J. Indones. Math. Soc.
Vol. 31, No. 4 (2025), pp. 1–7.

Pointwise Multipliers of Orlicz-Morrey Spaces

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Abstract. We investigate the space of pointwise multipliers of Orlicz-Morrey spaces. Using the Hölder inequality in Orlicz-Morrey spaces, we prove that the space of pointwise multipliers of Orlicz-Morrey spaces contains an Orlicz-Morrey space. We also prove a partial reverse inclusion of this result. In addition, we describe the space of pointwise multipliers of Orlicz-Morrey spaces by adding some growth conditions on Young functions. Our results can be viewed as an extension of the results on pointwise multipliers of Morrey spaces.

Key words and phrases: Morrey spaces, Orlicz spaces, Orlicz-Morrey spaces, pointwise multipliers.

1. INTRODUCTION

Suppose that we have two function spaces. One way to compare these spaces is to define a simple transformation from one into another function space that is bounded with respect to the norms of the function spaces. The transformation is the pointwise multiplier of a function from one function space to another, called a multiplier operator. The multiplier function lies in the third Lebesgue space in the multiplier transformation in two Lebesgue spaces. Explicitly, consider the multiplier operator $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ with $p > q$, and $T_g(f) = gf$, then $g \in L^r(\mathbb{R}^n)$ with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. In the case $p = q$, then $r = \infty$. For the case $p < q$, we have $g \equiv 0$, when the measure in $\mathbb{R}^n$ is a Lebesgue measure. But this is not the case with the multiplier operator for two Morrey spaces. This set of multiplier functions can only happen to certain Morrey spaces, either containing some Morrey spaces or the set is contained in a Morrey space. See [1, 2, 3] for the details.

The positive results also happen for multiplier operators in two different Orlicz spaces (see [4]). The set of multiplier functions is another Orlicz space with the

*Corresponding author
2020 Mathematics Subject Classification: 46E30.
Received: 05-03-2025, accepted: 13-06-2025.

Young function that can be derived from two given Young functions with a similar relation to Lebesgue spaces. We refer the reader to [5] and [6] for the characterization of pointwise multiplier operators in weak Orlicz spaces and weak Morrey spaces, respectively.

In this paper, we generalize this multiplier transformation to some Orlicz-Morrey spaces (see Definition 2.1). At this time, the results obtained are limited because the relation of the parameter in Lebesgue spaces and Morrey spaces cannot be directly extended in terms of Young functions. Nevertheless, we get several results about spaces of pointwise multipliers of Orlicz-Morrey spaces. Our first result is the fact that the space of pointwise multipliers from an Orlicz-Morrey space to another Orlicz-Morrey space contains the third Orlicz-Morrey spaces where their parameters are related by the usual assumption of the Hölder inequality in these spaces (see [7] and references therein). The partial reverse inclusion of this inclusion is our second result. Moreover, we prove a characterization of spaces of pointwise multipliers of Orlicz-Morrey spaces in terms of Orlicz-Morrey spaces by assuming additional growth conditions on Young functions (see Definition 2.2).

The remaining sections of this manuscript are organized as follows. We recall the definition of the space of pointwise multipliers and several facts about Orlicz-Morrey spaces in Section 2. Our main results and their proofs will be given in Section 3. Throughout this paper, the notation $A \lesssim B$ means the inequality $A \leq CB$ holds for some $C > 0$, independent of $A > 0$ and $B > 0$. Write $A \sim B$ if the inequalities $A \lesssim B$ and $B \lesssim A$ hold.

2. PRELIMINARIES

This section recalls some definitions, notations, and basic facts. Let X and Y be Banach spaces of measurable functions on $\mathbb{R}^n$. The space of pointwise multipliers from X to Y is defined by

$$\mathrm{PWM}(X,Y) := \{g : f \cdot g \in Y \text{ for all } f \in X\}.$$

For every $g \in \mathrm{PWM}(X,Y)$, we define

$$\|g\|_{\mathrm{PWM}(X,Y)} := \sup\left\{\frac{\|f \cdot g\|_Y}{\|f\|_X} : f \in X, f \neq 0\right\}.$$

Let us recall the definition of Orlicz spaces. The function $\Phi : [0,\infty) \to [0,\infty)$ is called a Young function if Φ is continuous, convex, and increasing with $\Phi(0) = 0$. Assume that Φ is not identically zero. The Orlicz space L^Φ is defined to be the set of all measurable functions f for which

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)\, dx < \infty,$$

for $\lambda > 0$. The norm in this space is defined by

$$\|f\|_{L^\Phi} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)\, dx \leq 1\right\}.$$

If $\Phi(t) := t^p$ with $1 \le p < \infty$, then $L^\Phi = L^p$. The characterization of the spaces of pointwise multipliers of Orlicz spaces can be found in [4]. The result of the characterization of spaces of pointwise multipliers on Musielak-Orlicz spaces can be seen in [8]. Related results in the setting of Musielak-Orlicz-Morrey spaces can be found in [9].

As mentioned in the first section, we will investigate the pointwise multipliers in Orlicz-Morrey spaces. These spaces are introduced in [10] and can act as a generalization of Morrey spaces. We recall the definition of these spaces below.

Definition 2.1. [10] *Let $B(x,r)$ be any ball centered at $x \in \mathbb{R}^n$ and $r > 0$. Define*

$$\|f\|_{\Phi;B(x,r)} := \inf\left\{\lambda > 0 : \frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi\left(\frac{|f(y)|}{\lambda}\right)\, dy \le 1\right\}.$$

Let $\varphi : (0,\infty) \to (0,\infty)$ be an increasing function and suppose that the function $t \mapsto t^{-n}\varphi(t)$ is decreasing. The Orlicz-Morrey space $\mathcal{M}_\Phi^\varphi$ is the space of all measurable functions f such that

$$\|f\|_{\mathcal{M}_\Phi^\varphi} := \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)\|f\|_{\Phi;B(x,r)} < \infty.$$

If $\Phi(t) := t^q$ and $\varphi(t) := t^{-\frac{n}{p}}$ where $1 \le q \le p < \infty$, then $\mathcal{M}_\Phi^\varphi$ is equal to the Morrey space

$$\mathcal{M}_q^p = \left\{f : \sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q\, dy\right)^{\frac{1}{q}} < \infty\right\}.$$

Remark that there are two other variants of Orlicz-Morrey spaces (see [11, 12, 13, 14]). In this paper, we only consider Definition 2.1.

We also assume the following growth conditions in describing spaces of pointwise multipliers of Orlicz-Morrey spaces in Theorem 3.3.

Definition 2.2. *The Young function Φ is said to satisfy the Δ'-condition if $\Phi(st) \lesssim \Phi(s)\Phi(t)$ for every $s,t \ge 0$. If there exists a constant $C > 0$ such that inequality $\Phi(s)\Phi(t) \le C\Phi(st)$ holds for every $s,t \ge 0$, then we call Φ satisfies the ∇'-condition.*

3. MAIN RESULTS

Our first result is that the space of a pointwise multiplier from an Orlicz-Morrey space to another Orlicz-Morrey space contains certain Orlicz-Morrey spaces.

Theorem 3.1. *Let Φ_0 and Φ_1 be Young functions. Let Φ_2 be a Young function satisfying the inequality*

$$\Phi_1(st) \lesssim \Phi_0(s) + \Phi_2(t)$$

for every $s,t \ge 0$. Suppose that φ_0 and φ_1 are positive and increasing functions on $(0,\infty)$ such that $t^{-n}\varphi_0(t)$ and $t^{-n}\varphi_1(t)$ are decreasing functions. If φ_2 is a function that satisfies

$$\varphi_1(r) \lesssim \varphi_0(r)\varphi_2(r)$$

for every $r > 0$, then

$$\mathcal{M}^{\varphi_2}_{\Phi_2} \subseteq \mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0}, \mathcal{M}^{\varphi_1}_{\Phi_1}).$$

Proof. Let $f \in \mathcal{M}^{\varphi_0}_{\Phi_0}$ and $g \in \mathcal{M}^{\varphi_2}_{\Phi_2}$. We will show that

$$T_g : f \mapsto g \cdot f$$

is a bounded operator from $\mathcal{M}^{\varphi_0}_{\Phi_0}$ to $\mathcal{M}^{\varphi_1}_{\Phi_1}$. We show that the inequality

$$\|f \cdot g\|_{\Phi_1;B(x,r)} \leq 2\|f\|_{\Phi_0;B(x,r)}\|g\|_{\Phi_2;B(x,r)} \tag{1}$$

holds for every $x \in \mathbb{R}^n$ and $r > 0$. Note that, for every $\varepsilon > 0$ we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi_0\left(\frac{|f(y)|}{\|f\|_{\Phi_0,B(x,r)} + \varepsilon}\right)\, dy \leq 1$$

and

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi_2\left(\frac{|g(y)|}{\|g\|_{\Phi_2,B(x,r)} + \varepsilon}\right)\, dy \leq 1.$$

By the relations of Φ_0, Φ_1, and Φ_2, we get

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi_1\left(\frac{|f(y)g(y)|}{2(\|f\|_{\Phi_0,B(x,r)} + \varepsilon)(\|g\|_{\Phi_2,B(x,r)} + \varepsilon)}\right)\, dy \leq 1.$$

Therefore, we obtain the inequality (1).

Finally, we have that the multiplier operator is bounded from $M^{\varphi_0}_{\Phi_0}$ into $\mathcal{M}^{\varphi_1}_{\Phi_1}$, then

$$g \in \mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0}, \mathcal{M}^{\varphi_1}_{\Phi_1}).$$

□

We now prove a partial reverse inclusion of Theorem 3.1, namely, the space of pointwise multipliers of Orlicz-Morrey spaces can be realized as a subset of Orlicz-Morrey spaces.

Theorem 3.2. *Let Φ_0 and Φ_1 be Young functions. Assume that φ_0 and φ_1 are positive, increasing on $(0,\infty)$, and also the functions $t^{-n}\varphi_0(t)$ and $t^{-n}\varphi_1(t)$ are decreasing. If φ_2 satisfies $\varphi_1(r) \sim \varphi_0(r)\varphi_2(r)$ for every $r > 0$, then*

$$\mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0}, \mathcal{M}^{\varphi_1}_{\Phi_1}) \subseteq \mathcal{M}^{\varphi_2}_{\Phi_1}.$$

Proof. Let $g \in \mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0}, \mathcal{M}^{\varphi_1}_{\Phi_1})$, $x \in \mathbb{R}^n$, and $r > 0$. Since $\chi_{B(x,r)} \in \mathcal{M}^{\varphi_0}_{\Phi_0}$, we have

$$\varphi_2(r)\|g\|_{\Phi_1;B(x,r)} \sim \frac{\varphi_1(r)}{\varphi_0(r)}\|g\|_{\Phi_1;B(x,r)} \leq \frac{\|g\chi_{B(x,r)}\|_{\mathcal{M}^{\varphi_1}_{\Phi_1}}}{\varphi_0(r)}. \tag{2}$$

Since $g \in \mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0}, \mathcal{M}^{\varphi_1}_{\Phi_1})$ and $\|\chi_{B(x,r)}\|_{\mathcal{M}^{\varphi_0}_{\Phi_0}} \sim \varphi_0(r)$, we see that

$$\|g\chi_{B(x,r)}\|_{\mathcal{M}^{\varphi_1}_{\Phi_1}} \lesssim \|g\|_{\mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0},\mathcal{M}^{\varphi_1}_{\Phi_1})}\varphi_0(r). \tag{3}$$

Combining the inequalities (2) and (3), we obtain

$$\varphi_2(r)\|g\|_{\Phi_1;B(x,r)} \lesssim \|g\|_{\mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0},\mathcal{M}^{\varphi_1}_{\Phi_1})}.$$

Therefore, $g \in \mathcal{M}^{\varphi_2}_{\Phi_2}$ with

$$\|g\|_{\mathcal{M}^{\varphi_2}_{\Phi_2}} \lesssim \|g\|_{\mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0},\mathcal{M}^{\varphi_1}_{\Phi_1})}.$$

This completes the proof. □

Under some additional growth assumptions on Young functions, we give the following characterization of the space of pointwise multipliers of Orlicz-Morrey spaces.

Theorem 3.3. *Let Φ_0 and Φ_1 be Young functions. Assume that Φ_0 and Φ_1 satisfy Δ' and ∇' conditions. Define φ_2 and Φ_2 by*

$$\varphi_2(t) := \frac{\varphi_1(t)}{\varphi_0(t)} \quad \text{and} \quad \Phi_2^{-1}(t) := \frac{\Phi_1^{-1}(t)}{\Phi_0^{-1}(t)}.$$

Assume that $\dfrac{\Phi_0 \circ \Phi_2^{-1}(\varphi_0(t))}{\varphi_2(t)}$ is increasing. If $\varphi_0, \varphi_1, \varphi_2, \Phi_0, \Phi_1$, and Φ_2 also satisfy the condition in Theorem 3.1, then

$$\mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0}, \mathcal{M}^{\varphi_1}_{\Phi_1}) = \mathcal{M}^{\varphi_2}_{\Phi_2}.$$

Proof of Theorem 3.3. The inclusion $\mathcal{M}^{\varphi_2}_{\Phi_2} \subseteq \mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0}, \mathcal{M}^{\varphi_1}_{\Phi_1})$ is a consequence of Theorem 3.1. To prove the reverse inclusion, let $g \in \mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0}, \mathcal{M}^{\varphi_1}_{\Phi_1})$, we will show that $g \in \mathcal{M}^{\varphi_2}_{\Phi_2}$. For every $L > 0$, define

$$g_L := g\chi_{B(0,L)\cap\{|g|\le L\}}.$$

Since $g_L \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $L^\infty_{\mathrm{c}}(\mathbb{R}^n) \subseteq \mathcal{M}^{\varphi_2}_{\Phi_2}(\mathbb{R}^n)$, we see that $g_L \in \mathcal{M}^{\varphi_2}_{\Phi_2}(\mathbb{R}^n)$. Observe that,

$$\|g_L\|_{\Phi_2;B(x,r)} \le \frac{\Phi_1(\Phi_2^{-1}(\|g \cdot (\Phi_2 \circ \Phi_0^{-1}(|g_L|)\chi_{B(x,r)})\|_{\mathcal{M}^{\varphi_1}_{\Phi_1}}))}{\Phi_1(\Phi_2^{-1}(\varphi_1(r)))}.$$

The assumption $g \in \mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0}, \mathcal{M}^{\varphi_1}_{\Phi_1})$ implies

$$\|g \cdot (\Phi_2 \circ \Phi_0^{-1}(|g_L|)\chi_{B(x,r)})\|_{\mathcal{M}^{\varphi_1}_{\Phi_1}} \le \|g\|_{\mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0},\mathcal{M}^{\varphi_1}_{\Phi_1})} \|\Phi_2 \circ \Phi_0^{-1}(|g_L|)\chi_{B(x,r)}\|_{\mathcal{M}^{\varphi_0}_{\Phi_0}}. \tag{4}$$

Using the Δ'-condition of Φ_0, we get

$$\|\Phi_2 \circ \Phi_0^{-1}(|g_L|)\chi_{B(x,r)}\|_{\mathcal{M}^{\varphi_0}_{\Phi_0}} \lesssim (\Phi_2 \circ \Phi_0^{-1})(\|g_L\chi_{B(x,r)}\|_{\mathcal{M}^{\psi}_{\Phi_2}}),$$

where $\psi := \Phi_2^{-1} \circ \Phi_0 \circ \varphi_0$. We calculate the right-hand side of the last inequality to obtain

$$\|g_L\chi_{B(x,r)}\|_{\mathcal{M}^{\psi}_{\Phi_2}} \lesssim \|g_L\|_{\mathcal{M}^{\varphi_2}_{\Phi_2}} \frac{\psi(r)}{\varphi_2(r)}.$$

Combining the previous estimates, we have

$$\varphi_2(r)\|g_L\|_{\Phi_2;B(x,r)} \lesssim (\Phi_1 \circ \Phi_2^{-1})(\|g\|_{\mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0},\mathcal{M}^{\varphi_1}_{\Phi_1})})(\Phi_1 \circ \Phi_0^{-1})(\|g_L\|_{\mathcal{M}^{\varphi_2}_{\Phi_2}}).$$

Taking supremum over all $x \in \mathbb{R}^n$ and $r > 0$ and then using the ∇'-condition of Φ_1, we get

$$\|g_L\|_{\mathcal{M}^{\varphi_2}_{\Phi_2}} \lesssim \Phi_1\big(\Phi_2^{-1}(\|g\|_{\mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0},\mathcal{M}^{\varphi_1}_{\Phi_1})})\Phi_0^{-1}(\|g\|_{\mathcal{M}^{\varphi_2}_{\Phi_2}})\big).$$

Combining the last inequality with the definition of Φ_2, we obtain

$$\|g_L\|_{\mathcal{M}^{\varphi_2}_{\Phi_2}} \lesssim \|g\|_{\mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0},\mathcal{M}^{\varphi_1}_{\Phi_1})}.$$

Thus, $g \in \mathcal{M}^{\varphi_2}_{\Phi_2}$ follows from the last inequality and Fatou's lemma. This shows that $\mathrm{PWM}(\mathcal{M}^{\varphi_0}_{\Phi_0}, \mathcal{M}^{\varphi_1}_{\Phi_1}) \subseteq \mathcal{M}^{\varphi_2}_{\Phi_2}$. □

4. CONCLUDING REMARKS

In this paper, we obtain three results about the space of pointwise multipliers between two Orlicz-Morrey spaces. Our first result is these spaces contain an Orlicz-Morrey space. The second result is a partial reverse inclusion of the first result, namely the space of pointwise multipliers of Orlicz-Morrey spaces, which can be recognized as a subset of an Orlicz-Morrey space. The last result is a characterization of pointwise multipliers of two Orlicz-Morrey spaces as a third Orlicz-Morrey space under some additional growth conditions of Young functions. The complete description of pointwise multipliers between Orlicz-Morrey spaces with weaker assumptions can be further investigated in future research.

Acknowledgement. This research is supported by PPMI-ITB 2021 Program.

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