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The Lieb and Nazarov Inequalities for Quaternion Linear Canonical S-Transform

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Abstract. This research introduces the quaternion linear canonical S-transform. This transform is an extension of the linear canonical S-transform within quaternion algebra. We recall the properties and present the natural link between the quaternion linear canonical transform and the quaternion linear canonical S-transform. We exploit these properties and relation to establish the Lieb and Nazarov inequalities to the quaternion linear canonical S-transform.

Key words and Phrases: quaternion linear canonical transform, S-transform, Lieb inequality, Nazarov inequality

1. Introduction

In recent years, the construction of numerous generalized transformations within quaternion algebra has received more attention by some researchers like the quaternion linear canonical wavelet transform [1], the quaternion Wigner-Ville distribution [2, 3, 4, 5], the quaternion shearlet transform [6], and the quaternion windowed Fourier transform [7]. Further, the authors of [8] have studied the linear canonical S-transform (LCST), which is the extension of the traditional S-transform in the framework of the linear canonical transform (LCT). It is known that several results of the traditional S-transform are modified in the LCST domain, such as modulation, translation, uncertainty inequalities and so on (see, e.g., [9, 10, 11]).

Recently, the authors in [12, 13] have studied the quaternion linear canonical S-transform and obtained the uncertainty relations. However, Lieb and Nazarov inequalities for this transformation have not been published to date. Therefore, in the present work, we will establish Lieb's inequality for the quaternion linear canonical

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S-transform. To achieve the result, we recall several results of the quaternion linear canonical S-transform and make a relation between the quaternion linear canonical transform and the quaternion linear canonical S-transform.

The structure of this work is organized as follows. In Section 2, some basic facts regarding quaternion algebra are collected. The definition of the quaternion Fourier transform (QFT) and useful properties are provided in Section 3. Section 4 briefly reviews the definition of the quaternion linear canonical transform (QLCT) and its connection with the quaternion Fourier transform (QFT). Lastly, Section 5 contains the introduction of the quaternion linear canonical S-transform (QLCST) and the derivation of Lieb and Nazarov inequalities related to the proposed transformation.

2. Quaternion Algebra

Let $\mathbb{H}$ be the associated algebra of real quaternion. The quaternion $p \in \mathbb{H}$ is represented as [14]

$$\mathbb{H} = \{p = p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3 \mid p_0, p_1, p_2, p_3 \in \mathbb{R}\}, \tag{1}$$

where the three different imaginary quaternion units $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ follow the multiplication rules:

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2}$$

Relation (2) shows that quaternion multiplication is not commutative. For every $p \in \mathbb{H}$,

$$p = p_0 + \boldsymbol{p} = Sc(p) + V(p), \tag{3}$$

where $Sc(p) = p_0$ and $V(p) = \boldsymbol{p} = \mathbf{i}\,p_1 + \mathbf{j}\,p_2 + \mathbf{k}\,p_3$ denote the scalar component and vector component, respectively.

The conjugate of quaternion $\bar{p}$ is defined as

$$\bar{p} = p_0 - \mathbf{i}\,p_1 - \mathbf{j}\,p_2 - \mathbf{k}\,p_3. \tag{4}$$

It satisfies the property

$$\overline{rp} = \bar{p}\bar{r}, \quad \forall r, p \in \mathbb{H}.$$

The scalar and vector parts of a quaternion p are

$$Sc(p) = \frac{1}{2}(p + \bar{p}) \quad \text{and} \quad \mathrm{V}(p) = \frac{1}{2}(p - \bar{p}). \tag{5}$$

The modulus of a quaternion p is defined by

$$|p| = \sqrt{p\bar{p}} = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}. \tag{6}$$

It is straightforward to verify that, for every $r, t, p \in \mathbb{H}$, the followings hold:

$$Sc(p) \leq |p|, \quad |\boldsymbol{p}| = |\mathrm{V}(p)| \leq |p|, \quad \text{and} \quad Sc(rpt) = Sc(rtp) = Sc(prt). \tag{7}$$

The inner product for two quaternion functions $f, g : \mathbb{R}^2 \longrightarrow \mathbb{H}$ is defined by

$$(f, g) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\, d\boldsymbol{x}, \quad d\boldsymbol{x} = dx_1 dx_2, \tag{8}$$

with symmetric scalar product

$$\langle f, g\rangle = Sc(f,g) = \frac{1}{2}\big[(f,g) + (g,f)\big]. \tag{9}$$

For $f = g$ in relation (9), we get $L^2(\mathbb{R}^2;\mathbb{H})$ norm, i.e.,

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \, d\boldsymbol{x}\right)^{1/2}. \tag{10}$$

3. Two-Sided Quaternion Fourier Transform

This part recalls the definition of two-sided quaternion Fourier transform (QFT). We list properties which will be needed in the next section. For more information about the QFT and its properties, we refer the reader to [15, 16, 17, 18, 19].

Definition 3.1. *Let $f \in L^2(\mathbb{R}^2;\mathbb{H})$, the two-sided quaternion Fourier transform of f is defined as*

$$\mathcal{F}_H\{f\}(\boldsymbol{w}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}2\pi w_1 x_1} f(\boldsymbol{x}) e^{-\mathbf{j}2\pi w_2 x_2} \, d\boldsymbol{x}. \tag{11}$$

The following shows that the function $f(\boldsymbol{x})$ in (11) can be recovered using its QFT.

Definition 3.2. *Let f in $L^1(\mathbb{R}^2;\mathbb{H})$ and $\mathcal{F}_H\{f\}$ in $L^1(\mathbb{R}^2;\mathbb{H})$. The inversion formula of the quaternion Fourier transform for f is calculated by*

$$\mathcal{F}_H^{-1}[\mathcal{F}_H\{f\}](\boldsymbol{x}) = f(\boldsymbol{x}) = \int_{\mathbb{R}^2} e^{\mathbf{i}2\pi w_1 x_1} \mathcal{F}_H\{f\}(\boldsymbol{w}) \, e^{\mathbf{j}2\pi w_2 x_2} \, d\boldsymbol{w}. \tag{12}$$

The next, from relation (12), one obtains

$$\mathcal{F}_H[\mathcal{F}_H\{f\}](\boldsymbol{x}) = f(-\boldsymbol{w}). \tag{13}$$

4. Quaternion Linear Canonical Transform (QLCT)

In the sequel, we introduce the two-sided quaternion linear canonical transform (shortly QLCT) and its relationship to the quaternion Fourier transform (QFT). For a detailed information on this transformation, the reader may consult to [20, 21].

Definition 4.1. *Let $A_1 = (a_1, b_1, c_1, d_1) = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $A_2 = (a_2, b_2, c_2, d_2) = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ belong to the special linear group $SL(2,\mathbb{R})$. The QLCT of a suitable function $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ is defined as*

$$\mathcal{L}^H_{A_1,A_2}\{f\}(\boldsymbol{w}) = \int_{\mathbb{R}^2} K_{A_1}(x_1, w_1) f(\boldsymbol{x}) K_{A_2}(x_2, w_2) \, d\boldsymbol{x}, \quad b_1 b_2 \neq 0, \tag{14}$$

and

$$\mathcal{L}^H_{A_1,A_2}\{f\}(\boldsymbol{w}) = \sqrt{d_1}\, e^{\mathbf{i}\left(\frac{c_1 d_1}{2}\right)w_1^2} f(d_1 w_1, d_2 w_2)\sqrt{d_2}\, e^{\mathbf{j}\left(\frac{c_2 d_2}{2}\right)w_2^2}, \quad b_1 b_2 = 0,$$

where

$$\begin{aligned} K_{A_1}(x_1,w_1) &= \frac{1}{\sqrt{2\pi b_1}}\, e^{\frac{\mathbf{i}}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1 w_1 + \frac{d_1}{b_1}w_1^2 - \frac{\pi}{2}\right)} \\ K_{A_2}(x_2,w_2) &= \frac{1}{\sqrt{2\pi b_2}}\, e^{\frac{\mathbf{j}}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2 w_2 + \frac{d_2}{b_2}w_2^2 - \frac{\pi}{2}\right)}. \end{aligned} \tag{15}$$

In this research, we always assume the QLCT with $b_1 b_2 \neq 0$. It is obvious that when $A_1 = A_2 = (a_i, b_i, c_i, d_i) = (0, 1, -1, 0)$ with $i = 1, 2$, the QLCT (14) changes to QFT

$$\begin{aligned} \mathcal{L}^H_{A_1,A_2}\{f\}(\boldsymbol{w}) &= \int_{\mathbb{R}^2} \frac{e^{-\mathbf{i}\frac{\pi}{4}}}{\sqrt{2\pi}} e^{-\mathbf{i}w_1 x_1}\, f(\boldsymbol{x}) e^{-\mathbf{j}w_2 x_2} \frac{e^{-\mathbf{j}\frac{\pi}{4}}}{\sqrt{2\pi}}\, d\boldsymbol{x} \\ &= \frac{e^{-\mathbf{i}\frac{\pi}{4}}}{\sqrt{2\pi}} \mathcal{F}_H\{f\}\left(\frac{\boldsymbol{w}}{2\pi}\right) \frac{e^{-\mathbf{j}\frac{\pi}{4}}}{\sqrt{2\pi}}, \end{aligned} \tag{16}$$

where $\mathcal{F}_H\{f\}$ is defined by (11).

Definition 4.2. *For any quaternion signal $f \in L^1(\mathbb{R}^2;\mathbb{H})$ with $\mathcal{L}^H_{A_1,A_2}\{f\} \in L^(\mathbb{R}^2;\mathbb{H})$, the inversion of the QLCT is described by*

$$\begin{aligned} f(\boldsymbol{x}) &= \left(\mathcal{L}^H_{A_1,A_2}\right)^{-1}\left[\mathcal{L}^H_{A_1,A_2}\{f\}\right](\boldsymbol{x}) \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi b_1}} e^{-\frac{\mathbf{i}}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1 w_1 + \frac{d_1}{b_1}w_1^2 - \frac{\pi}{2}\right)} \mathcal{L}^H_{A_1,A_2}\{f\}(\boldsymbol{w}) \\ &\quad \times \frac{1}{\sqrt{2\pi b_2}} e^{-\frac{\mathbf{j}}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2 w_2 + \frac{d_2}{b_2}w_2^2 - \frac{\pi}{2}\right)}\, d\boldsymbol{w}. \end{aligned} \tag{17}$$

According to the QLCT definition (14), we obtain

$$\begin{aligned} &\mathcal{L}^H_{A_1,A_2}\{f\}(\boldsymbol{w}) \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\frac{\mathbf{i}}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1 w_1 + \frac{d_1}{b_1}w_1^2 - \frac{\pi}{2}\right)} f(\boldsymbol{x}) \frac{1}{\sqrt{2\pi b_2}} e^{\frac{\mathbf{j}}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2 w_2 + \frac{d_2}{b_2}w_2^2 - \frac{\pi}{2}\right)}\, d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} \frac{e^{-\mathbf{i}\frac{\pi}{4}}}{\sqrt{2\pi b_1}} e^{\mathbf{i}\frac{d_1}{2b_1}w_1^2} e^{-\mathbf{i}\frac{x_1 w_1}{b_1}} e^{\mathbf{i}\frac{a_1}{2b_1}x_1^2} f(\boldsymbol{x}) \frac{e^{-\mathbf{j}\frac{\pi}{4}}}{\sqrt{2\pi b_2}} e^{\mathbf{j}\frac{d_2}{2b_2}w_2^2} e^{-\mathbf{j}\frac{x_2 w_2}{b_2}} e^{\mathbf{j}\frac{a_2}{2b_2}x_2^2}\, d\boldsymbol{x}. \end{aligned} \tag{18}$$

Hence,

$$\begin{aligned} &\sqrt{2\pi b_1}\, e^{\mathbf{i}\frac{\pi}{4}} e^{-\mathbf{i}\frac{d_1}{2b_1}w_1^2} \mathcal{L}^H_{A_1,A_2}\{f\}(\boldsymbol{w}) e^{-\mathbf{j}\frac{d_2}{2b_2}w_2^2} \sqrt{2\pi b_2}\, e^{\mathbf{j}\frac{\pi}{4}} \\ &\qquad = \int_{\mathbb{R}^2} e^{-\mathbf{i}\frac{x_1 w_1}{b_1}} e^{\mathbf{i}\frac{a_1}{2b_1}x_1^2} f(\boldsymbol{x}) e^{\mathbf{j}\frac{a_2}{2b_2}x_2^2} e^{-\mathbf{j}\frac{x_2 w_2}{b_2}}\, d\boldsymbol{x} \\ &\qquad = \mathcal{F}_H\{h\}\left(\frac{\boldsymbol{w}}{2\pi\boldsymbol{b}}\right) \\ &\qquad = \mathcal{F}_H\{h\}\left(\frac{w_1}{2\pi b_1}, \frac{w_2}{2\pi b_2}\right), \end{aligned} \tag{19}$$

where

$$h(\boldsymbol{x}) = e^{\mathbf{i}\frac{a_1}{2b_1}x_1^2} f(\boldsymbol{x}) e^{\mathbf{j}\frac{a_2}{2b_2}x_2^2}. \tag{20}$$

By equation (19), we can easily derive for every $f \in L^2(\mathbb{R}^2;\mathbb{H})$

$$\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \, d\boldsymbol{x} = \int_{\mathbb{R}^2} \left|\mathcal{L}^H_{A_1,A_2}\{f\}(\boldsymbol{w})\right|^2 d\boldsymbol{w}, \tag{21}$$

which is known as Parseval's formula for the QLCT.

5. Quaternion Linear Canonical S-Transform

In the sequel, we start with a definition of the quaternion linear canonical S-transform (QLCST). We collect its essential properties, which will be useful for deriving the main results in this research.

5.1. **Definition.**

Definition 5.1 (QLCST definition)**.** *Let $\phi \in L^2(\mathbb{R}^2;\mathbb{H})$ be a non-zero quaternion window function. The QLCST $\mathcal{S}^H_\phi$ for function $h \in L^2(\mathbb{R}^2;\mathbb{H})$ with respect to ϕ is defined by*

$$\mathcal{S}^H_\phi h(\boldsymbol{u},\boldsymbol{w}) = \int_{\mathbb{R}^2} K_{A_1}(x_1,w_1) h(\boldsymbol{x}) \overline{\phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w})}\, K_{A_2}(x_2,w_2)\, d\boldsymbol{x}. \tag{22}$$

Definition 5.2 (QLCST inversion formula)**.** *Let $\phi \in L^2(\mathbb{R}^2;\mathbb{H})$ be a quaternion window function that satisfies the following*

$$\int_{\mathbb{R}^2} |\phi(\boldsymbol{u},\boldsymbol{w})|^2 \, d\boldsymbol{u} = \phi_{\boldsymbol{u},\boldsymbol{w}}, \quad 0 < \phi_{\boldsymbol{u},\boldsymbol{w}} < \infty. \tag{23}$$

Then, every quaternion signal $h \in L^2(\mathbb{R}^2;\mathbb{H})$ can be reconstructed using inversion formula for the QLCST, that is,

$$h(\boldsymbol{x}) = \frac{1}{\phi_{\boldsymbol{u},\boldsymbol{w}}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{S}^H_\phi h(\boldsymbol{u},\boldsymbol{w}) \phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w}) K_{A_1}(x_1,w_1) K_{A_2}(x_2,w_2)\, d\boldsymbol{u}\, d\boldsymbol{w}. \tag{24}$$

According to the reference [13], we have the following facts:

$$\mathcal{S}^H_\phi h(\boldsymbol{u},\boldsymbol{w}) = \mathcal{L}^H_{A_1,A_2}\Big\{h(\boldsymbol{x})\overline{\phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w})}\Big\}. \tag{25}$$

This implies that

$$h(\boldsymbol{x})\overline{\phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w})} = \left(\mathcal{L}^H_{A_1,A_2}\right)^{-1}\left[\mathcal{S}^H_\phi h(\boldsymbol{u},\boldsymbol{w})\right]. \tag{26}$$

The following result will be useful for proving Lieb's inequality related to the QLCST in the next section.

Theorem 5.3 (QLCST sharp Hausdorff-Young relation [13]). *Let $r \in [1,2]$ and $\frac{1}{r}+\frac{1}{s}=1$. Then for all $f \in L^r(\mathbb{R}^2;\mathbb{H})$, we have*

$$\left(\int_{\mathbb{R}^2} \left|\mathcal{S}_\phi^H\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^s d\boldsymbol{w}d\boldsymbol{u}\right)^{\frac{1}{s}} \\ \leq C_r^2 (2\pi)^{\frac{2}{s}-1}(|b_1||b_2|)^{\frac{1}{s}-\frac{1}{2}} \left(\int_{\mathbb{R}^2} \left|\phi(\boldsymbol{u},\boldsymbol{w})\right|^r d\boldsymbol{u}\right)^{\frac{1}{r}} \|f\|_{L^r(\mathbb{R}^2;\mathbb{H})}, \tag{27}$$

where

$$C_r = \left(r^{\frac{1}{r}} s^{-\frac{1}{s}}\right)^{\frac{1}{2}}.$$

5.2. **Lieb's Inequality.**

Below, we utilize the sharp Hausdorff-Young inequality stated in Theorem 5.3 to derive Lieb's inequality for the QLCST. We then obtain the following result.

Theorem 5.4 (QLCST Lieb's inequality). *For any functions $f, \phi \in L^2(\mathbb{R}^2;\mathbb{H})$ and $2 \leq s < \infty$, one gets*

$$\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\mathcal{S}_\phi^H\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^s d\boldsymbol{w}d\boldsymbol{u}\right)^{\frac{1}{s}} \leq \left(\frac{2}{s}\right)^{\frac{2}{s}} \frac{\pi^{\frac{2}{s}-1}}{s^{\frac{2}{s}}} |b_1 b_2|^{\frac{1}{s}-\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} \|\phi\|_{L^2(\mathbb{R}^2;\mathbb{H})}.$$

Proof. From equation (72) in [13], it is easily seen that

$$\left(\int_{\mathbb{R}^2} \left|\mathcal{S}_\phi^H\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^s d\boldsymbol{w}\right)^{\frac{1}{s}} \leq C_r^2 (2\pi)^{\frac{2}{s}-1}(|b_1||b_2|)^{\frac{1}{s}-\frac{1}{2}} \left(\int_{\mathbb{R}^2} \left|f(\boldsymbol{x})\overline{\phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w})}\right|^r d\boldsymbol{x}\right)^{\frac{1}{r}} \\ = C_r^2 (2\pi)^{\frac{2}{s}-1}(|b_1||b_2|)^{\frac{1}{s}-\frac{1}{2}} \left(\left(|f(\boldsymbol{x})|^r * \left|\phi_w(\boldsymbol{x})\right|^r\right)(\boldsymbol{u})\right)^{\frac{1}{r}}, \tag{28}$$

where $\phi_w(\boldsymbol{x}) = \overline{\phi(\boldsymbol{x},\boldsymbol{w})}$. Here, the convolution operator for $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$ is given by

$$\big(f * g\big)(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{y}) g(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{y}. \tag{29}$$

Writing relation (28) above as

$$\int_{\mathbb{R}^2} \left|\mathcal{S}_\phi^H\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^s d\boldsymbol{w} \leq C_r^{2s}\left((2\pi)^{\frac{2}{s}-1}(|b_1||b_2|)^{\frac{1}{s}-\frac{1}{2}}\right)^s \left(\left(|f(\boldsymbol{x})|^r * \left|\phi_w(\boldsymbol{x})\right|^r\right)(\boldsymbol{u})\right)^{\frac{s}{r}}. \tag{30}$$

Integrating both sides of relation (30) with respect to the measure $d\boldsymbol{u}$ we immediately obtain

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\mathcal{S}_\phi^H\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^s d\boldsymbol{w}\, d\boldsymbol{u} \leq C_r^{2s} (2\pi)^{\frac{2}{s}-1}(|b_1||b_2|)^{\frac{1}{s}-\frac{1}{2}} \\ \times \left(\int_{\mathbb{R}^2} \left(|f(\boldsymbol{x})|^r * \left|\phi_w(\boldsymbol{x})\right|^r\right)(\boldsymbol{u}) d\boldsymbol{u}\right)^{\frac{s}{r}}. \tag{31}$$

Therefore,

$$\begin{aligned}
&\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\mathcal{S}_\phi^H\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^s d\boldsymbol{w}d\boldsymbol{u}\right)^{\frac{1}{s}}\\
&\quad\le C_r^2(2\pi)^{\frac{2}{s}-1}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}\left(\int_{\mathbb{R}^2}\left(\left(|f(\boldsymbol{x})|^r*\left|\phi_w(\boldsymbol{x})\right|^r\right)(\boldsymbol{u})\right)^{\frac{s}{r}}d\boldsymbol{u}\right)^{\frac{1}{s}}\\
&\quad= C_r^2(2\pi)^{\frac{2}{s}-1}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}\left(\int_{\mathbb{R}^2}\left(\left(|f(\boldsymbol{x})|^r*\left|\phi_w(\boldsymbol{x})\right|^r\right)(\boldsymbol{u})\right)^{\frac{s}{r}}d\boldsymbol{u}\right)^{\frac{1}{\frac{s}{r}}\cdot\frac{1}{r}}\\
&\quad= C_r^2(2\pi)^{\frac{2}{s}-1}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}\left(\left(\int_{\mathbb{R}^2}\left(\left(|f(\boldsymbol{x})|^r*\left|\phi_w(\boldsymbol{x})\right|^r\right)(\boldsymbol{u})\right)^{\frac{s}{r}}d\boldsymbol{u}\right)^{\frac{1}{\frac{s}{r}}}\right)^{\frac{1}{r}}\\
&\quad= C_r^2(2\pi)^{\frac{2}{s}-1}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}\left\|\left(|f(\boldsymbol{x})|^r*\left|\phi_w(\boldsymbol{x})\right|^r\right)\right\|_{L^{\frac{s}{r}}(\mathbb{R}^2;\mathbb{H})}^{\frac{1}{r}}. \qquad (32)
\end{aligned}$$

Since $f,\phi\in L^2(\mathbb{R}^2;\mathbb{H})$ then $|f|^r,|\phi_w|^r\in L^2(\mathbb{R}^2;\mathbb{H})$ and applying Young inequality with $(p,p,t)=(\frac{2}{r},\frac{2}{r},\frac{s}{r})$ and $\frac{1}{p}+\frac{1}{p}=\frac{1}{t}+1$, then we get

$$\|(|f(\boldsymbol{x})|^r*|\phi_w(\boldsymbol{x})|^r)\|_{L^t(\mathbb{R}^2;\mathbb{H})}\le C_p^{\frac{4}{r}}C_t^{\frac{2}{r}}\||f|^r\|_{L^p(\mathbb{R}^2;\mathbb{H})}^r\||\phi|^r\|_{L^p(\mathbb{R}^2;\mathbb{H})}^r. \qquad (33)$$

Now observe that

$$\begin{aligned}
C_r^2C_p^{\frac{4}{r}}C_t^{\frac{2}{r}} &= \left(\frac{r^{\frac{1}{r}}}{s^{\frac{1}{s}}}\right)\left(\frac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}\right)^{\frac{2}{r}}\left(\frac{t'^{\frac{1}{t'}}}{t^{\frac{1}{t}}}\right)^{\frac{1}{r}}\\
&= \left(\frac{r^{\frac{1}{r}}}{s^{\frac{1}{s}}}\right)\left(\frac{p^{\frac{2}{pr}}}{p'^{\frac{2}{p'r}}}\right)\left(\frac{t'^{\frac{1}{t'r}}}{t^{\frac{1}{tr}}}\right)\\
&= \left(\frac{r^{\frac{1}{r}}}{s^{\frac{1}{s}}}\right)\left(\frac{p}{p'^{\frac{2}{p'r}}}\right)\left(\frac{t'^{\frac{1}{t'r}}}{(\frac{s}{r})^{\frac{1}{s}}}\right)\\
&= \left(\frac{r^{\frac{1}{r}+\frac{1}{s}}}{s^{\frac{2}{s}}}\right)\left(\frac{2}{r}\right)\left(\frac{t'^{\frac{1}{t'r}}}{p'^{\frac{2}{p'r}}}\right)\\
&= \left(\frac{2r^{\frac{1}{r}+\frac{1}{s}-1}}{s^{\frac{2}{s}}}\right)\left(\frac{1}{2}\right)^{\frac{1}{r}-\frac{1}{s}}, \qquad (34)
\end{aligned}$$

where in the fourth inequality, we have chosen $p'=2t'$. Hence,

$$C_r^2C_p^{\frac{4}{r}}C_t^{\frac{2}{r}}=\frac{2^{\frac{1}{s}-\frac{1}{r}+1}r^{\frac{1}{r}+\frac{1}{s}-1}}{s^{\frac{2}{s}}}=\frac{2^{\frac{2}{s}}}{s^{\frac{2}{s}}}=\left(\frac{2}{s}\right)^{\frac{2}{s}}. \qquad (35)$$

Consequently,

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\mathcal{S}_{\phi}^{H}\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^{s} d\boldsymbol{w}\, d\boldsymbol{u}\right)^{\frac{1}{s}}\\
&\qquad\le \left(\frac{2}{s}\right)^{\frac{2}{s}}(2\pi)^{\frac{2}{s}-1}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}\left(\||f|^{r}\|_{L^p(\mathbb{R}^2;\mathbb{H})}\||\phi|^{r}\|_{L^p(\mathbb{R}^2;\mathbb{H})}\right)^{\frac{1}{r}}\\
&\qquad\le \left(\frac{2}{s}\right)^{\frac{2}{s}}(2\pi)^{\frac{2}{s}-1}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}\left(\left(\int_{\mathbb{R}^2}\left(\left|f(\boldsymbol{x})\right|^{r}\right)^{p}d\boldsymbol{x}\right)^{\frac{1}{p}}\right.\\
&\qquad\quad\left.\times\left(\int_{\mathbb{R}^2}\left(\left|\phi(\boldsymbol{x})\right|^{r}\right)^{p}d\boldsymbol{x}\right)^{\frac{1}{p}}\right)^{\frac{1}{r}}\\
&\qquad\le \left(\frac{2}{s}\right)^{\frac{2}{s}}(2\pi)^{\frac{2}{s}-1}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}\left(\left(\int_{\mathbb{R}^2}\left|f(\boldsymbol{x})\right|^{r\cdot\frac{2}{r}}d\boldsymbol{x}\right)^{\frac{r}{2}}\right.\\
&\qquad\quad\left.\times\left(\int_{\mathbb{R}^2}\left|\phi(\boldsymbol{x})\right|^{r\cdot\frac{2}{r}}d\boldsymbol{x}\right)^{\frac{r}{2}}\right)^{\frac{1}{r}}.
\end{aligned}
\tag{36}
$$

We finally arrive at

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\mathcal{S}_{\phi}^{H}\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^{s} d\boldsymbol{w}d\boldsymbol{u}\right)^{\frac{1}{s}}\\
&\qquad\le \left(\frac{2}{s}\right)^{\frac{2}{s}}(2\pi)^{\frac{2}{s}-1}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})}\|\phi\|_{L^2(\mathbb{R}^2;\mathbb{H})},
\end{aligned}
$$

which completes the proof. □

Remark 1.

- *As far as we know, Lieb inequality for the QLCST in Theorem 5.4 is derived using the sharp-Hausdorff Young inequality, so the proposed Lieb inequality is sharp.*
- *Similar to [22], the future research will explore that Gabor quaternion filter minimizes Lieb inequality involving the QLCST for $s = 2$.*

5.3. **Nazarov's Inequality.**

In the following we establish an analogue of Nazarov's inequality for the QLCST.

Theorem 5.5. *Let $\phi \in L^2(\mathbb{R}^2;\mathbb{H})$ be a non-zero quaternion window function. Let $f \in L^2(\mathbb{R}^2;\mathbb{H})$, and A, B be two subsets of $\mathbb{R}^2$ with finite measure, then there exists a constant $k > 0$ such that:*

$$
\begin{aligned}
&\phi_{\boldsymbol{u},\boldsymbol{w}}\int_{\mathbb{R}^2}\left|f(\boldsymbol{x})\right|^{2}d\boldsymbol{x}\\
&\le ke^{|A||B|}\left(\phi_{\boldsymbol{u},\boldsymbol{w}}\int_{\mathbb{R}^2\setminus A}\left|f(\boldsymbol{x})\right|^{2}d\boldsymbol{x}+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2\setminus Bb}\left|\mathcal{S}_{\phi}^{H}\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^{2}d\boldsymbol{w}\, d\boldsymbol{u}\right),
\end{aligned}
\tag{37}
$$

where $|A|$ and $|B|$ denote the Lebesgue measures of A and B.

Proof. Based on the Nazarov's inequality for the two-sided quaternion linear canonical transform, we have

$$\int_{\mathbb{R}^2} \left|f(\boldsymbol{x})\right|^2 d\boldsymbol{x} \leq ke^{|A||B|}\left(\int_{\mathbb{R}^2\setminus A} \left|f(\boldsymbol{x})\right|^2 d\boldsymbol{x} + \int_{\mathbb{R}^2\setminus Bb} \left|\mathcal{L}_A^H\{f\}(\boldsymbol{w})\right|^2 d\boldsymbol{w}\right). \tag{38}$$

An application of relation (25) into both sides of identity (38), we get

$$\begin{aligned}&\int_{\mathbb{R}^2} \left|\left(\mathcal{L}_A^H\right)^{-1}\left[\mathcal{L}_A^H\{f\}(\boldsymbol{x})\right]\right|^2 d\boldsymbol{x}\\ &\leq ke^{|A||B|}\left(\int_{\mathbb{R}^2\setminus A} \left|\left(\mathcal{L}_A^H\right)^{-1}\left[\mathcal{L}_A^H\{f\}(\boldsymbol{x})\right]\right|^2 d\boldsymbol{x} + \int_{\mathbb{R}^2\setminus Bb} \left|\mathcal{L}_A^H\{f\}(\boldsymbol{w})\right|^2 d\boldsymbol{w}\right). \end{aligned}\tag{39}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^2} \left|f(\boldsymbol{x})\overline{\phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w})}\right|^2 d\boldsymbol{x}\\ &\leq ke^{|A||B|}\left(\int_{\mathbb{R}^2\setminus A} \left|f(\boldsymbol{x})\overline{\phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w})}\right|^2 d\boldsymbol{x} + \int_{\mathbb{R}^2\setminus Bb} \left|\mathcal{S}_\phi^H\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^2 d\boldsymbol{w}\right). \end{aligned}\tag{40}$$

Integrating both sides of equation (40) with respect to $d\boldsymbol{u}$ we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|f(\boldsymbol{x})\right|^2\left|\overline{\phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w})}\right|^2 d\boldsymbol{x}\, d\boldsymbol{u}\\ &\leq ke^{|A||B|}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2\setminus A} \left|f(\boldsymbol{x})\overline{\phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w})}\right|^2 d\boldsymbol{x}d\boldsymbol{u} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2\setminus Bb} \left|\mathcal{S}_\phi^H\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^2 d\boldsymbol{w}d\boldsymbol{u}\right). \end{aligned}\tag{41}$$

Or, equivalently,

$$\begin{aligned}&\int_{\mathbb{R}^2} \left|f(\boldsymbol{x})\right|^2\left(\int_{\mathbb{R}^2}\left|\overline{\phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w})}\right|^2 d\boldsymbol{u}\right) d\boldsymbol{x}\\ &\leq ke^{|A||B|}\left(\int_{\mathbb{R}^2\setminus A} \left|f(\boldsymbol{x})\right|^2\left(\int_{\mathbb{R}^2}\left|\overline{\phi(\boldsymbol{u}-\boldsymbol{x},\boldsymbol{w})}\right|^2 d\boldsymbol{u}\right) d\boldsymbol{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2\setminus Bb} \left|\mathcal{S}_\phi^H\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^2 d\boldsymbol{w}d\boldsymbol{u}\right). \end{aligned}\tag{42}$$

Hence,

$$\begin{aligned}&\phi_{\boldsymbol{u},\boldsymbol{w}}\int_{\mathbb{R}^2} \left|f(\boldsymbol{x})\right|^2 d\boldsymbol{x}\\ &\qquad\leq ke^{|A||B|}\left(\phi_{\boldsymbol{u},\boldsymbol{w}}\int_{\mathbb{R}^2\setminus A} \left|f(\boldsymbol{x})\right|^2 d\boldsymbol{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2\setminus Bb} \left|\mathcal{S}_\phi^H\{f\}(\boldsymbol{u},\boldsymbol{w})\right|^2 d\boldsymbol{w}\, d\boldsymbol{u}\right),\end{aligned}$$

which finishes the proof. □

Now let us implement the above properties by providing a simple example.

Example 1. *Consider the Gaussian functions*

$$f(\boldsymbol{x}) = e^{-a|\boldsymbol{x}|^2}, \quad a > 0$$

and

$$\phi(\boldsymbol{x})=\begin{cases}1, & -1\leq x_1,x_2\leq 1\\ 0, & \text{elsewhere.}\end{cases}$$

Its quaternion linear canonical S-transform is derived as follows.

In view of equation (22), we obtain

$$\begin{aligned}\mathcal{S}_{\phi}^{H}\left\{f\right\}(\boldsymbol{u},\boldsymbol{w}) &= \int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1}}e^{\frac{\mathbf{i}}{2}\left(\frac{a_1}{b_1}x_1^2-\frac{2}{b_1}x_1w_1+\frac{d_1}{b_1}w_1^2-\frac{\pi}{2}\right)}e^{-a|x|^2}\overline{\phi(\boldsymbol{u}-x,\boldsymbol{w})}\\ &\quad\times\frac{1}{\sqrt{2\pi b_2}}e^{\frac{\mathbf{j}}{2}\left(\frac{a_2}{b_2}x_2^2-\frac{2}{b_2}x_2w_2+\frac{d_2}{b_2}w_2^2-\frac{\pi}{2}\right)}\,d\boldsymbol{x}\\ &= \frac{1}{\sqrt{2\pi b_1}}e^{\frac{\mathbf{i}}{2}\left(\frac{d_1}{b_1}w_1^2-\frac{\pi}{2}\right)}\int_{\mathbb{R}^2}e^{\frac{\mathbf{i}}{2}\left(\frac{a_1}{b_1}x_1^2-\frac{2}{b_1}x_1w_1\right)-ax_1^2}\overline{\phi(\boldsymbol{u}-x,\boldsymbol{w})}\\ &\quad\times e^{-ax_2^2+\frac{\mathbf{i}}{2}\left(\frac{a_2}{b_2}x_2^2-\frac{2}{b_2}x_2w_2\right)}\frac{1}{\sqrt{2\pi b_2}}e^{\frac{\mathbf{j}}{2}\left(\frac{d_2}{b_2}w_2^2-\frac{\pi}{2}\right)}\,d\boldsymbol{x}\\ &= \frac{1}{\sqrt{2\pi b_1}}e^{\frac{\mathbf{i}}{2}\left(\frac{d_1}{b_1}w_1^2-\frac{\pi}{2}\right)}\int_{u_1-1}^{u_1+1}e^{-ax_1^2+\frac{\mathbf{i}}{2}\left(\frac{a_1}{b_1}x_1^2-\frac{2}{b_1}x_1w_1\right)}\,dx_1\\ &\quad\times\int_{u_2-1}^{u_2+1}e^{-ax_2^2+\frac{\mathbf{j}}{2}\left(\frac{a_2}{b_2}x_2^2-\frac{2}{b_2}x_2w_2\right)}\,dx_2\frac{1}{\sqrt{2\pi b_2}}e^{\frac{\mathbf{j}}{2}\left(\frac{d_2}{b_2}w_2^2-\frac{\pi}{2}\right)}.\end{aligned}\tag{43}$$

Further we get

$$\begin{aligned}&\mathcal{S}_{\phi}^{H}\{f\}(\boldsymbol{u},\boldsymbol{w})\\ &= \frac{1}{\sqrt{2\pi b_1}}e^{\frac{\mathbf{i}}{2}\left(\frac{d_1}{b_1}w_1^2-\frac{\pi}{2}\right)}\int_{u_1-1}^{u_1+1}e^{-\left(a-\frac{\mathbf{i}a_1}{2b_1}\right)x_1^2-\frac{\mathbf{i}x_1w_1}{b_1}}\,dx_1\\ &\quad\times\int_{u_2-1}^{u_2+1}e^{-\left(a-\frac{\mathbf{j}a_2}{2b_2}\right)x_2^2-\frac{\mathbf{j}x_2w_2}{b_2}}\,dx_2\frac{1}{\sqrt{2\pi b_2}}e^{\frac{\mathbf{j}}{2}\left(\frac{d_2}{b_2}w_2^2-\frac{\pi}{2}\right)}\\ &= \frac{1}{\sqrt{2\pi b_1}}e^{\frac{\mathbf{i}}{2}\left(\frac{d_1}{b_1}w_1^2-\frac{\pi}{2}\right)}\int_{u_1-1}^{u_1+1}e^{-\left(a-\frac{\mathbf{i}a_1}{2b_1}\right)\left(x_1^2-\frac{\mathbf{i}\frac{w_1}{b_1}}{(a-\mathbf{i}\frac{a_1}{2b_1})}x_1\right)}\,dx_1\\ &\quad\times\int_{u_2-1}^{u_2+1}e^{-\left(a-\frac{\mathbf{j}a_2}{2b_2}\right)\left(x_2^2-\frac{\mathbf{j}\frac{w_2}{b_2}}{(a-\mathbf{j}\frac{a_2}{2b_2})}x_2\right)}\,dx_2\frac{1}{\sqrt{2\pi b_2}}e^{\frac{\mathbf{j}}{2}\left(\frac{d_2}{b_2}w_2^2-\frac{\pi}{2}\right)}\\ &= \frac{1}{\sqrt{2\pi b_1}}e^{\frac{\mathbf{i}}{2}\left(\frac{d_1}{b_1}w_1^2-\frac{\pi}{2}\right)}\int_{u_1-1}^{u_1+1}e^{-(a-\mathbf{i}\frac{a_1}{2b_1})\left(\left(x_1-\frac{\mathbf{i}\frac{w_1}{b_1}}{2(a-\mathbf{i}\frac{a_1}{b_1})}\right)^2-\left(\frac{\mathbf{i}\frac{w_1}{b_1}}{2(a-\mathbf{i}\frac{a_1}{2b_1})}\right)^2\right)}\,dx_1\\ &\quad\times\int_{u_2-1}^{u_2+1}e^{-\left(a-\mathbf{j}\frac{a_2}{2b_2}\right)\left(\left(x_2-\frac{\mathbf{j}\frac{w_2}{b_2}}{2(a-\mathbf{j}\frac{a_2}{b_2})}\right)^2-\left(\frac{\mathbf{j}\frac{w_2}{b_2}}{2(a-\mathbf{j}\frac{a_2}{2b_2})}\right)^2\right)}\,dx_2\frac{1}{\sqrt{2\pi b_2}}e^{\frac{\mathbf{j}}{2}\left(\frac{d_2}{b_2}w_2^2-\frac{\pi}{2}\right)}.\end{aligned}\tag{44}$$

Therefore,

$$
\begin{aligned}
&\mathcal{S}_{\phi}^{H}\{f\}(\boldsymbol{w}) \\
&= \frac{1}{\sqrt{2\pi b_1}} e^{\frac{\mathbf{i}}{2}\left(\frac{d_1}{b_1}w_1^2-\frac{\pi}{2}\right)-\frac{\left(\frac{w_1}{b_1}\right)^2}{4\left(a-\mathbf{i}\frac{a_1}{2b_1}\right)}} \int_{u_1-1}^{u_1+1} e^{-\left(a-\mathbf{i}\frac{a_1}{2b_1}\right)\left(x_1-\frac{\mathbf{i}\frac{w_1}{b_1}}{2\left(a-\mathbf{i}\frac{a_1}{b_1}\right)}\right)^2} dx_1 \\
&\quad \times \int_{u_2-1}^{u_2+1} e^{-\left(a-\mathbf{j}\frac{a_2}{2b_2}\right)\left(x_2-\frac{\mathbf{j}\frac{w_2}{b_2}}{2(a-\mathbf{j}\frac{a_2}{b_2})}\right)^2} dx_2 \frac{1}{\sqrt{2\pi b_2}} e^{\frac{\mathbf{j}}{2}\left(\frac{d_2}{b_2}w_2^2-\frac{\pi}{2}\right)-\frac{\left(\frac{w_2}{b_2}\right)^2}{4\left(a-\mathbf{j}\frac{a_2}{2b_2}\right)}} \\
&= \frac{1}{\sqrt{2\pi b_1}} e^{\frac{\mathbf{i}}{2}\left(\frac{d_1}{b_1}w_1^2-\frac{\pi}{2}\right)-\frac{w_1^2}{4b_1\left(a-\mathbf{i}\frac{a_1}{2b_1}\right)}} \int_{u_1-1}^{u_1+1} e^{-\left(\sqrt{a-\mathbf{i}\frac{a_1}{2b_1}}\left(x_1-\mathbf{i}\frac{w_1}{2b_1\left(a-\mathbf{i}\frac{a_1}{2b_1}\right)}\right)\right)^2} dx_1 \\
&\quad \times \int_{u_2-1}^{u_2+1} e^{-\left(\sqrt{a-\mathbf{j}\frac{a_2}{2b_2}}\left(x_2-\mathbf{j}\frac{w_2}{2b_2\left(a-\mathbf{j}\frac{a_2}{2b_2}\right)}\right)\right)^2} dx_2 \, \frac{1}{\sqrt{2\pi b_2}} e^{\frac{\mathbf{j}}{2}\left(\frac{d_2}{b_2}w_2^2-\frac{\pi}{2}\right)-\frac{w_2^2}{4b_2\left(a-\mathbf{j}\frac{a_2}{2b_2}\right)}}.
\end{aligned}
$$

This equation simplifies to

$$
\begin{aligned}
&\mathcal{S}_{\phi}^{H}\{f\}(\boldsymbol{u},\boldsymbol{w}) \\
&= \frac{1}{4\sqrt{2\pi b_1-\mathbf{i}a_1}} e^{\frac{\mathbf{i}}{2}\left(\frac{d_1}{b_1}w_1^2-\frac{\pi}{2}\right)-\frac{w_1^2}{4b_1\left(a-\mathbf{i}\frac{a_1}{2b_1}\right)}} \\
&\quad \times (\operatorname{erf}\left(\sqrt{a-\mathbf{i}\frac{a_1}{2b_1}}\left(u_1+1-\mathbf{i}\frac{w_1}{2b_1\left(a-\mathbf{i}\frac{a_1}{2b_1}\right)}\right)\right) \\
&\quad -\operatorname{erf}\left(\sqrt{a-\mathbf{i}\frac{a_1}{2b_1}}\left(u_1-1-\mathbf{i}\frac{w_1}{2b_1\left(a-\mathbf{i}\frac{a_1}{2b_1}\right)}\right)\right) \\
&\quad \times \operatorname{erf}\left(\sqrt{a-\mathbf{j}\frac{a_2}{2b_2}}\left(u_2+1-\mathbf{j}\frac{w_2}{2b_2\left(a-\mathbf{j}\frac{a_2}{2b_2}\right)}\right)\right) - \\
&\quad \operatorname{erf}\left(\sqrt{a-\mathbf{j}\frac{a_2}{2b_2}}\left(u_2-1-\mathbf{j}\frac{w_2}{2b_2\left(a-\mathbf{j}\frac{a_2}{2b_2}\right)}\right)\right) \frac{1}{\sqrt{2\pi b_2-\mathbf{j}a_2}} e^{\frac{\mathbf{j}}{2}\left(\frac{d_2}{b_2}w_2^2-\frac{\pi}{2}\right)-\frac{w_2^2}{4b_2\left(a-\mathbf{j}\frac{a_2}{2b_2}\right)}}.
\end{aligned}
$$

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