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Structural Properties and Reverse Topological Indices of Order GCD Graphs of Integers Modulo Ring

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Abstract. The order GCD graph of a ring, having the ring elements as the vertex set, and two distinct vertices are adjacent if and only if the greatest common divisor (gcd) of the order of both vertices is equal to the order of the product of these two vertices in the ring. This paper aims to analyze the reverse Sombor, Randić, and Harmonic indices of the order GCD graph where the set of vertices is integers modulo ring elements. The results provide new insights into the mathematical properties of reverse topological indices and their potential applications.

Key words and Phrases: Order GCD Graph; integers modulo ring; reverse Sombor index; reverse Randic index; reverse harmonic index.

1. Introduction

The reverse topological index is a modification of the conventional topological index, which is based on vertex degree. It uses the reverse vertex degree, defined as the maximum degree of the graph minus the degree of the given vertex, plus 1. The reverse topological index was first introduced by Ediz in 2015 [1]. In this article, the reverse second Zagreb index of chemical trees was introduced. The following year, Ediz studied the reverse Zagreb index of the Cartesian product of graphs [2], and in 2018, he further examined the reverse first beta Zagreb index [3]. As scientific knowledge continues to advance, the reverse topological index has become an intriguing subject of research. This is evident from several publications related to the reverse topological index and its significance in chemistry [4]. These studies inspired Swamy et al. (2020) to define the reverse Sombor index [5], and the

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reverse Euler Sombor index has been reported in [6]. Furthermore, investigations have examined various reverse topological indices of bistar graphs and the corona product [7].

The development of graph theory is no longer limited to simple graphs defined by vertex and edge sets. Still, it has expanded to encompass various algebraic structures, such as groups, rings, and fields, as seen in [8, 9] and offers background relevant to subgroup-induced graph structures [10]. Moreover, the constructed graph is extended to various graph types, as described in [11, 12]. This exploration not only enriches pure mathematics but also opens numerous applications in applied fields, including cryptography, coding theory, communication networks, and computer science. One example of a graph defined by group elements is the GCD graph. It was first formally introduced by Klotz and Sander in 2007 [13], when they discussed unitary Cayley graphs, and has since become an interesting new graph to study. Six years later, they showed the relationship between the GCD graph and the graph products. In 2024, Sarkar and Patra [14] introduced a new definition: the order GCD graph associated with the integer modulo group $\mathbb{Z}_n$. However, this paper did not mention whether $n = 2p^k$, where p is an odd prime, and $k \geq 2$ is an integer. This research gap warrants further examination, which is the urgency of this research.

Furthermore, in previous studies on reverse topological indices, the Order GCD graph has not been examined. There has been research on this graph [15], and in the context of the Sombor index for the group of integers modulo a prime power [16]. Therefore, this article aims to investigate several reverse topological indices of the order GCD graph derived from the ring of integers modulo $2p^k$, where p is an odd prime and $k \geq 2$ is an integer. Furthermore, the reverse topological indices studied include the Sombor, Randic, and Harmonic indices.

2. Preliminaries

Let G be a simple graph with $V(G)$ being the set of vertices and $E(G)$ being the edge set. Let uv be the edge between vertices u and v, and $\deg(u)$ is the number of vertices that are adjacent to the vertex u. The reverse vertex degree of vertex v in a graph G was first introduced by Ediz and Cancan in 2016 [2]. It is denoted by c_v and defined as $c_v = \Delta - \deg(v) + 1$, where Δ is the maximum degree in G. With the definition of reverse vertex degree, several reverse topological indices have been introduced, such as the second reverse Zagreb index [1], the first reverse beta Zagreb index [2], and the reverse Sombor index [3]. Among these reverse indices, Swamy et al. in 2020 defined the reverse Sombor index (RSO) of a graph G [5] as follows:

$$RSO\,(G) = \sum_{uv \in E(G)} \sqrt{(c_u)^2 + (c_v)^2}.$$

Following the same logic as [2], the reverse indices of Randic (RR) and Harmonic (RH) of a graph G are defined as follows:

$$RR\left(G\right)=\sum_{uv\in E(G)}\frac{1}{\sqrt{c_u\cdot c_v}},$$

$$RH\left(G\right)=\sum_{uv\in E(G)}\frac{2}{c_u+c_v}.$$

Throughout this paper, the order GCD graph of the integer modulo $2p^k$ is denoted as $\Gamma\left(\mathbb{Z}_{2p^k}\right)$. The set of vertices in this graph is all elements of the $\mathbb{Z}_{2p^k}$ ring and is denoted as $V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)$. Two distinct vertices $u\neq v\in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)$ are adjacent if and only if the greatest common divisor (gcd) of the order of both vertices is equal to the order of the product of these two vertices, or, in other words, $\gcd(o(u),o(v))=o(u\cdot v)$ [14].

3. Main Results

In this section, we examine the properties of the order GCD graph for $\mathbb{Z}_{2p^k}$ and discuss the computation of its reverse-based topological indices. The mentioned 3 indices are given as follows: reverse Sombor, reverse Randic, and reverse Harmonic indices.

3.1. Properties of Order GCD Graph.

Let $\mathbb{Z}_{2\cdot 3^2}$ ring, we get the order of elements of $\mathbb{Z}_{2\cdot 3^2}$ as $o(0)=1$, $o(1)=o(5)=o(7)=o(11)=o(13)=o(17)=18$, $o(2)=o(4)=o(8)=o(10)=o(14)=o(16)=9$, $o(3)=o(15)=6$, $o(6)=o(12)=3$, and $o(9)=2$. Now we list the gcd of every pair of elements of $\mathbb{Z}_{2\cdot 3^2}$ as seen in Table 1.

Table 1. GCD of Every Pair of Elements of $\mathbb{Z}_{2\cdot 3^2}$

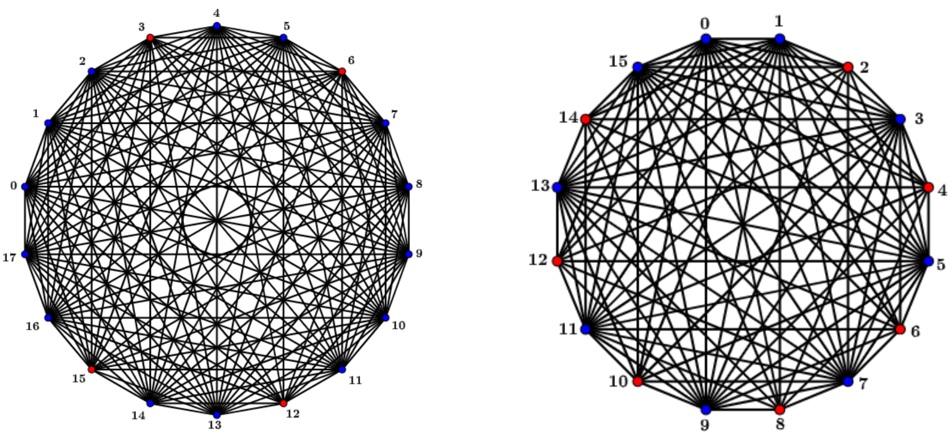
gcd	$o(0)$	$o(1)$	$o(2)$	$o(3)$	$o(4)$	$o(5)$	$o(6)$	$o(7)$	$o(8)$	$o(9)$	$o(10)$	$o(11)$	$o(12)$	$o(13)$	$o(14)$	$o(15)$	$o(16)$	$o(17)$
$o(0)$	–	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$o(1)$	1	–	9	6	9	18	3	18	9	2	9	18	3	18	9	6	9	18
$o(2)$	1	9	–	3	9	9	3	9	9	1	9	9	3	9	9	3	9	9
$o(3)$	1	6	3	–	3	6	3	6	3	2	3	6	3	6	3	6	3	6
$o(4)$	1	9	9	3	–	9	3	9	9	1	9	9	3	9	9	3	9	9
$o(5)$	1	18	9	6	9	–	3	18	9	2	9	18	3	18	9	6	9	18
$o(6)$	1	3	3	3	3	3	–	3	3	1	3	3	3	3	3	3	3	3
$o(7)$	1	18	9	6	9	18	3	–	9	2	9	18	3	18	9	6	9	18
$o(8)$	1	9	9	3	9	9	3	9	–	1	9	9	3	9	9	3	9	9
$o(9)$	1	2	1	2	1	2	1	2	1	–	1	2	1	2	1	2	1	2
$o(10)$	1	9	9	3	9	9	3	9	9	1	–	9	3	9	9	3	9	9
$o(11)$	1	18	9	6	9	18	3	18	9	2	9	–	3	18	9	6	9	18
$o(12)$	1	3	3	3	3	3	3	3	3	1	3	3	–	3	3	3	3	3
$o(13)$	1	18	9	6	9	18	3	18	9	2	9	18	3	–	9	6	9	18
$o(14)$	1	9	9	3	9	9	3	9	9	1	9	9	3	9	–	3	9	9
$o(15)$	1	6	3	6	3	6	3	6	3	2	3	6	3	6	3	–	3	6
$o(16)$	1	9	9	3	9	9	3	9	9	1	9	9	3	9	9	3	–	9
$o(17)$	1	18	9	6	9	18	3	18	9	2	9	18	3	18	9	6	9	–

Next, the order of the product between two distinct elements of the ring is as follows in Table 2.

TABLE 2. Order of the Product Between Two Distinct Elements of $\mathbb{Z}_{2\cdot 3^2}$

$o(u\cdot v)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	–	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	–	9	6	9	18	3	18	9	2	9	18	3	18	9	6	9	18
2	1	9	–	3	9	9	3	9	9	1	9	9	3	9	9	3	9	9
3	1	6	3	–	3	6	1	6	3	2	3	6	1	6	3	2	3	6
4	1	9	9	3	–	9	3	9	9	1	9	9	3	9	9	3	9	9
5	1	18	9	6	9	–	3	18	9	2	9	18	3	18	9	6	9	18
6	1	3	1	3	3	3	–	3	3	1	3	3	1	3	3	1	3	3
7	1	18	9	6	9	18	3	–	9	2	9	18	3	18	9	6	9	18
8	1	9	9	3	9	9	3	9	–	1	9	9	3	9	9	3	9	9
9	1	2	1	2	1	2	1	2	1	–	1	2	1	2	1	2	1	2
10	1	9	9	3	9	9	3	9	9	1	–	9	3	9	9	3	9	9
11	1	18	9	6	9	18	3	18	9	2	9	–	3	18	9	6	9	18
12	1	3	3	1	3	3	1	3	3	1	3	3	–	3	3	1	3	3
13	1	18	9	6	9	18	3	18	9	2	9	18	3	–	9	6	9	18
14	1	9	9	3	9	9	3	9	9	1	9	9	3	9	–	3	9	9
15	1	6	3	2	3	6	1	6	3	2	3	6	1	6	3	–	3	6
16	1	9	9	3	9	9	3	9	9	1	9	9	3	9	9	3	–	9
17	1	18	9	6	9	18	3	18	9	2	9	18	3	18	9	6	9	–

According to Tables 1 and 2, we obtain $\mathbb{Z}_{2\cdot 3^2}$ as shown in Figure 1a. By the same argument, we also get the order GCD graph $\Gamma\left(\mathbb{Z}_{2\cdot 2^3}\right)$ as seen in Figure 1b.



(A) Ring of $\mathbb{Z}_{2\cdot 3^2}$ (B) Ring of $\mathbb{Z}_{2\cdot 2^3}$

FIGURE 1. Order GCD Graph

Taking into account the illustrations in Figures 1a and 1b, it can be seen that $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ consists of two types of degree of vertex. However, the pattern of vertex degrees differs for odd and even prime numbers p. Theorems 3.1 and 3.2 support this distinction, as presented below.

Theorem 3.1. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of the integer modulo ring $2p^k$, where p is an odd prime and $k \geq 2$ is an integer. Suppose that $v \in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)$, then*

$$\deg(v) = \begin{cases} 2\left(p^k - p^{k-1} + 1\right) & , \textit{ for } v \in V_1 \\ 2p^k - 1 & , \textit{ for } v \notin V_1, \end{cases}$$

where $V_1 = \left\{p, 2p, 3p, \ldots, \left(2p^{k-1} - 1\right)p\right\} \setminus \left\{p^k\right\}$.

Proof. Let $V_1 = \left\{p, 2p, 3p, \ldots, \left(2p^{k-1} - 1\right)p\right\} \setminus \left\{p^k\right\}$. We consider three cases as follows:

(1) **Case 1:** For $u, v \in V_1$, let $u = mp$ and $v = np$ with $m, n \in \left\{1, 2, 3, \ldots, 2p^{k-1} - 1\right\}$. We get $o(u) = \dfrac{lcm\left(mp, 2p^k\right)}{mp} = 2p^{k-1}$ and $o(v) = \dfrac{lcm\left(np, 2p^k\right)}{np} = 2p^{k-1}$ which implies $\gcd\left(o(u), o(v)\right) = 2p^{k-1}$. Meanwhile, $u \cdot v = mnp^2$ yields $o(u \cdot v) = p^{k-2}$. Hence, $\gcd\left(o(u), o(v)\right) \neq o(u \cdot v)$. Therefore, u and v are not adjacent.
(2) **Case 2:** When $u, v \notin V_1$, then we have $\gcd(u, p^k) = 1$ and $\gcd(v, 2p^k) = 1$. Consequently, $o(u) = \dfrac{lcm(u, 2p^k)}{u} = \dfrac{u \cdot 2p^k}{u} = 2p^k$ and $o(v) = \dfrac{lcm(v, 2p^k)}{v} = 2p^k$. Since $u \cdot v \notin V_1$, in the same manner, we obtain $o(u \cdot v) = 2p^k$. Thus $\gcd(o(u), o(v)) = o(u \cdot v)$ which means u and v are adjacent.
(3) **Case 3:** Let $u \in V_1$ and $v \notin V_1$. According to the argument in the first and second cases, we obtain $\gcd(o(u), o(v)) = 2p^{k-1}$. On the other hand, $u \cdot v = rp$ for $r \in \left\{1, 2, 3, \ldots, 2p^{k-1} - 1\right\}$. Therefore, it suffices to prove that $o(u \cdot v) = 2p^{k-1}$ which means u and v are adjacent.

Based on the three cases above, it can be concluded that any vertex $u \notin V_1$ is adjacent to all vertices in V_1. Meanwhile, any two vertices v and w in V_1 are not adjacent. Furthermore, the number of elements in V_1 is given by $|V_1| = 2p^{k-1} - 2$. Since each vertex $u \notin V_1$ is adjacent to all vertices in V_1, its degree is given by $\deg(u) = 2p^k - 2$. On the other hand, since each vertex $v \in V_1$ is only adjacent to vertices outside V_1, its degree is equal to $|V|$ minus $|V_1|$, which results in $\deg(v) = p^k - p^{k-1} + 2$. $\square$

Theorem 3.2. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of integers modulo ring $2p^k$ for p is an even prime and $k \geq 2$ is an integer. Suppose that $v \in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)$, then*

$$\deg(v) = \begin{cases} 2\left(p^k - p^{k-1}\right) + 1 & , \textit{ for } v \in V_1 \\ 2p^k - 1 & , \textit{ for } v \notin V_1 \end{cases}$$

where $V_1 = \left\{p, 2p, 3p, \ldots, \left(2p^{k-1} - 1\right)p\right\}$.

Proof. The proof follows a similar approach to the proof of Theorem 3.1. $\square$

The following theorem presents the reverse vertex degrees of the GCD graph for odd prime p in Theorem 3.3 and even prime in Theorem 3.4.

Theorem 3.3. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of integers modulo ring $2p^k$ for odd prime p and $k \geq 2$ is an integer, then the reverse vertex degree of a vertex in $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is*

$$c_v = \begin{cases} 2\left(p^{k-1}-1\right) & , v \in V_1 \\ 1 & , v \notin V_1, \end{cases}$$

where $V_1 = \left\{p, 2p, 3p, \ldots, \left(2p^{k-1}-1\right)p\right\} \setminus \left\{p^k\right\}$

Proof. Based on Theorem 3.1, we obtain $\Delta = 2p^k - 1$. Furthermore, by the definition of the reverse degree, we have $c_u = 2\left(p^{k-1}-1\right)$ and $c_v = 1$ for $u \in V_1$ and $v \notin V_1$, respectively. $\square$

Theorem 3.4. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of integers modulo ring $2p^k$ for even prime p and $k \geq 2$ is an integer, then the reverse vertex degree of a vertex in $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is*

$$c_v = \begin{cases} 2p^{k-1}-1 & , v \in V_1 \\ 1 & , v \notin V_1, \end{cases}$$

where $V_1 = \left\{p, 2p, 3p, \ldots, \left(2p^{k-1}-1\right)p\right\}.$

Proof. It is proven based on Theorem 3.2 and the definition of the reverse degree. $\square$

Furthermore, Theorems 3.1 and 3.2 imply the following theorem about the computation of the number of edges in $\Gamma(\mathbb{Z}_{2p^k})$.

Theorem 3.5. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of integers modulo ring $2p^k$ for odd prime p and $k \geq 2$ be an integer. Then the number of edges in $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is*

$$\left|E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)\right| = \left(p^k - p^{k-1} + 1\right)\left(2p^k + 2p^{k-1} - 3\right)$$

Proof. Let V_1 be defined as $\left\{p, 2p, 3p, \ldots, \left(2p^{k-1}-1\right)p\right\} \setminus \{p^k\}$. Based on Theorem 3.1, the degrees of the vertices are given by $\deg(v_1) = 2\left(p^k - p^{k-1} + 1\right)$ and $\deg(v_2) = 2p^k - 1$ for $v_1 \in V_1$ and $v_2 \notin V_1$. Furthermore, the number of vertices in V_1 is $|V_1| = 2p^{k-1} - 2$. This means that the number of vertices with degree $\deg(v_1)$ is $2p^{k-1} - 2$. Consequently, the number of vertices with degree $\deg(v_2)$ is given by the total number of vertices minus $|V_1|$, Since the number of edges in $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is equal to half of the sum of all vertex degrees [17], we obtain the following result based on Theorem 3.1:

$$\left|E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)\right| = \frac{1}{2} \sum_{v \in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \deg(v)$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\sum_{v_1\in V_1\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \deg(v_1) + \sum_{v_2\notin V_1\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \deg(v_2)\right)\\
&= \frac{1}{2}\left(2\left(p^{k-1}-1\right)\left(2\left(p^k-p^{k-1}+1\right)\right)+\left(2\left(p^k-p^{k-1}+1\right)\right)\left(2p^k-1\right)\right)\\
&= \left(p^k-p^{k-1}+1\right)\left(2\left(p^{k-1}-1\right)+\left(2p^k-1\right)\right)\\
&= \left(p^k-p^{k-1}+1\right)\left(2p^k+2p^{k-1}-3\right).
\end{aligned}
$$

□

Theorem 3.6. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of integers modulo ring $2p^k$ for even prime p and $k \geq 2$ be an integer. Then the number of edges in $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is*

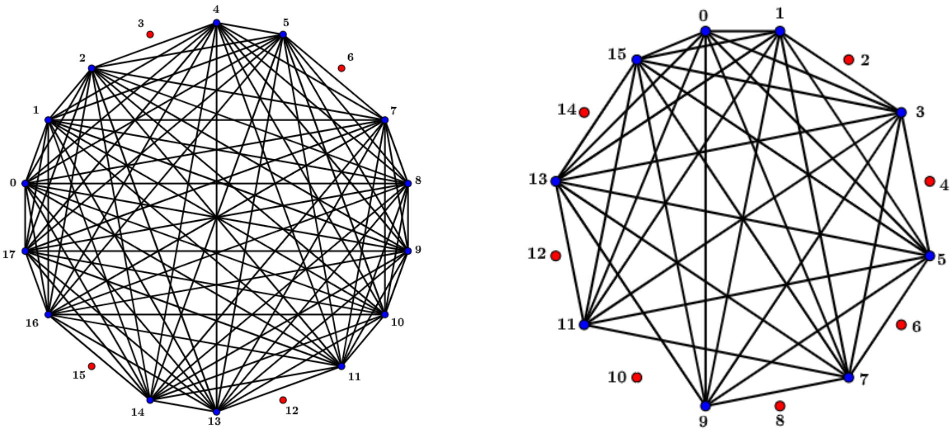
$$
\left|E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)\right| = \left(2\left(p^k-p^{k-1}\right)+1\right)\left(p^k+p^{k-1}-1\right).
$$

Proof. Let $V_1 = \left\{p, 2p, 3p, \ldots, \left(2p^{k-1}-1\right)p\right\}$. Using the same reasoning as in the proof of Theorem 3.5, we obtain $|V_1| = 2p^{k-1}-1$ and $|V \backslash V_1| = 2(p^k-p^{k-1})+1$. Based on Theorem 3.2, we derive the following result:

$$
\begin{aligned}
\left|E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)\right| &= \frac{1}{2}\sum_{v\in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \deg(v)\\
&= \frac{1}{2}\left(\sum_{v_1\in V_1\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \deg(v_1) + \sum_{v_2\notin V_1\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \deg(v_2)\right)\\
&= \frac{1}{2}\left(2p^{k-1}-1\right)\left(2\left(p^k-p^{k-1}\right)+1\right)+\left(2\left(p^k-p^{k-1}\right)+1\right)\left(2p^k-1\right)\\
&= \frac{1}{2}\left(2\left(p^k-p^{k-1}\right)+1\right)\left(\left(2p^{k-1}-1\right)+\left(2p^k-1\right)\right)\\
&= \left(2\left(p^k-p^{k-1}\right)+1\right)\left(p^k+p^{k-1}-1\right).
\end{aligned}
$$

□

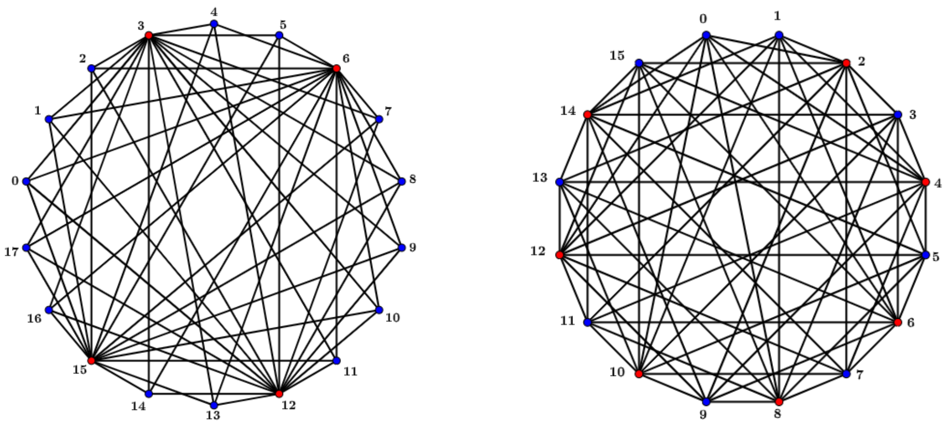
Observe the red vertices in Figures 1a and 1b. If the edges incident to these vertices are removed, a complete subgraph K_{14} is formed for $\Gamma(\mathbb{Z}_{2\cdot 3^2})$ and a complete subgraph K_9 is formed for $\Gamma(\mathbb{Z}_{2\cdot 3^2})$. This can be seen in Figure 2.



(A) Ring of $\mathbb{Z}_{2\cdot 3^2}$ (B) Ring of $\mathbb{Z}_{2\cdot 2^3}$

FIGURE 2. Complete Subgraph of Order GCD Graph

However, if the edges forming the complete graph are removed from the graph $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ pin Figure 1a, the resulting subgraph is a complete bipartite subgraph $K_{4,14}$ as shown in Figure 3a. The partition of the set of vertices V_1 and V_2 with all the members of each set which $|V_1| = 4$ and $|V_2| = 14$, where $V_1 \cap V_2 = \varnothing$. Similarly, for the graph in Figure 1b, the resulting subgraph is the complete bipartite graph $K_{9,7}$, as illustrated in Figure 3b.



(A) Ring of $\mathbb{Z}_{2\cdot 3^2}$ (B) Ring of $\mathbb{Z}_{2\cdot 2^3}$

FIGURE 3. Complete Bipartite Subgraph of $\mathbb{Z}_{2\cdot 2^3}$

Therefore, a theorem regarding the subgraphs of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ can be established. This is formally stated in Theorem 3.7 and Theorem 3.8, as follows.

Theorem 3.7. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of ring of integers modulo $2p^k$ where p an odd prime and $k \geq 2$ be an integer, then a complete subgraph $K_{2(p^k-p^{k-1}+1)}$ and a complete bipartite subgraph $K_{2(p^{k-1}-1),2(p^k-p^{k-1}+1)}$ are formed.*

Proof. Let $V_1 = \left\{p, 2p, 3p, \ldots, \left(2p^{k-1}-1\right)p\right\} \backslash \{p^k\}$. From the explanation in Theorem 3.1, it is clear that two distinct vertices that are not in V_1, that is, $v_1 \neq v_2 \notin V_1$ or equivalently $v_1 \neq v_2 \in V \backslash V_1$ are adjacent. Therefore, the subgraph induced by the set of vertex $V \backslash V_1$ forms a complete graph. Furthermore, the number of vertices in $V \backslash V_1$ is $|V \backslash V_1| = 2\left(p^k - p^{k-1} + 1\right)$. Thus, one of the subgraphs of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is the complete subgraph $K_{2(p^k-p^{k-1}+1)}$.

Observe that if the edges forming the complete subgraph are removed from $\Gamma\left(\mathbb{Z}_{2p^k}\right)$, it is evident that any vertex $u_1 \in V_1$ is adjacent to any vertex $v_1 \notin V_1$. According to Theorem 3.1, any two distinct vertices $u_1 \neq u_2 \in V_1$ are not adjacent. Therefore, a complete bipartite subgraph is formed with two partition sets: V_1 dan $V \backslash V_1$. Thus, we obtain the complete bipartite subgraph $K_{2(p^{k-1}-1),2(p^k-p^{k-1}+1)}$. □

Theorem 3.8. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of ring of integers modulo $2p^k$ where p an even prime and $k \geq 2$ be an integer, then a complete subgraph $K_{2(p^k-p^{k-1})+1}$ and a complete bipartite subgraph $K_{2p^{k-1}-1,2(p^k-p^{k-1})+1}$ are formed.*

Proof. Let $V_1 = \left\{p, 2p, 3p, \ldots, \left(2p^{k-1}-1\right)p\right\}$, then $|V_1| = 2p^{k-1} - 1$, Furthermore, we obtain $|V \backslash V_1| = 2\left(p^k - p^{k-1}\right) + 1$. Using the same reasoning as in the proof of Theorem 3.7, we derive the complete subgraph $K_{2(p^k-p^{k-1})+1}$ and the complete bipartite subgraph $K_{2(p^k-p^{k-1})+1,2p^{k-1}-1}$. □

The number of edges in the subgraphs formed from Theorem 3.7 and Theorem 3.8 is stated, respectively, in Theorem 3.9 and Theorem 3.10 below.

Theorem 3.9. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of ring of integers modulo $2p^k$ where p is an odd prime and $k \geq 2$ is an integer. The number of edges in the complete subgraph $K_{2(p^k-p^{k-1}+1)}$ and the complete bipartite subgraph $K_{2(p^{k-1}-1),2(p^k-p^{k-1}+1)}$ are respectively*

$$\left|E\left(\Gamma\left(K_{2(p^k-p^{k-1}+1)}\right)\right)\right| = 2\left(p^k - p^{k-1} + 1\right)\left(p^k - p^{k-1} + \frac{1}{2}\right),$$

and

$$\left|E\left(\Gamma\left(K_{2(p^{k-1}-1),2(p^k-p^{k-1}+1)}\right)\right)\right| = 4\left(p^{k-1} - 1\right)\left(p^k - p^{k-1} + 1\right).$$

Proof. Since $K_{2(p^k-p^{k-1}+1)}$ is a complete subgraph, the number of vertices in the subgraph is $2(p^k - p^{k-1} + 1)$, and each vertex has a degree of $2(p^k - p^{k-1} + 1) - 1$. Furthermore, since the number of edges in a graph is equal to half of the sum of the degrees of all vertices [17], we obtain

$$\begin{aligned}\left|E\left(\Gamma\left(K_{2(p^k-p^{k-1}+1)}\right)\right)\right| &= \frac{1}{2} \cdot \left(2\left(p^k - p^{k-1} + 1\right)\right)\left(2\left(p^k - p^{k-1} + 1\right) - 1\right) \\ &= \left(p^k - p^{k-1} + 1\right)\left(2\left(p^k - p^{k-1} + 1\right) - 1\right)\end{aligned}$$

$$= 4\left(p^k - p^{k-1} + 1\right)\left(p^k - p^{k-1} + \frac{1}{2}\right).$$

The complete bipartite graph is partitioned into two vertex sets, namely V_1 and $V\backslash V_1$. Moreover, based on the explanations in the proofs of Theorems 3.1 and 3.7, we obtain $|V_1| = 2p^{k-1} - 2$ dan $|V\backslash V_1| = 2\left(p^k - p^{k-1} + 1\right)$. In this case, $|V_1|$ can be interpreted as the degree of any vertex $v_2 \in V\backslash V_1$, and similarly, $|V\backslash V_1|$ can be interpreted as the degree of any vertex $v_1 \in V_1$. Therefore, the number of edges in the complete bipartite graph is given by

$$\begin{aligned}
\left|E\left(\Gamma\left(K_{2(p^{k-1}-1),2(p^k-p^{k-1}+1)}\right)\right)\right| &= \frac{1}{2}\left(\sum_{v_1\in V_1}\deg(v_1) + \sum_{v_2\in V\backslash V_1}\deg(v_2)\right)\\
&= \frac{1}{2}\left(|V_1|\cdot\deg(v_1) + |V\backslash V_1|\cdot\deg(v_2)\right)\\
&= \frac{1}{2}\left[\left(2p^{k-1}-2\right)\cdot 2\left(p^k - p^{k-1}+1\right)\right.\\
&\quad \left. +2\left(p^k - p^{k-1}+1\right)\cdot\left(2p^{k-1}-2\right)\right]\\
&= 2\left(p^{k-1}-1\right)\left(p^k - p^{k-1}+1\right).
\end{aligned}$$

□

Theorem 3.10. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of the ring of integers modulo $2p^k$ where p an even prime and $k \geq 2$ be an integer. The number of edges in the complete subgraph $K_{2(p^k-p^{k-1}+1)}$ and the complete bipartite subgraph $K_{2(p^{k-1}-1),2(p^k-p^{k-1}+1)}$ are respectively*

$$\left|E\left(\Gamma\left(K_{2(p^k-p^{k-1}+1)}\right)\right)\right| = \left(2\left(p^k - p^{k-1}\right)+1\right)\left(p^k - p^{k-1}\right)$$

and

$$\left|E\left(\Gamma\left(K_{2p^{k-1}-1,2(p^k-p^{k-1}+1)}\right)\right)\right| = \left(2p^{k-1}-1\right)\cdot\left(2\left(p^k - p^{k-1}\right)+1\right).$$

Proof. Using a similar explanation as in Theorem 3.10 and the information from Theorem 3.10, we obtain

$$\begin{aligned}
\left|E\left(\Gamma\left(K_{2(p^k-p^{k-1})+1}\right)\right)\right| &= \frac{1}{2}\cdot\left(2\left(p^k - p^{k-1}\right)+1\right)\left(2\left(p^k - p^{k-1}\right)\right)\\
&= \left(2\left(p^k - p^{k-1}\right)+1\right)\left(p^k - p^{k-1}\right)
\end{aligned}$$

and

$$\left|E\left(\Gamma\left(K_{2p^{k-1}-1,2(p^k-p^{k-1})+1}\right)\right)\right| = \left(2p^{k-1}-1\right)\cdot\left(2\left(p^k - p^{k-1}\right)+1\right).$$

□

3.2. **Reverse-Based Topological Indices of Order GCD Graph.**

This part investigates the reverse-based topological indices of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$. In particular, we first derive the reverse Sombor index, which is presented in Theorem 3.11.

Theorem 3.11. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of the ring of integers modulo $2p^k$, where p an odd prime and $k \geq 2$ be an integer, then the reverse Sombor index is*

$$RSO\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) = 2\left(p^k - p^{k-1} + 1\right)\left(2\left(p^{k-1}-1\right)\sqrt{4\left(p^{k-1}-1\right)^2 + 1} + \left(p^k - p^{k-1} + \frac{1}{2}\right)\sqrt{2}\right).$$

Proof. Let $V_1 = \left\{p, 2p, 3p, \ldots, \left(p^{k-1}-1\right)p\right\} \backslash \{p^k\}$ where $u_1 \in V_1$ and $v_1, v_2 \notin V_1$. From Theorem 3.3, we obtain $c_{u_1} = 2\left(p^{k-1}-1\right)$ and $c_{v_1} = c_{v_2} = 1$. The number of edges connecting vertices u_1 and v_1 is equal to the number of edges in the complete bipartite subgraph conforming Theorem 3.9), which is $4\left(p^{k-1}-1\right)\left(p^k - p^{k-1}+1\right)$. Next, the number of edges connecting vertices v_1 and v_2 is equal to the number of edges in the complete subgraph by Theorem 3.9), which is $2\left(p^k - p^{k-1} + 1\right)\left(p^k - p^{k-1} + \frac{1}{2}\right)$. Thus, the general formula for the Reverse Sombor index of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is given by

$$\begin{aligned} RSO\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= \sum_{uv \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \sqrt{(c_u)^2 + (c_v)^2} \\ &= \sum_{u_1v_1 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \sqrt{(c_{u_1})^2 + (c_{v_1})^2} + \sum_{v_1v_2 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \sqrt{(c_{v_1})^2 + (c_{v_2})^2}. \end{aligned}$$

By substituting the values obtained earlier, then

$$\begin{aligned} RSO\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= 4\left(p^{k-1}-1\right)\left(p^k - p^{k-1} + 1\right)\sqrt{\left(2\left(p^{k-1}-1\right)\right)^2 + 1^2} \\ &\quad + 2\left(p^k - p^{k-1} + 1\right)\left(p^k - p^{k-1} + \frac{1}{2}\right)\sqrt{1^2 + 1^2} \\ &= 2\left(p^k - p^{k-1} + 1\right)\left[2\left(p^{k-1}-1\right)\sqrt{4\left(p^{k-1}-1\right)^2 + 1} \right. \\ &\quad \left. + \left(p^k - p^{k-1} + \frac{1}{2}\right)\sqrt{2}\right], \end{aligned}$$

and the proof is completed. □

The following theorem presents the reverse Randic index, thereby complementing the previously discussed results.

Theorem 3.12. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of the ring of integers modulo $2p^k$ where p an odd prime and $k \geq 2$ be an integer, then the reverse Randic index is*

$$RR\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) = 2\left(p^k - p^{k-1} + 1\right)\left(\sqrt{2\left(p^{k-1}-1\right)} + \left(p^k - p^{k-1} + \frac{1}{2}\right)\right).$$

Proof. With the same argument as in the proof of Theorem 3.11, we have

$$RR\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) = \sum_{uv \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \frac{1}{\sqrt{c_u \cdot c_v}}$$

$$
\begin{aligned}
&= \sum_{u_1 v_1 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \frac{1}{\sqrt{c_{u_1} \cdot c_{v_1}}} + \sum_{v_1 v_2 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \frac{1}{\sqrt{c_{v_1} \cdot c_{v_2}}} \\
&= 4\left(p^{k-1}-1\right)\left(p^k - p^{k-1}+1\right) \frac{1}{\sqrt{2\left(p^{k-1}-1\right)\cdot 1}} \\
&\ + 2\left(p^k - p^{k-1}+1\right)\left(p^k - p^{k-1} + \frac{1}{2}\right) \frac{1}{\sqrt{1\cdot 1}} \\
&= 2\left(p^k - p^{k-1}+1\right)\left(\sqrt{2\left(p^{k-1}-1\right)} + \left(p^k - p^{k-1} + \frac{1}{2}\right)\right).
\end{aligned}
$$

Thus, the general formula for the Reverse Randić index of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is

$$
RR\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) = 2\left(p^k - p^{k-1}+1\right)\left(\sqrt{2\left(p^{k-1}-1\right)} + \left(p^k - p^{k-1} + \frac{1}{2}\right)\right).
$$

□

Next, we present the reverse harmonic index, which further extends the sequence of reverse-based topological indices discussed above.

Theorem 3.13. *Let $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ be the order GCD graph of the ring of integers modulo $2p^k$ where p an odd prime and $k \geq 2$ be an integer, then the reverse harmonic index is*

$$
RH\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) = 2\left(p^k - p^{k-1}+1\right)\left(4\left(p^{k-1}-1\right)\frac{1}{2p^{k-1}-1} + \left(p^k - p^{k-1} + \frac{1}{2}\right)\right).
$$

Proof. Under the same assumptions as in the proof of Theorem 3.11, we obtain

$$
\begin{aligned}
RH\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= \sum_{uv \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \frac{2}{c_u + c_v} \\
&= \sum_{u_1 v_1 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \frac{2}{c_{u_1} + c_{v_1}} + \sum_{v_1 v_2 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \frac{2}{c_{v_1} + c_{v_2}} \\
&= 4\left(p^{k-1}-1\right)\left(p^k - p^{k-1}+1\right) \frac{2}{2\left(p^{k-1}-1\right)+1} \\
&\ + 2\left(p^k - p^{k-1}+1\right)\left(p^k - p^{k-1} + \frac{1}{2}\right) \frac{2}{1+1} \\
&= 4\left(p^{k-1}-1\right)\left(p^k - p^{k-1}+1\right) \frac{2}{2p^{k-1}-1} \\
&\ + 2\left(p^k - p^{k-1}+1\right)\left(p^k - p^{k-1} + \frac{1}{2}\right) \\
&= 2\left(p^k - p^{k-1}+1\right)\left(4\left(p^{k-1}-1\right)\frac{1}{2p^{k-1}-1} + \left(p^k - p^{k-1} + \frac{1}{2}\right)\right).
\end{aligned}
$$

Thus, the general formula for the reverse Harmonic index of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is

$$RH\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)=2\left(p^k-p^{k-1}+1\right)\left(4\left(p^{k-1}-1\right)\frac{1}{2p^{k-1}-1}+\left(p^k-p^{k-1}+\frac{1}{2}\right)\right).$$

□

In the order GCD graph $\Gamma(\mathbb{Z}_{2\cdot3^2})$, see Figure 1a, it is evident that $\deg(3)=\deg(6)=\deg(12)=\deg(15)=14$. In other words, the other vertices have a degree of 17. Based on the definition of the reverse degree, we obtain $c_3=c_6=c_{12}=c_{15}=4$ while the other vertices have a reverse degree of 1. The number of edges adjacent to vertices $3,6,12$, and 15 is 56. Next, according to Theorem 3.5, we find that $|E(\Gamma(\mathbb{Z}_{2\cdot3^2}))|=147$. Thus, the number of edges adjacent to vertices other than $3,6,12$, and 15 is 91. Therefore, based on the definition of the Reverse Sombor Index, we get

$$RSO\left(\Gamma\left(\mathbb{Z}_{2\cdot3^2}\right)\right)=91\cdot\sqrt{1^2+1^2}+56\cdot\sqrt{1^2+4^2}=91\sqrt{2}+56\sqrt{17}.$$

Next, by applying Theorem 3.11, we have

$$\begin{aligned}RSO\left(\Gamma\left(\mathbb{Z}_{2\cdot3^2}\right)\right)&=2\left(3^2-3^{2-1}+1\right)\left(2\left(3^{2-1}-1\right)\sqrt{4\left(3^{2-1}-1\right)^2+1}\right.\\&\quad\left.+\left(3^2-3^{2-1}+\frac{1}{2}\right)\sqrt{2}\right)\\&=14\left(4\sqrt{67}+\frac{13}{2}\sqrt{2}\right)\\RSO\left(\Gamma\left(\mathbb{Z}_{2\cdot3^2}\right)\right)&=56\sqrt{67}+91\sqrt{2}.\end{aligned}$$

Using the same argument as in the calculation of the reverse Sombor index above, and based on the definition of the reverse Randic index, we have the following

$$RR\left(\Gamma\left(\mathbb{Z}_{2\cdot3^2}\right)\right)=91\cdot\frac{1}{\sqrt{1\cdot1}}+56\cdot\frac{1}{\sqrt{1\cdot4}}=119.$$

According to 3.12, we can state that

$$RR\left(\Gamma\left(\mathbb{Z}_{2\cdot3^2}\right)\right)=2\left(3^2-3^{2-1}+1\right)\left(\sqrt{2\left(3^{2-1}-1\right)}+\left(3^2-3^{2-1}+\frac{1}{2}\right)\right)=119.$$

The reverse harmonic Index of $\Gamma\left(\mathbb{Z}_{2\cdot3^2}\right)$ based on its definition is given by

$$RH\left(\Gamma\left(\mathbb{Z}_{2\cdot3^2}\right)\right)=91\cdot\frac{2}{1+1}+56\cdot\frac{2}{1+4}=113\frac{2}{5}.$$

Moreover, by Theorem 3.13, it can be written as

$$\begin{aligned}RH\left(\Gamma\left(\mathbb{Z}_{2\cdot3^2}\right)\right)&=2\left(3^2-3^{2-1}+1\right)\left(4\left(3^{2-1}-1\right)\frac{1}{2\cdot3^{2-1}-1}+\left(3^2-3^{2-1}+\frac{1}{2}\right)\right)\\&=14\left(8\cdot\frac{1}{5}+\frac{13}{2}\right)\\&=113\frac{2}{5}.\end{aligned}$$

Regardless of whether the reverse index values are computed using the definition or the theorem, the results are the same.

4. Conclusion

In this article, we have described the properties of the order GCD graph of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ and the formulation of reverse-based indices referred to as reverse Sombor, Randic, and Harmonic indices. We have calculated these indices for two cases: odd and even primes p. Future work may extend this study to finite abelian p-groups and to fuzzy subgroups of finite abelian groups, as discussed in recent literature. These frameworks offer promising tools for developing more general subgroup-based and fuzzy graph models, thereby enhancing the scope and relevance of the results within algebraic graph theory.

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