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Necessary Condition for Boundedness of Stein-Weiss Operator on Orlicz-Morrey Spaces

Sofihara Al Hazmy[1], Al Azhary Masta[2*], Siti Fatimah[3], and Harmanus Batkunde[4]

[1,2,3]Mathematics Study Program, Universitas Pendidikan Indonesia, Indonesia,
[1]sofiharaalhazmy@upi.edu, [2]alazhari.masta@upi.edu, [3]sitifatimah@upi.edu
[4]Department of Mathematics, Pattimura University, Indonesia,
harmanus.batkunde@lecturer.unpatti.ac.id.

Abstract. This study aims to find necessary conditions for the boundedness of Stein-Weiss Operator on Orlicz-Morrey spaces. It is well known that the Orlicz-Morrey space is the generalization of the Lebesque space. In particular, by considering power function as Young's function and zero as Morrey Space index, the Orlicz-Morrey space is a Lebesgue space itself. Orlicz-Morrey space and several operators in the space have been studied intensively by several researchers. In this study, we find a necessary condition for the boundedness of the Stein-Weiss operator on Orlicz-Morrey spaces. The technique to achieve our purpose is by substituting dilation of a radial function on a ball into inequality of boundedness assumption, and some basic properties of Young's function. Due to the properties of the dilation, the result will be presented as an inequality involving suitable parameters. As a result we get the necessary condition. As a discussion, we try to find several examples that satisfy the inequalities. The most important, the result shows that there is a significant factor to see the boundedness of the Stein-Weiss operator on Orlicz-Morrey spaces. Since Stein-Weiss operator is a generalization of fractional integral operator, the result shows that it generalize the boundedness of fractional integral on Orlicz space.

Key words and Phrases: Stein-Weiss Operator, Orlicz-Morrey Spaces.

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1. INTRODUCTION

Let $\lambda > 0$, the fractional integral operator T_λ is an operator on a function space given by

$$T_\lambda f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}} dy.$$

In one dimension, Hardy and Littlewood proved that $\frac{1}{q} = \frac{1}{p} - \lambda$ is a necessary condition for the boundedness of T_λ from the Lebesgue space $L^p([0,\infty)$ to the Lebesgue space $L^q([0,\infty))$. Then, Stein-Weiss extended their result in a weighted n-dimensional space. They show that

$$\frac{1}{q} = \frac{1}{p} + \frac{\alpha+\beta-\lambda}{n} \tag{1}$$

is a necessary condition for the boundedness of T_λ from the weighted Lebesgue space $L^p\left(\mathbb{R}^n, |\cdot|^\alpha\right)$ to the weighted Lebesgue space $L^q\left(\mathbb{R}^n, |\cdot|^\beta\right)$.

The Stein-Weiss operator is a generalization of the fractional integral operator, and was first introduced by Stein-Weiss [1]. That is,

$$T_{\alpha,\lambda,\beta} = \int_{\mathbb{R}^n} \frac{f(y)}{|y|^\alpha |x-y|^{n-\lambda}|x|^\beta}\, dy. \tag{2}$$

Several authors have studied further about quasilinear operators on several classical spaces, such as Orlicz Spaces, Morrey Spaces, Orlicz-Morrey Spaces, etc. Nakai [2] and Hashimoto et al. [3] obtained the sufficient conditions for the boundedness of generalized fractional integral operators on an integrated version of Orlicz spaces. Guliyev and Deringoz [4] derived the necessary and sufficient conditions for the boundedness of the fractional maximal operator on generalized Orlicz-Morrey spaces. Hästö [5] investigates the properties of the maximal operator on generalized Orlicz spaces. Cianchi [6] have acquired the necessary and sufficient conditions for boundedness of several classical operators on Orlicz spaces, and the results are very sharp. In Theorem 1, 2, 3 [6], he found the necessary and sufficient condition for the weak and strong boundedness of maximal operator, fractional integral operator, and singular integral on Orlicz spaces. In addition, Sawano et al. [7] have found the results of hard work for inequalities of fractional operators on Orlicz-Morrey spaces. Those are available in Theorems 2.8, 2.9, 2.10, and Corollary 2.11. Salim et. al. [8], [9], [10] extend Adams [11] results about the boundedness of rough fractional integral operator and Stein-Weiss operator on Morrey spaces. Fatimah et. al. [12] have provided a necessary condition for boundedness of Stein-Weiss operator on Orlicz space, they prove that $t^{\frac{\alpha+\beta-\lambda}{n}}\phi^{-1}(t) \lesssim \psi^{-1}(t)$ is necessary condition for boundedness of Stein-Weiss operator $T_{\alpha,\lambda,\beta}$ on Orlicz space $L^\phi(\mathbb{R}^n)$ to $L^\psi(\mathbb{R}^n)$. We denote $f(t) \lesssim g(t)$ if there exist a constant $C > 0$ such that $f(t) \le Cg(t)$ for any $t > 0$. In this research, we want to generalize Fatimah's[12] result. The results in [13, 14] motivate further investigation into the necessary condition in more generalized settings.

To describe the Orlicz-Morrey spaces, we need to recall several basic definitions as followings.

Definition 1.1 (Young's Function and Its Inverse). *A function* $\phi : [0, \infty) \to [0, \infty)$ *is said to be a Young's function if* ϕ *is convex function on* $[0, \infty)$*,* $\phi(0) = 0$*, and* $\lim_{t\to\infty} \phi(t) = \infty$*. The inverse of a Young's function* ϕ *is given by* $\phi^{-1}(s) = \inf\{t > 0 \mid \phi(t) > s\}$*.*

Young's function $\phi : [0, \infty) \to [0, \infty)$ at Definition 1.1 is surjective, so the set $\{t > 0 \mid \phi(t) > s\}$ is not empty for any $s \in [0, \infty)$, which implies its inverse at Definition 1.1 is well-defined.

Definition 1.2 (Luxemburg's Norm). *Suppose that* $\phi : [0, \infty) \to [0, \infty)$ *is a Young's function, and* $f : \mathbb{R}^n \to \mathbb{R}$ *is a measurable function. Luxemburg norm of* f *over* ϕ *is given by*

$$\|f\|_{L^\phi(\mathbb{R}^n)} = \inf\left\{b > 0 \;\middle|\; \int_{\mathbb{R}^n} \phi\left(\frac{|f(x)|}{b}\right)\, dx \le 1\right\}.$$

Definition 1.3 (Orlicz Spaces). *Suppose that* $\phi : [0, \infty) \to [0, \infty)$ *is Young's function (Definition 1.1). The set*

$$L^\phi(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \textit{ measurable}, \ \|f\|_{L^\phi(\mathbb{R}^n)} < \infty\}$$

is called Orlicz space. $\|.\|_{L^\phi(\mathbb{R}^n)}$ *denote the Luxemburg norm as described in Definition 1.2.*

Definition 1.4 (Morrey Spaces). *Suppose that* $0 \le \gamma \le n$ *and* $1 \le p < \infty$*. The set*

$$L^{p,\gamma}(\mathbb{R}^n) = \left\{f : \mathbb{R}^n \to \mathbb{R} \mid f \textit{ measurable}, \ \|f\|_{L^{p,\gamma}(\mathbb{R}^n)} < \infty\right\}$$

is called Morrey space, which

$$\|f\|_{L^{p,\gamma}(\mathbb{R}^n)} = \sup_{a\in\mathbb{R}^n,\, r>0} \left(\frac{1}{r^\gamma} \int_{B(a,r)} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Both Orlicz and Morrey spaces are generalization of Lebesgue space, by choosing $\phi(t) = t^p$ and $\gamma = 0$ implies $L^\phi(\mathbb{R}^n) = L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Definition 1.5 (Orlicz-Morrey Spaces). *Suppose that* $\phi : [0, \infty) \to [0, \infty)$ *is Young's function and* $0 \le \gamma \le n$*. The set*

$$L^{\phi,\gamma}(\mathbb{R}^n) = \left\{f : \mathbb{R}^n \to \mathbb{R} \mid f \textit{ measurable}, \ \|f\|_{L^{\phi,\gamma}(\mathbb{R}^n)} < \infty\right\}$$

is called Orlicz-Morrey space, which

$$\|f\|_{L^{\phi,\gamma}(\mathbb{R}^n)} = \sup_{a\in\mathbb{R}^n,\, r>0} \inf\left\{b > 0 \;\middle|\; \frac{1}{r^\gamma} \int_{B(a,r)} \phi\left(\frac{|f(x)|}{b}\right)\, dx \le 1\right\}.$$

Notice that $L^{\phi,0}(\mathbb{R}^n) = L^\phi(\mathbb{R}^n)$, and by choosing $\phi(t) = t^p$ we get $L^{\phi,\gamma}(\mathbb{R}^n) = L^{p,\gamma}(\mathbb{R}^n)$. These show that Orlicz-Morrey space is generalization of both Orlicz and Morrey spaces. Useful elements of Orlicz-Morrey spaces and its properties will be provided in Section 2.

2. Preliminaries of Orlicz-Morrey Spaces

Orlicz-Morrey space is not empty. Theorem 2.1 shows that a radial power function on unit ball is an element of any Orlicz-Morrey space.

Theorem 2.1. *Suppose that $\phi : [0, \infty) \to [0, \infty)$ is a Young's function, $0 \leq \gamma \leq n$, and $\alpha \geq 0$. Let $f : \mathbb{R}^n \to \mathbb{R}$ which $f(x) = |x|^\alpha \chi_{B(0,1)}(x)$, then $f \in L^{\phi,\gamma}(\mathbb{R}^n)$.*

Proof. By Definition 1.5,

$$\begin{aligned}
\|f\|_{L^{\phi,\gamma}(\mathbb{R}^n)} &= \sup_{a\in\mathbb{R}^n,\ r>0} \inf\left\{ b>0 : \frac{1}{r^\gamma}\int_{B(a,r)} \phi\left(\frac{|x|^\alpha \chi_{B(0,1)}}{b}\right) dx \leq 1 \right\} \\
&= \sup_{a\in\mathbb{R}^n,\ r>0} \inf\left\{ b>0 : \frac{1}{r^\gamma}\int_{B(a,r)\cap B(0,1)} \phi\left(\frac{|x|^\alpha}{b}\right) dx \leq 1 \right\} \\
&\leq \sup_{a\in\mathbb{R}^n,\ r>0} \inf\left\{ b>0 : \frac{\phi\left(\frac{1}{b}\right)|B(a,r)\cap B(0,1)|}{r^\gamma} \leq 1 \right\} \\
&= \sup_{a\in\mathbb{R}^n,\ r>0} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{|B(a,r)\cap B(0,1)|}\right)} \\
&= \sup_{r>0} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{|B(0,r)\cap B(0,1)|}\right)}.
\end{aligned}$$

The last equality is obtained since ϕ^{-1} is increasing and $|B(a,r) \cap B(0,1)| \leq |B(0,r) \cap B(0,1)|$. Notice that, for $r > 1$,

$$\begin{aligned}
\phi^{-1}\left(\frac{r^\gamma}{|B(0,r)\cap B(0,1)|}\right) &= \phi^{-1}\left(\frac{r^\gamma}{|B(0,1)|}\right) \\
&\geq \phi^{-1}\left(\frac{1^\gamma}{|B(0,1)\cap B(0,1)|}\right)
\end{aligned}$$

implies

$$\frac{1}{\phi^{-1}\left(\frac{1^\gamma}{|B(0,1)\cap B(0,1)|}\right)} \geq \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{|B(0,r)\cap B(0,1)|}\right)}.$$

So we have

$$\begin{aligned}
\sup_{r>0} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{|B(0,r)\cap B(0,1)|}\right)} &= \sup_{0<r\leq 1} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{|B(0,r)\cap B(0,1)|}\right)} \\
&= \sup_{0<r\leq 1} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{|B(0,r)|}\right)} \\
&= \sup_{0<r\leq 1} \frac{1}{\phi^{-1}\left(\frac{1}{a_n r^{n-\gamma}}\right)} \\
&\leq \frac{1}{\phi^{-1}\left(\frac{1}{a_n}\right)} < \infty.
\end{aligned}$$

This proves $f \in L^{\phi,\gamma}(\mathbb{R}^n)$. $\square$

Theorem 2.2. *Suppose that* $\phi : [0,\infty) \to [0,\infty)$ *is a Young's function,* $0 \le \gamma \le n$, $\alpha \ge 0$ *and* f *be a function in theorem 2.1. For any* $t > 0$, *define* $g_t(x) = f(tx)$, *then*

$$\|g_t\|_{L^{\phi,\gamma}(\mathbb{R}^n)} \le \frac{1}{\phi^{-1}\left(\frac{t^{n-\gamma}}{a_n}\right)},$$

which $a_n = |B(0,1)|$.

Proof. By Theorem 2.1, $f(x) = |x|^\alpha \chi_{B(0,1)}(x)$ is element of Orlicz-Morrey space $L^{\phi,\gamma}(\mathbb{R}^n)$, then $g_t(x) = f(tx)$ is also element of Orlicz-Morrey space $L^{\phi,\gamma}(\mathbb{R}^n)$. Notice that

$$\begin{aligned}
\int_{B(a,r)} \phi\left(\frac{|g_t(x)|}{b}\right) dx &= \int_{B(a,r)} \phi\left(\frac{|tx|^\alpha \chi_{B(0,1)}(tx)}{b}\right) dx \\
&= t^{-n} \int_{B(ta,tr)} \phi\left(\frac{|x|^\alpha \chi_{B(0,1)}(x)}{b}\right) dx \\
&= t^{-n} \int_{B(ta,tr)\cap B(0,1)} \phi\left(\frac{|x|^\alpha}{b}\right) dx \\
&\le t^{-n} |B(ta,tr) \cap B(0,1)| \, \phi\left(\frac{1}{b}\right).
\end{aligned}$$

Then we have,

$$\begin{aligned}
\|g_t\|_{L^{\phi,\gamma}(\mathbb{R}^n)} &\le \sup_{a\in\mathbb{R}^n, r>0} \inf\left\{ b > 0 : \frac{t^{-n} |B(ta,tr) \cap B(0,1)| \, \phi\left(\frac{1}{b}\right)}{r^\gamma} \le 1 \right\} \\
&= \sup_{a\in\mathbb{R}^n, r>0} \inf\left\{ b > 0 : \phi\left(\frac{1}{b}\right) \le \frac{r^\gamma}{t^{-n} |B(ta,tr) \cap B(0,1)|} \right\} \\
&= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{t^{-n}|B(ta,tr)\cap B(0,1)|}\right)}. \qquad (3)
\end{aligned}$$

Since ϕ^{-1} is increasing and $|B(ta,tr) \cap B(0,1)| \le |B(0,tr) \cap B(0,1)|$, we have

$$\sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{t^{-n}|B(ta,tr)\cap B(0,1)|}\right)} = \sup_{r>0} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{t^{-n}|B(0,tr)\cap B(0,1)|}\right)}.$$

For $tr > 1$, then $\frac{r^\gamma}{t^{-n}|B(0,tr)\cap B(0,1)|} = \frac{r^\gamma}{t^{-n}|B(0,1)|} > \frac{\left(\frac{1}{t}\right)^\gamma}{t^{-n}|B(0,1)\cap B(0,1)|}$, we have

$$\sup_{r>0} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{t^{-n}|B(0,tr)\cap B(0,1)|}\right)} = \sup_{tr\le 1} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{t^{-n}|B(0,tr)\cap B(0,1)|}\right)}.$$

For $tr < 1$, then $\frac{r^\gamma}{t^{-n}|B(0,tr)\cap B(0,1)|} = \frac{r^\gamma}{t^{-n}|B(0,tr)|} = \frac{1}{a_n r^{n-\gamma}} > \frac{1}{a_n t^{n-\gamma}} = \frac{\left(\frac{1}{t}\right)^\gamma}{t^{-n}|B(0,1)\cap B(0,1)|}$, we have

$$\sup_{tr\leq 1} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{t^{-n}|B(0,tr)\cap B(0,1)|}\right)} = \sup_{tr=1} \frac{1}{\phi^{-1}\left(\frac{r^\gamma}{t^{-n}|B(0,tr)\cap B(0,1)|}\right)}$$
$$= \frac{1}{\phi^{-1}\left(\frac{1}{a_n r^{n-\gamma}}\right)} \tag{4}$$

From (3) and (4) we have

$$\|g_t\|_{L^{\phi,\gamma}(\mathbb{R}^n)} \leq \frac{1}{\phi^{-1}\left(\frac{t^{n-\gamma}}{a_n}\right)}.$$

The proof is complete. □

Our objective is to find the necessary condition for the boundedness of the Stein-Weiss operator on Orlicz-Morrey space. There are at least three big steps to achieve our objective. First, the boundedness of Stein-Weiss Operator on Orlicz-Morrey spaces will be assumed. Second, we will substitute specific function

$$f(x) = |x|^\alpha \chi_{B(0,1)}(x)$$

and its dilation

$$g_t f(x) = f(tx)$$

into the inequality

$$\|T_{\alpha,\lambda,\beta} f\|_{L^{\phi_2,\gamma_2}(\mathbb{R}^n)} \lesssim \|f\|_{L^{\phi_1,\gamma_1}(\mathbb{R}^n)}.$$

Third, we will estimate $\|f\|_{L^{\phi_1,\gamma_1}(\mathbb{R}^n)}$ from above and $\|T_{\alpha,\lambda,\beta} f\|_{L^{\phi_2,\gamma_2}(\mathbb{R}^n)}$ from below to get some relation between $\phi_1,\ \phi_2, \gamma_1, \gamma_2, \alpha, \lambda, \beta$, which is it will be our necessary condition. The relation between the parameters will be presented in conclusion of Theorem 3.1.

3. MAIN RESULTS

In this section, we provide a necessary condition for the boundedness of the Stein–Weiss operator on Orlicz–Morrey spaces, presented as follows.

Theorem 3.1. *Suppose that $\phi_1,\ \phi_2$ are Young's function and $a_n = |B(0,1)|$. Let $\alpha,\ \beta \geq 0,\ 0 \leq \gamma_1 \leq n$, and $0 \leq \gamma_2 \leq n$. If $T_{\alpha,\lambda,\beta}$ is bounded from $L^{\phi_1,\gamma_1}(\mathbb{R}^n) \to L^{\phi_2,\gamma_2}(\mathbb{R}^n)$ then*

$$t^{\alpha+\beta-\lambda} \phi_1^{-1}\left(\frac{t^{n-\gamma_1}}{a_n}\right) \lesssim \phi_2^{-1}\left(\frac{t^{n-\gamma_2}}{a_n}\right). \tag{5}$$

Proof. Suppose that $T_{\alpha,\lambda,\beta}$ is bounded from $L^{\phi_1,\gamma_1}(\mathbb{R}^n) \to L^{\phi_2,\ \gamma_2}(\mathbb{R}^n)$, then

$$\|T_{\alpha,\lambda,\beta} h\|_{L^{\phi_2,\gamma_2}(\mathbb{R}^n)} \lesssim \|h\|_{L^{\phi_1,\gamma_1}(\mathbb{R}^n)},$$

for every $h \in L^{\phi_1,\gamma_1}(\mathbb{R}^n)$. By Theorem 2.1 $f(x) = |x|^\alpha \chi_{B(0,1)}(x)$ is element of Orlicz-Morrey space $L^{\phi_1,\gamma_1}(\mathbb{R}^n)$, then $g_t(x) = f(tx)$ is also element of Orlicz-Morrey space $L^{\phi_1,\gamma_1}(\mathbb{R}^n)$ for any $t > 0$. Therefore,

$$\|T_{\alpha,\lambda,\beta} g_t\|_{L^{\phi_2,\gamma_2}(\mathbb{R}^n)} \lesssim \|g_t\|_{L^{\phi_1,\gamma_1}(\mathbb{R}^n)}, \tag{6}$$

for any $t > 0$. First, we will estimate the right hand side of Inequality (6). By Theorem 2.2, we have

$$\|g_t\|_{L^{\phi_1,\gamma_1}(\mathbb{R}^n)} \leq \frac{1}{\phi_1^{-1}\left(\frac{t^{n-\gamma_1}}{a_n}\right)}.$$

Next, we will estimate $\|T_{\alpha,\lambda,\beta} f\|_{L^{\phi_2,\gamma_2}(\mathbb{R}^n)}$ from below. By using change of variables method, we have

$$T_{\alpha,\lambda,\beta} g_t(x) = \int_{\mathbb{R}^n} \frac{g_t(y)}{|y|^\alpha |x-y|^{n-\lambda} |x|^\beta}\, dy = t^{\alpha+\beta-\lambda} T_{\alpha,\lambda,\beta} f(tx).$$

Therefore,

$$\|T_{\alpha,\lambda,\beta} g_t\|_{L^{\phi_2,\gamma_2}(\mathbb{R}^n)} = t^{\alpha+\beta-\lambda} \sup_{a\in\mathbb{R}^n, r>0} \inf \left\{ b > 0 : \frac{t^{-n}}{r^{\gamma_2}} \int_{B(ta,tr)} \phi_2\left(\frac{|T_{\alpha,\lambda,\beta} f(x)|}{b}\right) dx \leq 1 \right\}.$$

Notice that,

$$T_{\alpha,\lambda,\beta} f(x) = \int_{B(0,1)} \frac{1}{|x-y|^{n-\lambda}|x|^\beta}\, dy.$$

Since the value of $\frac{1}{|x|^\beta}$ is radially decreasing, we have

$$\int_{B(ta,tr)} \phi_2\left(\frac{\int_{B(0,1)} \frac{1}{|x-y|^{n-\lambda}|x|^\beta}\, dy}{b}\right) dx \leq \int_{B(0,tr)} \phi_2\left(\frac{\int_{B(0,1)} \frac{1}{|x-y|^{n-\lambda}|x|^\beta}\, dy}{b}\right) dx.$$

Therefore,

$$\begin{aligned}
&\sup_{a\in\mathbb{R}^n, r>0} \inf \left\{ b > 0 : \frac{t^{-n}}{r^{\gamma_2}} \int_{B(ta,tr)} \phi_2\left(\frac{\int_{B(0,1)} \frac{1}{|x-y|^{n-\lambda}|x|^\beta}\, dy}{b}\right) dx \leq 1 \right\} \\
&\quad = \sup_{r>0} \inf \left\{ b > 0 : \frac{t^{-n}}{r^{\gamma_2}} \int_{B(0,tr)} \phi_2\left(\frac{\int_{B(0,1)} \frac{1}{|x-y|^{n-\lambda}|x|^\beta}\, dy}{b}\right) dx \leq 1 \right\} \\
&\quad \geq \sup_{0<r<\frac{1}{2t}} \inf \left\{ b > 0 : \frac{t^{-n}}{r^{\gamma_2}} \int_{B(0,tr)} \phi_2\left(\frac{\int_{B(0,1)} \frac{1}{|x-y|^{n-\lambda}|x|^\beta}\, dy}{b}\right) dx \leq 1 \right\}.
\end{aligned}$$

Also, if $0 < tr < \frac{1}{2}$ and $x \in B(0,tr)$ we have $B(x,tr) \subset B(0,1)$. Therefore,

$$\begin{aligned}
\int_{B(0,tr)} \phi_2 \left(\frac{|T_{\alpha,\lambda,\beta} f(x)|}{b} \right) dx &\geq \int_{B(0,tr)} \phi_2 \left(\frac{\int_{B(x,\ tr)} \frac{1}{|x-y|^{n-\lambda}}\ dy}{b^1 |x|^\beta} \right)\ dx \\
&= \int_{B(0,tr)} \phi_2 \left(\frac{a_n (tr)^\lambda}{\lambda b |x|^\beta} \right) dx \\
&\geq a_n (tr)^n \phi_2 \left(\frac{a_n (tr)^{\lambda-\beta}}{\lambda b} \right).
\end{aligned}$$

So, we have,

$$\begin{aligned}
\|T_{\alpha,\lambda,\beta} g_t\|_{L^{\phi_2,\gamma_2}(\mathbb{R}^n)} &\geq t^{\alpha+\beta-\lambda} \sup_{0<r<\frac{1}{2t}} \inf \left\{ b > 0 : \frac{t^{-n} a_n (tr)^n \phi_2 \left(\frac{a_n (tr)^{\lambda-\beta}}{\lambda b} \right)}{r^{\gamma_2}} \leq 1 \right\} \\
&= t^{\alpha+\beta-\lambda} \sup_{0<r<\frac{1}{2t}} \frac{a_n (tr)^{\lambda-\beta}}{\lambda \phi_2^{-1} \left(\frac{1}{a_n r^{n-\gamma_2}} \right)} \\
&\geq t^{\alpha+\beta-\lambda} \sup_{\frac{1}{4t}<r<\frac{1}{2t}} \frac{a_n (tr)^{\lambda-\beta}}{\lambda \phi_2^{-1} \left(\frac{1}{a_n r^{n-\gamma_2}} \right)} \\
&\gtrsim t^{\alpha+\beta-\lambda} \sup_{\frac{1}{4t}<r<\frac{1}{2t}} \frac{1}{\phi_2^{-1} \left(\frac{1}{a_n r^{n-\gamma_2}} \right)} \\
&\gtrsim t^{\alpha+\beta-\lambda} \frac{1}{\phi_2^{-1} \left(\frac{(4t)^{n-\gamma_2}}{a_n} \right)}.
\end{aligned}$$

Since $T_{\alpha,\lambda,\beta}$ is bounded from $L^{\phi_1,\gamma_1}(\mathbb{R}^n) \to L^{\phi_2,\ \gamma_2}(\mathbb{R}^n)$, then

$$t^{\alpha+\beta-\lambda} \frac{1}{\phi_2^{-1} \left(\frac{(4t)^{n-\gamma_2}}{a_n} \right)} \lesssim \frac{1}{\phi_1^{-1} \left(\frac{t^{n-\gamma_1}}{a_n} \right)}.$$

Since ϕ_2^{-1} concave and $4^{n-\gamma_2} \geq 1$, we have $\phi_2^{-1} \left(\frac{(4t)^{n-\gamma_2}}{a_n} \right) \lesssim \phi_2^{-1} \left(\frac{t^{n-\gamma_2}}{a_n} \right)$. So

$$t^{\alpha+\beta-\lambda} \phi_1^{-1} \left(\frac{t^{n-\gamma_1}}{a_n} \right) \lesssim \phi_2^{-1} \left(\frac{t^{n-\gamma_2}}{a_n} \right).$$

The proof is complete. $\square$

Theorem 3.1 in [12] generalized Theorem 1 pg. 3796. Specifically, for $\gamma_1 = \gamma_2 = 0$, we have

$$t^{\alpha+\beta-\lambda} \phi_1^{-1} \left(\frac{t^n}{a_n} \right) \lesssim \phi_2^{-1} \left(\frac{t^n}{a_n} \right).$$

By replacing t with $(a_n t)^{\frac{1}{n}}$ we have the inequality in Theorem 1 in [12], which is a necessary condition for boundedness of Stein-Weiss Operator $T_{\alpha,\lambda,\beta}$ from Orlicz space $L^{\phi_1}(\mathbb{R}^n)$ to Orlicz space $L^{\phi_2}(\mathbb{R}^n)$.

A necessary condition of the boundedness of Stein-Weiss operator in Morrey spaces can be recovered from Theorem 3.1 by substituting $\phi_1(t) = t^{p_1}$ and $\phi_2(t) = t^{p_2}$ into (5) we have

$$t^{\alpha+\beta-\lambda+\frac{n-\gamma_1}{p_1}} \lesssim t^{\frac{n-\gamma_2}{p_2}}$$

for every $t > 0$. Consequently,

$$\alpha + \beta - \lambda + \frac{n-\gamma_1}{p_1} = \frac{n-\gamma_2}{p_2}. \tag{7}$$

Which means, (7) is a necessary condition for boundedness of $T_{\alpha.\lambda,\beta}$ from $L^{p_1,\gamma_1}(\mathbb{R}^n)$ to $L^{p_2,\gamma_2}(\mathbb{R}^n)$.

4. CONCLUDING REMARKS

A necessary condition for boundedness of Stein-Weiss operator is already provided in Theorem 3.1. The idea behind the construction of Theorem 3.1 consists of the following three main steps:

(1) boundedness assumption of $T_{\alpha,\lambda,\beta}$ from $L^{\phi_1,\gamma_1}(\mathbb{R}^n) \to L^{\phi_2,\gamma_2}(\mathbb{R}^n)$;
(2) constructing $g_t f(x) = f(tx)$, with $f(x) = |x|^\alpha \chi_{B(0,1)}(x)$, to be substituted into the boundedness assumption;
(3) estimate $\|f\|_{L^{\phi_1,\gamma_1}(\mathbb{R}^n)}$ from above, and estimate $\|T_{\alpha,\lambda,\beta} f\|_{L^{\phi_2,\gamma_2}(\mathbb{R}^n)}$ from below.

In case of $\gamma_1 = \gamma_2 = 0$, Inequality (5) become a necessary condition for boundedness $T_{\alpha,\lambda,\beta} : L^{\phi_1}(\mathbb{R}^n) \to L^{\phi_2}(\mathbb{R}^n)$. While in case of $\phi_1(t) = t^{p_1}$ and $\phi_2(t) = t^{p_2}$, we have a necessary condition for boundedness $T_{\alpha,\lambda,\beta} : L^{p_1,\gamma_1}(\mathbb{R}^n) \to L^{p_2,\gamma_2}(\mathbb{R}^n)$. For further research, we are taking interest in sufficient condition for boundedness of $T_{\alpha,\lambda,\beta} : L^{\phi_1,\gamma_1}(\mathbb{R}^n) \to L^{\phi_2,\gamma_2}(\mathbb{R}^n)$.

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