J. Indones. Math. Soc.
Vol. 21, No. 1 (2015), pp. 45–58.

FUZZY TRANSLATIONS OF FUZZY H-IDEALS IN BCK/BCI-ALGEBRAS

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Abstract. In this paper, the concepts of fuzzy translation to fuzzy H-ideals in BCK/BCI-algebras are introduced. The notion of fuzzy extensions and fuzzy multiplications of fuzzy H-ideals with several related properties are investigated. Also, the relationships between fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy H-ideals are investigated.

Key words and Phrases: Fuzzy ideal, fuzzy H-ideal, fuzzy translation, fuzzy extension, fuzzy multiplication.

Abstrak. Dalam paper ini, diperkenalkan konsep pergeseran fuzzy pada fuzzy H-ideals dalam BCK/BCI-algebra. Kemudian diperiksa ide perluasan dan perkalian fuzzy dari fuzzy H-ideal dengan beberapa sifat terkait. Diperiksa juga hubungan antara pergeseran fuzzy, perluasan fuzzy, dan perkalian fuzzy dari fuzzy H-ideal.

Kata kunci: Fuzzy ideal, fuzzy H-ideal, pergeseran fuzzy, perluasan fuzzy, perkalian fuzzy.

2010 Mathematics Subject Classification: 06F35, 03G25, 08A72.
Received: 04-03-2014, revised: 15-02-2015, accepted: 22-02-2015.

1. INTRODUCTION

Fuzzy set theory, which was introduced by Zadeh [24], is the oldest and most widely reported component of present day soft computing, allowing the design of more flexible information processing systems [18], with applications in different areas, such as artificial intelligence, multiagent systems, machine learning, knowledge discovery, information processing, statistics and data analysis, system modeling, control system, decision sciences, economics, medicine and engineering, as shown in the recent literature collected by Dubois et al. [2, 3]. Fuzzy logic provides a precise formalization and the effective means for the mechanization of the human capabilities of approximate reasoning and decision making in an environment of imperfect information, allowing the performance of a wide variety of physical and mental tasks without any measurements and any computations [25].

In [8, 9], BCK-algebras and BCI-algebras are abbreviated to two Boolean algebras. The former was raised in 1966 by Imai and Iseki, and the latter was primitives in the same year due to Iseki. In 1991, Xi [23] applied the concept of fuzzy sets to BCK-algebras. In 1993, Jun [10] and Ahmad [1] applied it to BCI-algebras. After that Jun, Meng, Liu and several researchers investigated further properties of fuzzy BCK-algebras and fuzzy ideals (see [4, 6, 7, 14-20]). In 1999, Khalid and Ahmad [12] introduced fuzzy H-ideals in BCI-algebras. In 2010, Satyanarayana et al. [19] introduced intuitionistic fuzzy H-ideals in BCK-algebras. Lee et al. [13] and Jun [11] discussed fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy subalgebras and ideals in BCK/BCI-algebras. They investigated relations among fuzzy translations, fuzzy extensions and fuzzy multiplications.

In this paper, fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy H-ideals in BCK/BCI-algebras are discussed. Relations among fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy H-ideals in BCK/BCI-algebras are also investigated.

2. Introduction

In this section, some elementary aspects that are necessary for this paper are included.
By a BCI-algebra we mean an algebra X with a constant 0 and a binary operation "$*$" satisfying the following axioms for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$
(ii) $(x * (x * y)) * y = 0$
(iii) $x * x = 0$
(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$.

If a BCI-algebra X satisfies $0 * x = 0$, for all $x \in X$, then we say that X is a BCK-algebra. Any BCK-algebra X satisfies the following axioms for all

$x, y, z \in X$:

(1) $(x * y) * z = (x * z) * y$

(2) $((x * z) * (y * z)) * (x * y) = 0$

(3) $x * 0 = x$

(4) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0.$

Throughout this paper, X always means a BCK/BCI-algebra without any specification.

A non-empty subset S of X is called a subalgebra of X if $x*y \in S$ for any $x, y \in S$.

A nonempty subset I of X is called an ideal of X if it satisfies

(I_1) $0 \in I$ and

(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

A non-empty subset I of X is said to be an H-ideal [12] of X if it satisfies (I_1) and

(I_3) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$, for all $x, y, z \in X$.

A BCI-algebra is said to be associative [5] if $(x*y)*z = x*(y*z)$, for all $x, y, z \in X$.

A fuzzy set $\mu : X \to [0, 1]$ is called a fuzzy subalgebra of X if it satisfies the inequality $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

A fuzzy set μ in X is called a fuzzy ideal [1, 23] of X if it satisfies

(F_1) $\mu(0) \geq \mu(x)$ and

(F_2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$, for all $x, y \in X$.

A fuzzy set μ in X is called a fuzzy H-ideal [12] of X if it satisfies (F_1) and

(F_3) $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$, for all $x, y, z \in X$.

Example 2.1. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	4	3	4
2	2	2	0	5	5	3
3	3	3	3	0	0	0
4	4	3	4	1	0	1
5	5	5	3	2	2	0

Let μ be a fuzzy subset of X defined by $\mu(0) = t_0$, $\mu(1) = t_1$ and $\mu(x) = t_2$ for all $x \in X \setminus \{0, 1\}$, where $t_0 > t_1 > t_2$ and $t_0, t_1, t_2 \in [0, 1]$. Routine calculations show that μ is a fuzzy H-ideal of X. Since every fuzzy H-ideal is a fuzzy ideal, therefore it is also a fuzzy ideal.

3. Main Results

Throughout this paper, we take $\top = 1 - \sup\{\mu(x) \mid x \in X\}$ for any fuzzy set μ of X.

Definition 3.1. [13] Let μ be a fuzzy subset of X and let $\alpha \in [0, \top]$. A mapping $\mu_\alpha^T : X \to [0, 1]$ is called a fuzzy α-translation of μ if it satisfies $\mu_\alpha^T(x) = \mu(x) + \alpha$, for all $x \in X$.

Theorem 3.2. *If μ is a fuzzy H-ideal of X, then the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X, for all $\alpha \in [0, \top]$.*

PROOF. Assume that μ is a fuzzy H-ideal of X and let $\alpha \in [0, \top]$. Then, $\mu_\alpha^T(0) = \mu(0) + \alpha \geq \mu(x) + \alpha = \mu_\alpha^T(x)$ and

$$\begin{aligned} \mu_\alpha^T(x * z) &= \mu(x * z) + \alpha \geq \min\{\mu(x * (y * z)), \mu(y)\} + \alpha \\ &= \min\{\mu(x * (y * z)) + \alpha, \mu(y) + \alpha\} \\ &= \min\{\mu_\alpha^T(x * (y * z)), \mu_\alpha^T(y)\} \end{aligned}$$

for all $x, y, z \in X$. Hence, the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X.□

Theorem 3.3. *Let μ be a fuzzy subset of X such that the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X for some $\alpha \in [0, \top]$. Then, μ is a fuzzy H-ideal of X.*

PROOF. Assume that μ_α^T is a fuzzy H-ideal of X for some $\alpha \in [0, \top]$. Let $x, y, z \in X$ then $\mu(0) + \alpha = \mu_\alpha^T(0) \geq \mu_\alpha^T(x) = \mu(x) + \alpha$ and so $\mu(0) \geq \mu(x)$. Now, we have

$$\begin{aligned} \mu(x * z) + \alpha &= \mu_\alpha^T(x * z) \geq \min\{\mu_\alpha^T(x * (y * z)), \mu_\alpha^T(y)\} \\ &= \min\{\mu(x * (y * z)) + \alpha, \mu(y) + \alpha\} \\ &= \min\{\mu(x * (y * z)), \mu(y)\} + \alpha \end{aligned}$$

which implies that $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$. Hence, μ is a fuzzy H-ideal of X. □

We now discuss the relation between fuzzy subalgebras and fuzzy α-translation μ_α^T of μ for the fuzzy H-ideals.

Theorem 3.4. *If the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X, for all $\alpha \in [0, \top]$ then it must be a fuzzy subalgebra of X.*

PROOF. Let the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X. Then, we have $\mu_\alpha^T(x * z) \geq \min\{\mu_\alpha^T(x * (y * z)), \mu_\alpha^T(y)\}$. Substituting y for z we get

$$\begin{aligned} \mu_\alpha^T(x * y) &\geq \min\{\mu_\alpha^T(x * (y * y)), \mu_\alpha^T(y)\} \\ &= \min\{\mu_\alpha^T(x * 0), \mu_\alpha^T(y)\} \\ &= \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}. \end{aligned}$$

Therefore, μ_α^T is a fuzzy subalgebra of X. □

Proposition 3.5. *Let μ be a fuzzy subset of X such that the fuzzy α-translation μ_α^T of μ is a fuzzy ideal of X for $\alpha \in [0, \top]$. If $(x * a) * b = 0$, for all $a, b, x \in X$, then $\mu_\alpha^T(x) \geq \min\{\mu_\alpha^T(a), \mu_\alpha^T(b)\}$.*

PROOF. Let $a, b, x \in X$ be such that $(x * a) * b = 0$. Then,

$$\begin{aligned} \mu_\alpha^T(x) &\geq \min\{\mu_\alpha^T(x * a), \mu_\alpha^T(a)\} \\ &\geq \min\{\min\{\mu_\alpha^T((x * a) * b), \mu_\alpha^T(b)\}, \mu_\alpha^T(a)\} \\ &= \min\{\min\{\mu_\alpha^T(0), \mu_\alpha^T(b)\}, \mu_\alpha^T(a)\} \\ &= \min\{\mu_\alpha^T(b), \mu_\alpha^T(a)\} \text{ since } \mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(\mathrm{b}) \\ &= \min\{\mu_\alpha^T(a), \mu_\alpha^T(b)\} \end{aligned}$$

The proof is complete. □

The following can easily be proved by induction.

Corollary 3.6. *Let μ be a fuzzy subset of X such that the fuzzy α-translation μ_α^T of μ is a fuzzy ideal of X for $\alpha \in [0, \top]$. If $(\cdots((x * a_1) * a_2) * \cdots) * a_n = 0$, for all $x, a_1, a_2, \ldots, a_n \in X$, then $\mu_\alpha^T(x) \geq \min\{\mu_\alpha^T(a_1), \mu_\alpha^T(a_2), \ldots, \mu_\alpha^T(a_n)\}$.*

We now give a condition for the fuzzy α-translation μ_α^T of μ which is a fuzzy ideal of X to be a fuzzy H-ideal of X.

Theorem 3.7. *Let μ be a fuzzy subset of X such that the fuzzy α-translation μ_α^T of μ is a fuzzy ideal of X for $\alpha \in [0, \top]$. If it satisfies the condition $\mu_\alpha^T(x*y) \geq \mu_\alpha^T(x)$, for all $x, y \in X$, then the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X.*

Proof. Let the fuzzy α-translation μ_α^T of μ is a fuzzy ideal of X. For any $x, y, z \in X$, we have

$$\begin{aligned} \mu_\alpha^T(x*z) &\geq \min\{\mu_\alpha^T((x*z)*(y*z)), \mu_\alpha^T(y*z)\} \\ &= \min\{\mu_\alpha^T((x*(y*z))*z), \mu_\alpha^T(y*z)\} \\ &\geq \min\{\mu_\alpha^T(x*(y*z)), \mu_\alpha^T(y)\}. \end{aligned}$$

Hence, the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X for some $\alpha \in [0, \top]$. □

Theorem 3.8. *If μ be a fuzzy subset of associative BCI/BCK-algebra X such that the fuzzy α-translation μ_α^T of μ is a fuzzy ideal of X for $\alpha \in [0, \top]$, then the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X.*

Proof. Let the fuzzy α-translation μ_α^T of μ is a fuzzy ideal of X. For any $x, y, z \in X$, we have

$$\begin{aligned} \mu_\alpha^T(x*z) &\geq \min\{\mu_\alpha^T((x*z)*y), \mu_\alpha^T(y)\} \\ &= \min\{\mu_\alpha^T((x*y)*z), \mu_\alpha^T(y)\} \\ &= \min\{\mu_\alpha^T(x*(y*z)), \mu_\alpha^T(y)\} \end{aligned}$$

Hence, the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X. □

Theorem 3.9. *If μ be a fuzzy subset of X such that the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X for $\alpha \in [0, \top]$, then the set $I_\mu := \{x \in X \mid \mu_\alpha^T(x) = \mu_\alpha^T(0)\}$ is an H-ideal of X.*

Proof. Obviously, $0 \in I_\mu$. Let $x, y, z \in X$ be such that $x*(y*z) \in I_\mu$ and $y \in I_\mu$. Then, $\mu_\alpha^T(x*(y*z)) = \mu_\alpha^T(0) = \mu_\alpha^T(y)$ and so $\mu_\alpha^T(x*z) \geq \min\{\mu_\alpha^T(x*(y*z)), \mu_\alpha^T(y)\} = \mu_\alpha^T(0)$. Since, μ_α^T of μ is a fuzzy H-ideal of X, we conclude that $\mu_\alpha^T(x*z) = \mu_\alpha^T(0)$. This implies that $\mu(x*z) + \alpha = \mu(0) + \alpha$ or, $\mu(x*z) = \mu(0)$ so that $x*z \in I_\mu$. Therefore, I_μ is an H-ideal of X. □

Proposition 3.10. *If the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X, then it is order reversing.*

Proof. Let $x, y \in X$ be such that $x \leq y$. Then, $x * y = 0$ and hence

$$\begin{aligned}\mu_\alpha^T(x) &= \mu_\alpha^T(x*0) \geq \min\{\mu_\alpha^T(x*(y*0)), \mu_\alpha^T(y)\} \\ &= \min\{\mu_\alpha^T(x*y), \mu_\alpha^T(y)\} = \min\{\mu_\alpha^T(0), \mu_\alpha^T(y)\} \\ &= \mu_\alpha^T(y).\end{aligned}$$

This completes the proof. □

The characterizations of fuzzy α-translation μ_α^T of μ are given by the following theorem.

Theorem 3.11. *Let μ be a fuzzy subset of X such that the fuzzy α-translation μ_α^T of μ is a fuzzy ideal of X, then the following assertions are equivalent:*

(i) μ_α^T is a fuzzy H-ideal of X,

(ii) $\mu_\alpha^T(x*y) \geq \mu_\alpha^T(x*(0*y))$, for all $x, y \in X$,

(iii) $\mu_\alpha^T((x*y)*z) \geq \mu_\alpha^T(x*(y*z))$, for all $x, y, z \in X$.

Proof. $(i) \Rightarrow (ii)$ Let μ_α^T is a fuzzy H-ideal of X. Then, for all $x, y \in X$ we have $\mu_\alpha^T(x*y) \geq \min\{\mu_\alpha^T(x*(0*y)), \mu_\alpha^T(0)\} = \mu_\alpha^T(x*(0*y))$. Therefore, the inequality (ii) is satisfied.

$(ii) \Rightarrow (iii)$ Assume that (ii) is satisfied. For all $x, y, z \in X$, we have $((x*y)*(0*z))*(x*(y*z)) = ((x*y)*(x*(y*z)))*(0*z) \leq ((y*z)*y)*(0*z) = ((y*y)*z)*(0*z) = (0*z)*(0*z) = 0$. It follows from Proposition 3.10 that $\mu_\alpha^T((x*y)*(0*z))*(x*(y*z)) \geq \mu_\alpha^T(0)$. Since μ_α^T is a fuzzy H-ideal of X, we have $\mu_\alpha^T((x*y)*(0*z))*(x*(y*z)) = \mu_\alpha^T(0)$.

Using (ii) we get

$$\begin{aligned}\mu_\alpha^T((x*y)*z) &\geq \mu_\alpha^T((x*y)*(0*z)) \\ &= \min\{\mu_\alpha^T(((x*y)*(0*z))*(x*(y*z))), \mu_\alpha^T(x*(y*z))\} \\ &= \min\{\mu_\alpha^T(0), \mu_\alpha^T(x*(y*z))\} \\ &= \mu_\alpha^T(x*(y*z)).\end{aligned}$$

Therefore, inequality (iii) is also satisfied.

$(iii) \Rightarrow (i)$ Assume that (iii) is valid. For all $x, y, z \in X$, we have

$$\begin{aligned}\mu_\alpha^T(x*z) &\geq \min\{\mu_\alpha^T((x*z)*y), \mu_\alpha^T(y)\} \\ &= \min\{\mu_\alpha^T((x*y)*z), \mu_\alpha^T(y)\} \\ &\geq \min\{\mu_\alpha^T(x*(y*z)), \mu_\alpha^T(y)\}.\end{aligned}$$

Therefore, μ_α^T is a fuzzy H-ideal of X. Hence, the assertion (i) holds. The proof is complete. □

Next we give another characterizations of fuzzy α-translation μ_α^T of μ in the following theorem.

Theorem 3.12. *Let μ be a fuzzy subset of X such that the fuzzy α-translation μ_α^T of μ is a fuzzy ideal of X, then the following assertions are equivalent:*

(i) μ_α^T is a fuzzy H-ideal of X,

(ii) $\mu_\alpha^T((x*z)*y) \geq \mu_\alpha^T((x*z)*(0*y))$, for all $x, y, z \in X$,

(iii) $\mu_\alpha^T(x*y) \geq \min\{\mu_\alpha^T((x*z)*(0*y)), \mu_\alpha^T(z)\}$, for all $x, y, z \in X$.

PROOF. $(i) \Rightarrow (ii)$ Same as above theorem.
$(ii) \Rightarrow (iii)$ Assume that (ii) is valid. For all $x, y, z \in X$, we have $\mu_\alpha^T(x * y) \geq \min\{\mu_\alpha^T((x * y) * z), \mu_\alpha^T(z)\} = \min\{\mu_\alpha^T((x * z) * y), \mu_\alpha^T(z)\} \geq \min\{\mu_\alpha^T((x * z) * (0 * y)), \mu_\alpha^T(z)\}$. Therefore, (iii) is satisfied.
$(iii) \Rightarrow (i)$ Assume that (iii) is valid. Therefore, for all $x, y, z \in X$, we have $\mu_\alpha^T(x * y) \geq \min\{\mu_\alpha^T((x * z) * (0 * y)), \mu_\alpha^T(z)\}$. Putting $z = 0$ we get $\mu_\alpha^T(x * y) \geq \min\{\mu_\alpha^T((x * 0) * (0 * y)), \mu_\alpha^T(0)\} = \min\{\mu_\alpha^T(x * (0 * y)), \mu_\alpha^T(0)\} = \mu_\alpha^T(x * (0 * y))$. It follows from Theorem 3.11 that μ_α^T is a fuzzy H-ideal of X. The proof is complete. □

Definition 3.13. [13] Let μ_1 and μ_2 be fuzzy subsets of X. If $\mu_1 \leq \mu_2$, for all $x \in X$, then we say that μ_2 is a fuzzy extension of μ_1.

Definition 3.14. *Let μ_1 and μ_2 be fuzzy subsets of X. Then,, μ_2 is called a fuzzy H-ideal extension of μ_1 if the following assertions are valid:*

(i) μ_2 is a fuzzy extension of μ_1.

(ii) If μ_1 is a fuzzy H-ideal of X, then μ_2 is a fuzzy H-ideal of X.

From the definition of fuzzy α-translation, we get $\mu_\alpha^T(x) \geq \mu(x)$, for all $x \in X$. Therefore, we have the following theorem.

Theorem 3.15. *Let μ be a fuzzy H-ideal of X and $\alpha \in [0, \top]$. Then, the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal extension of μ.*

A fuzzy H-ideal extension of a fuzzy H-ideal μ may not be represented as a fuzzy α-translation of μ, that is, the converse of the Theorem 3.15 is not true in general as seen in the following example.

Example 3.16. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let μ be a fuzzy subset of X defined by

X	0	1	2	3	4
μ	0.8	0.7	0.6	0.5	0.5

Then, μ is a fuzzy H-ideal of X. Let ν be a fuzzy subset of X given by

X	0	1	2	3	4
ν	0.82	0.78	0.65	0.51	0.51

Then, ν is a fuzzy H-ideal extension of μ. But it is not the fuzzy α-translation μ_α^T of μ, for all $\alpha \in [0, \top]$.

Clearly, the intersection of fuzzy H-ideal extensions of a fuzzy subset μ of X is a fuzzy H-ideal extension of μ. But the union of fuzzy H-ideal extensions of a fuzzy subset μ of X is not a fuzzy H-ideal extension of μ as seen in the following example.

Example 3.17. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

Let μ be a fuzzy subset of X defined by

X	0	1	2	3	4
μ	0.74	0.52	0.52	0.52	0.52

Then, μ is a fuzzy H-ideal of X. Let ν and δ be a fuzzy subset of X given by

X	0	1	2	3	4
ν	0.87	0.77	0.54	0.54	0.77
δ	0.89	0.62	0.68	0.62	0.62

respectively. Then, ν and δ are fuzzy H-ideal extensions of μ. Obviously, the union $\nu \cup \delta$ is a fuzzy extension of μ, but it is not a fuzzy H-ideal extension of μ since $(\nu \cup \delta)(3 * 0) = (\nu \cup \delta)(3) = 0.62 \ngeq 0.68 = \min\{(\nu \cup \delta)(1), (\nu \cup \delta)(2)\} = \min\{(\nu \cup \delta)(3 * (2 * 0)), (\nu \cup \delta)(2)\}$.

For a fuzzy subset μ of X, $\alpha \in [0, \top]$ and $t \in [0, 1]$ with $t \geq \alpha$, let

$$U_\alpha(\mu; t) := \{x \in X \mid \mu(x) \geq t - \alpha\}.$$

If μ is a fuzzy H-ideal of X, then it is clear that $U_\alpha(\mu; t)$ is an H-ideal of X, for all $t \in Im(\mu)$ with $t \geq \alpha$. But, if we do not give a condition that μ is a fuzzy H-ideal of X, then $U_\alpha(\mu; t)$ is not an H-ideal of X as seen in the following example.

Example 3.18. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra in Example 3.17 and μ be a fuzzy subset of X defined by*

X	0	1	2	3	4
μ	0.78	0.62	0.43	0.62	0.62

Since $\mu(3 * 1) = \mu(3) = 0.43 \ngeq 0.62 = \min\{\mu(3), \mu(0)\} = \min\{\mu(3 * (0 * 1)), \mu(0)\}$, we have μ is not a fuzzy H-ideal of X. For $\alpha = 0.18$ and $t = 0.63$, we obtain $U_\alpha(\mu; t) = \{0, 1, 3, 4\}$ which is not an H-ideal of X since $3 * (0 * 1) = 3 \in \{0, 1, 3, 4\}$, but $3 * 1 = 2 \notin \{0, 1, 3, 4\}$.

Theorem 3.19. *Let μ be a fuzzy subset of X and $\alpha \in [0, \top]$. Then, the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X if and only if $U_\alpha(\mu; t)$ is an H-ideal of X, for all $t \in Im(\mu)$ with $t > \alpha$.*

Proof. Suppose that μ_α^T is a fuzzy H-ideal of X and $t \in Im(\mu)$ with $t > \alpha$. Since $\mu_\alpha^T(0) \geq \mu_\alpha^T(x)$, for all $x \in X$, we have $\mu(0)+\alpha = \mu_\alpha^T(0) \geq \mu_\alpha^T(x) = \mu(x)+\alpha \geq t$ for $x \in U_\alpha(\mu;t)$. Hence, $0 \in U_\alpha(\mu;t)$. Let $x, y, z \in X$ such that $x*(y*z), y \in U_\alpha(\mu;t)$. Then, $\mu(x*(y*z)) \geq t-\alpha$ and $\mu(y) \geq t-\alpha$ i.e., $\mu_\alpha^T(x*(y*z)) = \mu(x*(y*z))+\alpha \geq t$ and $\mu_\alpha^T(y) = \mu(y)+\alpha \geq t$. Since μ_α^T is a fuzzy H-ideal. So, we have $\mu(x*z)+\alpha = \mu_\alpha^T(x*z) \geq \min\{\mu_\alpha^T(x*(y*z)), \mu_\alpha^T(y)\} \geq t$ that is, $\mu(x*z) \geq t-\alpha$ so that $x*z \in U_\alpha(\mu;t)$. Therefore, $U_\alpha(\mu;t)$ is an H-ideal of X.

Conversely, suppose that $U_\alpha(\mu;t)$ is an H-ideal of X, for all $t \in Im(\mu)$ with $t > \alpha$. If there exists $a \in X$ such that $\mu_\alpha^T(0) < \beta \leq \mu_\alpha^T(a)$, then $\mu(a) \geq \beta - \alpha$ but $\mu(0) < \beta - \alpha$. This shows that $a \in U_\alpha(\mu;t)$ and $0 \notin U_\alpha(\mu;t)$. This is a contradiction, and $\mu_\alpha^T(0) \geq \mu_\alpha^T(x)$, for all $x \in X$.

Now we assume that there exist $a, b \in X$ such that $\mu_\alpha^T(a*c) < \gamma \leq \min\{\mu_\alpha^T(a*(b*c)), \mu_\alpha^T(b)\}$. Then, $\mu(a*(b*c)) \geq \gamma-\alpha$ and $\mu(b) \geq \gamma-\alpha$ but $\mu(a*c) < \gamma-\alpha$. Hence, $a*(b*c) \in U_\alpha(\mu;t)$ and $b \in U_\alpha(\mu;t)$ but $a*c \notin U_\alpha(\mu;t)$ which is a contradiction. Thus, $\mu_\alpha^T(x*z) \geq \min\{\mu_\alpha^T(x*(y*z)), \mu_\alpha^T(y)\}$, for all $x, y, z \in X$. Consequently, μ_α^T is a fuzzy H-ideal of X. □

If we put $t \leq \alpha$ in the sufficient part of the Theorem 3.19 then we get $U_\alpha(\mu;t) = X$.

Theorem 3.20. *Let μ be a fuzzy H-ideal of X and let $\alpha, \beta \in [0, \top]$. If $\alpha \geq \beta$, then the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal extension of the fuzzy β-translation μ_β^T of μ.*

Proof. Straightforward. □

Now, for every fuzzy H-ideal μ of X and $\beta \in [0, \top]$, the fuzzy β-translation μ_β^T of μ is a fuzzy H-ideal of X. If ν is a fuzzy H-ideal extension of μ_β^T, then there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and $\nu(x) \geq \mu_\alpha^T$, for all $x \in X$. Hence, we have the following theorem.

Theorem 3.21. *Let μ be a fuzzy H-ideal of X and $\beta \in [0, \top]$. For every fuzzy H-ideal extension ν of the fuzzy β-translation μ_β^T of μ, there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and ν is a fuzzy H-ideal extension of the fuzzy α-translation μ_α^T of μ.*

Let us illustrate the Theorem 3.21 using the following example.

Example 3.22. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:*

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

Let μ be a fuzzy subset of X defined by

X	0	1	2	3	4
μ	0.6	0.4	0.3	0.4	0.3

Then, μ is a fuzzy H-ideal of X and $\top = 0.4$. If we take $\beta = 0.17$, then the fuzzy β-translation μ_β^T of μ is given by

X	0	1	2	3	4
μ_β^T	0.77	0.57	0.47	0.57	0.47

Let ν be a fuzzy subset of X defined by

X	0	1	2	3	4
ν	0.83	0.67	0.55	0.67	0.55

Then, ν is a fuzzy H-ideal extension of the fuzzy β-translation μ_β^T of μ. But ν is not a fuzzy α-translation of μ, for all $\alpha \in [0, \top]$. If we take $\alpha = 0.21$ then $\alpha = 0.21 > 0.17 = \beta$ and the fuzzy α-translation μ_α^T of μ is given as follows:

X	0	1	2	3	4
μ_α^T	0.81	0.61	0.51	0.61	0.51

Note that $\nu(x) \geq \mu_\alpha^T(x)$, for all $x \in X$, and hence ν is a fuzzy H-ideal extension of the fuzzy α-translation μ_α^T of μ.

Definition 3.23. [13] Let μ be a fuzzy subset of X and $\gamma \in [0, 1]$. A fuzzy γ-multiplication of μ, denoted by μ_γ^m, is defined to be a mapping $\mu_\gamma^m : X \to [0, 1]$, $x \mapsto \mu(x).\gamma$

For any fuzzy subset μ of X, a fuzzy 0-multiplication μ_0^m of μ is clearly a fuzzy H-ideal of X.

Theorem 3.24. *If μ is a fuzzy H-ideal of X, then the fuzzy γ-multiplication of μ is a fuzzy H-ideal of X, for all $\gamma \in [0, 1]$.*

PROOF. Straightforward. □

Theorem 3.25. *Let μ be a fuzzy subset of X. Then, μ is a fuzzy H-ideal of X if and only if the fuzzy γ-multiplication μ_γ^m of μ is a fuzzy H-ideal of X, for all $\gamma \in [0, 1]$.*

PROOF. Necessity follows from Theorem 3.24. Let $\gamma \in (0, 1]$ be such that μ_γ^m is a fuzzy H-ideal of X. Then, $\mu(0).\gamma = \mu_\gamma^m(0) \geq \mu_\gamma^m(x) = \mu(x).\gamma$ which implies that $\mu(0) \geq \mu(x)$, for all $x \in X$. Also, for $x, y, z \in X$, we have,

$$\begin{aligned} \mu(x * z).\gamma &= \mu_\gamma^m(x * z) \geq min\{\mu_\gamma^m(x * (y * z)), \mu_\gamma^m(y)\} \\ &= min\{\mu(x * (y * z)).\gamma, \mu(y).\gamma\} \\ &= min\{\mu(x * (y * z)), \mu(y)\}.\gamma \end{aligned}$$

which implies that $\mu(x * z) \geq min\{\mu(x * (y * z)), \mu(y)\}$, for all $x, y, z \in X$. Hence, μ is a fuzzy H-ideal of X. □

Theorem 3.26. *Let μ be a fuzzy subset of X, $\alpha \in [0, 1]$ and $\gamma \in (0, 1]$. Then, every fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal extension of the fuzzy γ-multiplication μ_γ^m of μ.*

Proof. For all $x \in X$, we have $\mu_\alpha^T(x) = \mu(x) + \alpha \geq \mu(x) \geq \mu(x).\gamma = \mu_\gamma^m(x)$ and so μ_α^T is a fuzzy extension of μ_γ^m. Assume that μ_γ^m is a fuzzy H-ideal of X. Then, by Theorem 3.25, μ is a fuzzy H-ideal of X. It follows from Theorem 3.2 that the fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal of X, for all $\alpha \in [0, \top]$. Therefore, every fuzzy α-translation μ_α^T of μ is a fuzzy H-ideal extension of fuzzy γ-multiplication μ_γ^m of μ. □

The following example shows that Theorem 3.26 is not valid for $\gamma = 0$.

Example 3.27. *Let $(\mathbb{Z}, *, 0)$ be a BCI-algebra, where $\mathbb{Z}$ is the set of all integers and $*$ is the minus operation. Define a fuzzy subset $\mu : \mathbb{Z} \to [0,1]$ by*

$$\mu(x) := \begin{cases} 0.4 & \text{if } x > 2 \\ 0.6 & \text{if } x \leq 2 \end{cases}$$

If we take $\gamma = 0$, then $\mu_0^m(x * z) = 0 = \min\{\mu_0^m(x * (y * z)), \mu_0^m(y)\}$, for all $x, y, z \in \mathbb{Z}$ that is, μ_0^m is a fuzzy H-ideal of $\mathbb{Z}$. But

$$\begin{aligned} \mu_\alpha^T(3 * 0) &= \mu_\alpha^T(3) = 0.4 + \alpha < 0.6 + \alpha \\ &= \min\{\mu(3 * (1 * 0)), \mu(1)\} + \alpha \\ &= \min\{\mu(3 * (1 * 0)) + \alpha, \mu(1) + \alpha\} \\ &= \min\{\mu_\alpha^T(3 * (1 * 0)), \mu_\alpha^T(1)\} \end{aligned}$$

for all $\alpha \in [0, 0.4]$, which shows that μ_α^T is not a fuzzy H-ideal of $\mathbb{Z}$. Hence, μ_α^T is not a fuzzy H-ideal extension of μ_0^m, for all $\alpha \in [0, 0.4]$.

Let us illustrate Theorem 3.26 using the following examples.

Example 3.28. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let μ be a fuzzy subset of X defined by

X	0	1	2	3	4
μ	0.8	0.6	0.6	0.4	0.4

Then, μ is a fuzzy H-ideal of X. If we take $\gamma = 0.15$, then the fuzzy 0.15-multiplication $\mu_{0.15}^m$ of μ is given by

X	0	1	2	3	4
$\mu_{0.15}^m$	0.12	0.09	0.09	0.06	0.06

Then, $\mu_{0.15}^m$ is a fuzzy H-ideal of X. Also, for any $\alpha \in [0, 0.2]$, the fuzzy α-translation μ_α^T of μ is given as follows:

X	0	1	2	3	4
μ_α^T	$0.8 + \alpha$	$0.6 + \alpha$	$0.6 + \alpha$	$0.4 + \alpha$	$0.4 + \alpha$

Then, μ_α^T is a fuzzy extension of $\mu_{0.15}^m$ and μ_α^T is always a fuzzy H-ideal of X, for all $\alpha \in [0, 0.2]$. Therefore, μ_α^T is a fuzzy H-ideal extension of $\mu_{0.15}^m$, for all $\alpha \in [0, 0.2]$.

Example 3.29. *Let* $X = \{0, 1, 2, 3, 4, 5\}$ *be a* BCI*-algebra with the following Cayley table:*

$*$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Let μ be a fuzzy subset of X defined by

X	0	1	2	3	4	5
μ	0.7	0.5	0.5	0.3	0.3	0.3

Then, μ is a fuzzy H-ideal of X. If we take $\gamma = 0.25$, then the fuzzy 0.25-multiplication $\mu_{0.25}^m$ of μ is given by

X	0	1	2	3	4	5
$\mu_{0.25}^m$	0.175	0.125	0.125	0.075	0.075	0.075

Then, $\mu_{0.25}^m$ is a fuzzy H-ideal of X. Also, for any $\alpha \in [0, 0.3]$, the fuzzy α-translation μ_α^T of μ is given as follows:

X	0	1	2	3	4	5
μ_α^T	$0.7+\alpha$	$0.5+\alpha$	$0.5+\alpha$	$0.3+\alpha$	$0.3+\alpha$	$0.3+\alpha$

Then, μ_α^T is a fuzzy extension of $\mu_{0.25}^m$ and μ_α^T is always a fuzzy H-ideal of X, for all $\alpha \in [0, 0.3]$. Therefore, μ_α^T is a fuzzy H-ideal extension of $\mu_{0.25}^m$, for all $\alpha \in [0, 0.3]$.

4. Concluding Remarks

In this paper, the notion of translation of fuzzy H-ideals in BCK/BCI-algebra are introduced and investigated some of their useful properties. We have shown that the fuzzy α-translation of a fuzzy H-ideal is a fuzzy H-ideal extension but the converse is not true. It is also shown that intersection of fuzzy H-ideal extensions of a fuzzy subset is a fuzzy H-ideal extension but the union of fuzzy H-ideal extensions of a fuzzy subset is not a fuzzy H-ideal extension. The relationships are discussed between fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy H-ideals in BCK/BCI-algebras.

It is our hope that this work would other foundations for further study of the theory of BCK/BCI-algebras. In our future study of fuzzy structure of BCK/BCI-algebra, may be the following topics should be considered: (i) to find translation of fuzzy a-ideals in BCK/BCI-algebra, (ii) to find translation of fuzzy

p-ideals in BCK/BCI-algebra, (iii) to find the relationship between translations of fuzzy H-ideals, a-ideals and p-ideals in BCK/BCI-algebra.

Acknowledgement. The authors would like to express sincere appreciation to the referees for their valuable suggestions and comments helpful in improving this paper.

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