}\right)}$ is a supernilpotent radical, it follows from Lemma 2.4 that $\overline{R} \in \widetilde{l}_{\overline{S}/\beta\left(\overline{S}\right)}$ and so $\widetilde{l}_{\overline{R}} \subseteq \widetilde{l}_{\overline{S}/\beta\left(\overline{S}\right)}$. Moreover, since $\overline{l}_R$ is a supernilpotent atom containing the nonzero semiprime ring $\overline{R}$, it follows from Lemma 3.4 (iii) that $\overline{l}_R = \overline{l}_{\overline{R}}$. So, $R \in \overline{l}_{\overline{R}}$. But, since N-radicals are supernilpotent, it follows that $\overline{l}_{\overline{R}} \subseteq \widetilde{l}_{\overline{R}}$. Consequently, $R \in \widetilde{l}_{\overline{R}}$ which gives $\widetilde{l}_R \subseteq \widetilde{l}_{\overline{R}}$. So, as $\widetilde{l}_{\overline{R}} \subseteq \widetilde{l}_R$, it follows that $\widetilde{l}_{\overline{R}} = \widetilde{l}_{\overline{R}}$. Moreover, since $T < A \in \widetilde{l}_R$ and N-radicals are left hereditary and homomorphically closed, we have $T/\beta\left(T\right) \in \widetilde{l}_R$ which implies that $\widetilde{l}_{T/\beta(T)} \subseteq \widetilde{l}_A$. Thus, recapping all our observations, we have $\widetilde{l}_R = \widetilde{l}_{\overline{R}} \subseteq \widetilde{l}_{\overline{S}/\beta\left(\overline{S}\right)} = \widetilde{l}_{T/\beta(T)} \subseteq \widetilde{l}_A$, which ends the proof.

□

As a direct consequence of Theorem 3.5, we get the following corollary:

Corollary 3.6. *If R is a nonzero $*$-ring, then $\widetilde{l}_R$ is an N-atom. Furthermore, if R is a nonzero $**$-ring, then $\widetilde{l}_R$ is also an N-atom.*

Proof. Let R be a $*-$ring. Since the class $*$ of all $*-$rings is strictly contained in the class $**$ of all $**-$rings [7], R is a $**-$ring. It follows from Theorem 6 in [7] that $\overline{l}_R$ is a supernilpotent atom. Moreover, it follows from Theorem 3.5 that $\widetilde{l}_R$ is an N-atom. By using the same argument, we can directly see $\widetilde{l}_R$ is an N-atom if R is a $**-$ring. □

Theorem 3.5 also implies that some nonzero semiprime rings do not generate N-atoms, as our next results show. To identify such rings, recall that a semiprime ring R is called prime essential [6] if for every nonzero prime ideal P of R, we have $P \cap I \neq 0$ whenever I is a nonzero two-sided ideal of R.

Lemma 3.7. *[6] Let S be a linearly ordered set with no greatest element, with least element e and which is such that every interval $[x,y]$, $x<y$, has cardinality κ. Then S is a semi-group with multiplication defined by $xy = max\{x,y\}$. Let A be a nonzero semiprime ring.*

(i). The semigroup ring $A(S)$ is a subdirect product of copies of A.
(ii). $A(S)$ is semiprime.
(iii). If Q is a prime ideal of $A(S)$, then $P := \{a \in A : ae \in Q\}$ is a prime ideal of A and $A(S)/Q \simeq A/P$.
(iv). If I is a nonzero accessible subring of $A(S)$, then the cardinality $|I|$ of I is at least κ.

Remark 3.1. *[6] For every cardinal $\kappa > 1$, the set $W(k)$ of all finite words made from a (well-ordered) alphabet of cardinality κ, lexicographically ordered, is a linearly ordered set satisfying the assumptions of Lemma 3.7.*

Theorem 3.8. *For every nonzero semiprime ring A and any cardinal $\kappa > |A|$, the radical $\overline{l}_{A(W(k))}$ is not a supernilpotent atom.*

Proof. By Lemma 3.7, $A(W(k))$ is a subdirect product of copies of A. This means that $A(W(k))$ contains a family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals such that $\cap_{\lambda\in\Lambda} I_\lambda = \{0\}$ and the factor ring $(A(W(k)))/I_\lambda$ is isomorphic to A for every $\lambda \in \Lambda$. If all $I_\lambda = A$, then $\{0\} = \cap_{\lambda\in\Lambda} I_\lambda = A$ which is not the case. So there exists at least one $\lambda_0 \in \Lambda$ such that $I_{\lambda_0} \neq A$. Then $(A(W(k)))/I_{\lambda_0} \simeq A$ which means that A is a nonzero homomorphic image of $A(W(k))$. This implies that $A \in \overline{l}_{A(W(k))}$ since $A(W(k)) \in \overline{l}_{A(W(k))}$ and radical classes are homomorphically closed. Consequently, $\overline{l}_A \subseteq \overline{l}_{A(W(k))}$. Moreover, $\beta \subsetneqq \overline{l}_A$ because $A \in \overline{l}_A \cap \mathcal{S}(\beta)$. So, if $\overline{l}_{A(W(k))}$ were a supernilpotent atom, then we would have $\overline{l}_A = \overline{l}_{A(W(k))}$ which implies that $A(W(k)) \in \overline{l}_A$. Then, it follows from Proposition 3.1 and Proposition 3.2 that $A(W(k))$ contains a nonzero accessible subring $R \simeq S/K$, for some accessible subring S of A. But then $|R| = |S/K| \leq |S| \leq |A| < \kappa$ which contradicts part (iv) of Lemma 3.7. □

Another example of a nonzero prime essential ring which does not generate a supernilpotent atom can be found in [8]. Namely,

Corollary 3.9. *[8] Let $\mathbb{Q}$ be the field of all rational numbers. Then the set $B := \{b_i : i \in \mathbb{Q}\}$ forms a semigroup with respect to the multiplication given by $b_i b_j := b_{\max(i,j)}$. Let $R := \mathbb{Z}_2[B]$ be the semigroup ring of the semigroup B over the two element field $\mathbb{Z}_2$. Then $\overline{l}_R$ is not a supernilpotent atom. Consequently, $\widetilde{l}_R$ is not an N-atom.*

In the next result, we refer to [20] and [21] to consider a concept from analysis, namely a Tychonoff space, that is, a completely regular Hausdorff space. As a further example of a prime essential ring that does not generate a supernilpotent atom or a special atom, we obtain the following corollary.

Corollary 3.10. *Let X be a Tychonoff space that does not contain any isolated points. Define a set $C(X)$ on $X, C(X) = \{f | f$ is a continuous real-valued function defined on $X\}$ and let κ be any cardinal such that $\kappa > |C(X)|$. Then the radical $\bar{l}_{C(X)(W(k))}$ is not a supernilpotent atom. Moreover, $\widetilde{l}_{C(X)(W(k))}$ is not an N-atom.*

Proof. It follows from Proposition 7 in [20] that $C(X)$ is a prime essential ring. Thus $C(X)$ is a semiprime ring. On the other hand, it follows from Theorem 3.8 that $\bar{l}_{C(X)(W(k))}$ is not a supernilpotent atom and $\widetilde{l}_{C(X)(W(k))}$ is not an N-atom. $\square$

An example of Tychonoff space can be seen in [20]. Furthermore, there exist prime rings R that generate supernilpotent atoms $\bar{l}_R \subsetneq \widetilde{l}_R$, as our next two examples show. Therefore, in general, $\widetilde{l}_R$ is not a supernilpotent atom.

Example 3.11. *Let F be a field and let $R := M_2(F)$ be the ring of all 2×2 matrices with entries from F. Then R is a nonzero simple prime ring and, it follows from [3] that R generates a supernilpotent atom $\bar{l}_R = l_{(\beta \cup \{R\})}$. Since every N-radical is supernilpotent, it therefore follows that $\bar{l}_R \subseteq \widetilde{l}_R$. Moreover, $\widetilde{l}_R$ is left hereditary since every N-radical is left hereditary. But $\bar{l}_R$ is not left hereditary since, for example, $\begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix} <_l \begin{pmatrix} F & F \\ F & F \end{pmatrix} \in \bar{l}_R$ and the ring $\begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix} \notin \bar{l}_R$ because its nonzero homomorphic image $\begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix} / \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} \simeq F$ is not in $\beta \cup \{R\}$. Therefore, $\bar{l}_R \subsetneq \widetilde{l}_R$.*

Example 3.12. *Let $R := \mathbb{Z}_2$ be the two-element field. As before, R generates the supernilpotent atom $\bar{l}_R = l_{(\beta \cup \{\mathbb{Z}_2\})}$. Clearly, $\bar{l}_R$ is left and right hereditary since $\mathbb{Z}_2$ is a commutative ring and β is left and right hereditary. However, $\bar{l}_R \neq \widetilde{l}_R$ since, otherwise, being an N-radical, $\bar{l}_R$ would be left and right strong. Then, since $\mathbb{Z}_2 \in \bar{l}_R$ and $\mathbb{Z}_2 \simeq \begin{pmatrix} \mathbb{Z}_2 & 0 \\ 0 & 0 \end{pmatrix} <_r \begin{pmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & 0 \end{pmatrix}$, the right strongness of $\bar{l}_R$ implies that the ideal $\begin{pmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & 0 \end{pmatrix}$ of $\begin{pmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & 0 \end{pmatrix}$ generated by $\begin{pmatrix} \mathbb{Z}_2 & 0 \\ 0 & 0 \end{pmatrix}$ is in $\bar{l}_R$. Then, since $\begin{pmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & 0 \end{pmatrix} <_l \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$, the left strongness of $\bar{l}_R$ implies that the ideal $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ of $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ generated by $\begin{pmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & 0 \end{pmatrix}$ is in $\bar{l}_R$. But it is not so because, being a noncommutative prime ring, the ring $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ is not isomorphic to $\mathbb{Z}_2$. However, the radical $\widetilde{l}_R$, being an N-radical, is left and right strong. So, $\bar{l}_R \subsetneq \widetilde{l}_R$.*

In the proofs of the following two theorems (Theorem 3.13 and Theorem 3.14, we require Theorem 1 in [3], which states that if A is a nonzero $*$-ring, then $\bar{l}_A$ is a supernilpotent atom. Furthermore, we also make use of Theorem 1 in [5], which asserts that if A is a nonzero $*$-ring, then $\mathcal{U}(\pi \setminus \pi_A) = \widehat{l}_A$ is a special atom.

The next theorem shows the existence of a nonzero prime essential ring R such that the smallest $N-$radical $\widetilde{l}_R$ containing R is an $N-$atom and the smallest special radical $\widehat{l}_R$ is a special atom.

Theorem 3.13. *Let A be a nonzero semiprime ring and let $R := A(W(k))$, where $W(k)$ is the set of all finite words made from a (well-ordered) alphabet of cardinality κ. If A is a $*-$ring, then the smallest $N-$radical $\widetilde{l}_R$ containing R is an $N-$atom and the smallest special radical $\widehat{l}_R$ containing R is a special atom.*

Proof. It follows from the proof of Theorem 3.8 that $A(W(k)) \in \overline{l}_A$. Therefore, $\overline{l}_{A(W(k))} \subseteq \overline{l}_A$. It follows from Theorem 1 in [3] that $\overline{l}_A$ is a supernilpotent atom. Hence, $\overline{l}_{A(W(k))} = \overline{l}_A$. In the other words, $\overline{l}_R = \overline{l}_{A(W(k))}$ is a supernilpotent atom. Furthermore, it follows from Theorem 3.5 that $\widetilde{l}_{A(W(k))} = \widetilde{l}_R$ is an $N-$atom.

On the other hand, arguing as in the proof of Theorem 3.8, we get $A \in \overline{l}_{A(W(k))}$ which implies that $\overline{l}_A \subseteq \overline{l}_{A(W(k))}$. Now, since A is a nonzero $*-$ring, it follows from Theorem 1 in [5] that $\widehat{l}_A$ is a special atom and $\widehat{l}_A = \mathcal{U}(\pi \setminus \pi_A)$, where π is the class of all prime rings, π_A is the essential closure of the hereditary closure of A, and $\mathcal{U}(\pi \setminus \pi_A)$ is the upper radical determined by the class $\pi \setminus \pi_A$. Thus, $A(W(k)) \in \mathcal{U}(\pi \setminus \pi_A)$ since part (iii) of Lemma 3.7 guarantees that every nonzero prime homomorphic image of $A(W(k))$ is isomorphic to some nonzero prime homomorphic image of A and the only nonzero prime homomorphic image of A is A because A is a nonzero $*-$ring. Consequently, $\widehat{l}_{A(W(k))} \subseteq \widehat{l}_A$ and so $\widehat{l}_R = \widehat{l}_{A(W(k))} = \widehat{l}_A$. $\square$

The existence of a nonzero prime essential ring that generates a special atom, a supernilpotent atom, and an $N-$atom shown in Theorem 3.13 shows that Question 1 posed in [8] has a positive answer. The following theorem describes a prime essential ring that does not generate a supernilpotent atom and a special atom.

Theorem 3.14. *Let $\mathbb{Z}$ be the ring of integers. For every cardinal $\kappa > |\mathbb{Z}|$ the special radical $\widehat{l}_{\mathbb{Z}(W(k))}$ is not a special atom and the supernilpotent radical $\overline{l}_{\mathbb{Z}(W(k))}$ is not a supernilpotent atom.*

Proof. Arguing as in the proof of Theorem 3.8, we get $\mathbb{Z} \in \widehat{l}_{\mathbb{Z}(W(k))}$ which implies that $\widehat{l}_{\mathbb{Z}} \subseteq \widehat{l}_{\mathbb{Z}(W(k))}$. Moreover, since $\mathbb{Z}_p \simeq \mathbb{Z}/p\mathbb{Z} \in \widehat{l} \cap \mathcal{S}(\beta)$ for every prime number p, it follows that $\beta \subsetneqq \widehat{l}_{\mathbb{Z}_p}$. But, since $\mathbb{Z}_p$ is a $*-$ring, it follows from Theorem 1 in [5] that $\widehat{l}_{\mathbb{Z}_p} = \mathcal{U}(\pi \setminus \{\mathbb{Z}_p\})$. This implies that $\widehat{l}_{\mathbb{Z}_p} \neq \widehat{l}_{\mathbb{Z}}$ as otherwise, we would have $\mathbb{Z} \in (\pi \setminus \{\mathbb{Z}_p\})$ and then $\mathbb{Z}_q \simeq \mathbb{Z}/q\mathbb{Z} \in \mathcal{U}(\pi \setminus \{\mathbb{Z}_p\})$ for a prime number $q \neq p$ which is not the case as $\mathbb{Z}_q \in \pi \setminus \{\mathbb{Z}_p\}$. Thus $\beta \subsetneqq \widehat{l}_{\mathbb{Z}_p} \subsetneq \widehat{l}_{\mathbb{Z}(W(k))}$ which clearly shows that $\widehat{l}_{\mathbb{Z}(W(k))}$ is not a special atom. $\square$

Our final result describes a case when the converse of Theorem 3.5 does hold. For that, we need the following two observations:

Lemma 3.15. *Let A and T be nonzero semiprime commutative rings. Then, if $T \in \widetilde{l}_A$, then $T \in \overline{l}_A$.*

Proof. Clearly, for every commutative ring A, $L < A$ if and only if $L \lhd A$. Hence, if T is a semiprime commutative ring such that $T \in \widetilde{l}_A$, then $\overline{T} \in \widetilde{l}_A$ for every nonzero semiprime homomorphic image $\overline{T}$ of T. Then, it follows from Theorem 2.2, that $\overline{T}$ contains a left ideal S such that $S/\beta(S) \simeq J/I$, for some $0 \neq J = J_1 < J_2 < ... < J_m = A$. But, since both rings $\overline{T}$ and A are commutative, this means that $S \lhd \overline{T}$ and $0 \neq J = J_1 \lhd J_2 \lhd ... \lhd J_m = A$. Moreover, $\beta(S) = 0$ since $\overline{T}$ is semiprime. In other words, every nonzero semiprime homomorphic image of T contains a nonzero ideal $S \in H(IR) = \{A : A \simeq S/K$, for some accessible subring S of $R\}$.

which means that $T \in \overline{l}_A$ because $\overline{l}_A = l_{(H(IR)\cup\beta)}$. □

Lemma 3.16. *[4] Let L be a left ideal of a ring R and $a_R(L) := \{r \in R : rL = 0\}$. If $a_R(L) = 0$ and L is a commutative ring, then R is also commutative.*

Let A be a nonzero commutative semiprime ring. As the final result of this paper, we give a necessary and sufficient condition for the smallest $N-$radical $\widetilde{l}_A$ containing A to be an N-atom in the following theorem.

Theorem 3.17. *Let A be a nonzero commutative semiprime ring. Then $\overline{l}_A$ is a supernilpotent atom if and only if $\widetilde{l}_A$ is an N-atom.*

Proof. Assume that A is a nonzero commutative semiprime ring such that $\widetilde{l}_A$ is an N-atom. Let $T \in \overline{l}_A$ be a nonzero prime ring. Then, T contains a nonzero ideal $I \in HIR$ and $a_T(I) = 0$ since T is prime. Moreover, I is commutative because every ring in HIR is commutative since R is commutative. So, it follows from Lemma 3.16 that T is a commutative ring. Moreover, since $\overline{l}_A \subseteq \widetilde{l}_A$, it follows that $T \in \widetilde{l}_A$. Then $\beta \subsetneq \widetilde{l}_T \subseteq \widetilde{l}_A$ which gives $\widetilde{l}_T = \widetilde{l}_A$, since $\widetilde{l}_A$ is an N-atom. This shows that $A \in \widetilde{l}_T$. But, as both rings T and A are commutative, this, in view of Lemma 3.15, means that $A \in \overline{l}_T$. Consequently, $\overline{l}_A \subseteq \overline{l}_T$. Moreover, $\overline{l}_T \subseteq \overline{l}_A$ since $T \in \overline{l}_A$. Hence $\overline{l}_A = \overline{l}_T$ which, in view of Lemma 3.4 (v), means that $\overline{l}_A$ is a supernilpotent atom.

The remaining part of the proof follows directly from Theorem 3.5. □

4. CONCLUDING REMARKS

Based on the results that are explained in the previous section, we may infer that there exist prime essential rings which do not generate $N-$atoms. On the other hand, the existence of the $*-$ring which was introduced in [5] is important since we can construct a prime essential ring which generates an $N-$atom as shown in Theorem 3.13. Furthermore, let A be a nonzero semiprime ring. As the main result of this research, we infer that the smallest $N-$radical $\widetilde{l}_A$ containing A is an

N-atom if the smallest supernilpotent radical $\overline{l}_A$ containing A is a supernilpotent atom.

Conversely, if the smallest $N-$radical $\widetilde{l}_A$ containing A is an N-atom, we can conclude that the smallest supernilpotent radical $\overline{l}_A$ containing A is a supernilpotent atom. Furthermore, the smallest special radical $\widehat{l}_A$ containing A is a special atom. This gives a positive answer for Question 1 posed in [8].

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