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Coupled Fractional Fourier Transform: Properties and Uncertainty Principle

Mawardi Bahri[1*]

[1]Department of Mathematics, Hasanuddin University, Indonesia

Abstract. The coupled fractional Fourier transform is a general form of the fractional Fourier transform. In the present work, we demonstrate basic properties of the proposed transformation, such as linearity, shifting and modulation. We also propose convolution and correlation theorems and then explore a generalized uncertainty principle for the coupled fractional Fourier transform. These crucial results are modifications of the corresponding properties pertaining to the fractional Fourier transform and the conventional Fourier transform.

Key words and Phrases: coupled fractional Fourier transform; uncertainty principle; fractional Fourier transform.

1. INTRODUCTION

The fractional Fourier transform (FrFT) [1, 2, 3, 4] is an effective mathematical tool that has been applied in quantum mechanics, neural networks, differential equations, optics, radar, sonar, and other communication systems [5, 6, 7, 8, 9]. It can be viewed as an expansion of the Fourier transform (FT). In the recent work, the authors of [10, 11] have proposed an extended form of the FrFT to the coupled fractional Fourier transform (CFrFT). The utility of the CFrFT kernel in constructing the other general transformations, such as the short-time coupled fractional Fourier transform and the coupled fractional Wigner-Ville distribution were discussed in [12, 13, 14, 15]. Especially, the authors of [10] studied in detail various uncertainty inequalities associated with the CFrFT. However, some properties of the CFrFT like convolution and correlation theorems have not yet studied.

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*Corresponding author : mawardibahri@gmail.com

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Therefore, in the present research we first establish some basic results pertaining to the CFrFT, which are missing in the existing literature. We then study convolution and correlation operators related to the generalized transformation. Finally, we also explore an uncertainty inequality associated with the proposed CFrFT.

The paper is outlined as follows. In Section 2, we recall the definition of the FrFT and some existing results related to the transformation. The definition of the CFrFT is also introduced in this section. Section 3 is devoted to definition and properties of the CFrFT. We also present convolution and correlation theorems related to the CFrFT. Section 4 deals with the derivation of an uncertainty principles associated with the CFrFT. Lastly, Section 5 contains a brief conclusion of the work.

2. Preliminaries

First of all, we introduce the basic results related to the FrFT and the CFrFT. We begin with the following definition.

Definition 2.1. *The space $L^r(\mathbb{R}^2)$ is the set of measurable functions on $\mathbb{R}^2$ satisfying*

$$\|f\|_{L^r(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{t})|^r \, d\boldsymbol{t}\right)^{1/r} < \infty, \quad 1 \leq r < \infty. \tag{1}$$

Here $\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2, d\boldsymbol{t} = dt_1 dt_2$. Also for every $f, g \in L^r(\mathbb{R}^2)$ with $f = g$ if and only if $f = g$ a.e.

In particular, when $r = \infty$ we define $L^\infty(\mathbb{R}^2)$-norm

$$\|f\|_{L^\infty(\mathbb{R}^2)} = \text{ess sup}_{\boldsymbol{t}\in\mathbb{R}^2} |f(\boldsymbol{t})|, \tag{2}$$

and if f is continuous, then

$$\|f\|_{L^\infty(\mathbb{R}^2)} = \sup_{\boldsymbol{t}\in\mathbb{R}^2} |f(\boldsymbol{t})|. \tag{3}$$

The inner product of $L^2(\mathbb{R}^2)$ is defined by

$$\langle f, g\rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} f(\boldsymbol{t})\overline{g(\boldsymbol{t})} \, d\boldsymbol{t}. \tag{4}$$

Definition 2.2. *Let function $f \in L^1(\mathbb{R}^2)$. The 2-D FrFT with parameter ϑ is defined by*

$$\mathfrak{F}_\vartheta\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{t}) K_\vartheta(\boldsymbol{\omega}, \boldsymbol{t}) \, d\boldsymbol{t}, \tag{5}$$

where $K_\vartheta(\boldsymbol{\omega},\boldsymbol{t})$ *is given by*

$$K_\vartheta(\boldsymbol{\omega},\boldsymbol{t})=\begin{cases} A_\vartheta e^{-i(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)\frac{\cot\vartheta}{2}+i\boldsymbol{t}\cdot\boldsymbol{\omega}\csc\vartheta}, & \vartheta\neq n\pi\\ \frac{1}{2\pi}e^{i\boldsymbol{t}\cdot\boldsymbol{\omega}\csc\vartheta}, & \vartheta=\frac{\pi}{2}\\ \delta(\boldsymbol{t}-\boldsymbol{\omega}), & \vartheta=2n\pi\\ \delta(\boldsymbol{t}+\boldsymbol{\omega}), & \vartheta=(2n+1)\pi, n\in\mathbb{Z}. \end{cases}\tag{6}$$

Here, $\delta(\boldsymbol{t}-\boldsymbol{\omega})=\delta(t_1-\omega_1)\delta(t_2-\omega_2)$ *and*

$$A_\vartheta=\frac{1-i\cot\vartheta}{2\pi},\quad \overline{A_\vartheta}=\frac{1+i\cot\vartheta}{2\pi}.\tag{7}$$

It is not difficult to check that the FrFT kernel fulfills the following basic properties:

$$\overline{K_\vartheta(\boldsymbol{\omega},\boldsymbol{t})}=K_{-\vartheta}(\boldsymbol{\omega},\boldsymbol{t}),$$

and

$$\int_{\mathbb{R}^2}K_\vartheta(\boldsymbol{\omega},\boldsymbol{t})\overline{K_\vartheta(\boldsymbol{\omega}',\boldsymbol{t})}\,d\boldsymbol{t}=\delta(\boldsymbol{\omega}-\boldsymbol{\omega}'),$$

where $\overline{K_\vartheta(\boldsymbol{\omega},\boldsymbol{t})}$ is the conjugation of $K_\vartheta(\boldsymbol{\omega},\boldsymbol{t})$.

Definition 2.3. *Let* $f\in L^1(\mathbb{R}^2)$ *and* $\mathfrak{F}_\vartheta\{f\}\in L^1(\mathbb{R}^2)$. *The inverse formula of the 2-D FrFT for the function* f *is given by*

$$\begin{aligned} f(\boldsymbol{t})&=\int_{\mathbb{R}^2}\mathfrak{F}_\vartheta\{f\}(\boldsymbol{\omega})\overline{K_\vartheta(\boldsymbol{\omega},\boldsymbol{t})}\,d\boldsymbol{\omega}\\ &=\int_{\mathbb{R}^2}\mathfrak{F}_\vartheta\{f\}(\boldsymbol{\omega})\overline{A_\vartheta}e^{i(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)\frac{\cot\vartheta}{2}-i\boldsymbol{t}\cdot\boldsymbol{\omega}\csc\vartheta}\,d\boldsymbol{\omega}. \end{aligned}\tag{8}$$

Formula (8) tells us how to generate the original function from its FrFT. Next, we introduce a definition of the CFrFT.

Definition 2.4. *The CFrFT of any function* f *belongs to* $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ *is defined as* [10, 11]

$$\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})=\int_{\mathbb{R}^2}f(\boldsymbol{t})K_{\alpha,\beta}(\boldsymbol{\omega},\boldsymbol{t})d\boldsymbol{t},\tag{9}$$

where

$$K_{\alpha,\beta}(\boldsymbol{\omega},\boldsymbol{t})=d(\gamma)e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\boldsymbol{\omega}\right)}.\tag{10}$$

In this case,

$$\gamma=\frac{\alpha+\beta}{2},\quad \delta=\frac{\alpha-\beta}{2},\quad a(\gamma)=\frac{\cot\gamma}{2},\quad b(\gamma,\delta)=\frac{\cos\delta}{\sin\gamma},$$
$$c(\gamma,\delta)=\frac{\sin\delta}{\sin\gamma},\quad d(\gamma)=\frac{ie^{-i\gamma}}{2\pi\sin\gamma},\quad M=\begin{pmatrix} b(\gamma,\delta) & c(\gamma,\delta)\\ -c(\gamma,\delta) & b(\gamma,\delta)\end{pmatrix},$$

with $\alpha,\beta\in\mathbb{R}$ *and* $\alpha+\beta\notin 2\pi\mathbb{Z}$.

Relation (9) may be expressed in the form

$$\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})=\int_{\mathbb{R}^2}d(\gamma)\left(f(\boldsymbol{t})e^{-ia(\gamma)|\boldsymbol{t}|^2}\right)e^{-ia(\gamma)|\boldsymbol{\omega}|^2}e^{i\boldsymbol{t}\cdot M\boldsymbol{\omega}}d\boldsymbol{t}.\tag{11}$$

Denoting

$$g(\boldsymbol{t}) = f(\boldsymbol{t})e^{-ia(\gamma)|\boldsymbol{t}|^2}, \tag{12}$$

we easily get

$$|g(\boldsymbol{t})| = |f(\boldsymbol{t})|. \tag{13}$$

Now formula (11) may be represented as

$$\begin{aligned}(d(\gamma))^{-1}\, e^{ia(\gamma)|\boldsymbol{\omega}|^2}\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} g(\boldsymbol{t})e^{i\boldsymbol{t}\cdot M\boldsymbol{\omega}}d\boldsymbol{t}\\ &= \mathfrak{F}\{g\}(M\boldsymbol{\omega}),\end{aligned} \tag{14}$$

which explains the direct interaction between the CFrFT and the 2-D Fourier transformation. Here $\mathfrak{F}\{f\}(\boldsymbol{\omega})$ denotes the 2-D Fourier transform of $f \in L^2(\mathbb{R}^2)$ given by (see [16, 17, 18])

$$\mathfrak{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{t})e^{i\boldsymbol{t}\cdot\boldsymbol{\omega}}d\boldsymbol{t}, \tag{15}$$

where $\boldsymbol{t}, \boldsymbol{\omega} \in \mathbb{R}^2, \boldsymbol{t}\cdot\boldsymbol{\omega} = t_1\omega_1 + t_2\omega_2$.

The original function can be obtained in terms of the CFrFT using the definition as below.

Definition 2.5. *For any $\mathfrak{F}_{\alpha,\beta}\{f\}$ and f in $L^1(\mathbb{R}^2)$, the inverse of the CFrFT is calculated by*

$$\begin{aligned}f(\boldsymbol{t}) &= \int_{\mathbb{R}^2} \mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})\overline{K_{\alpha,\beta}(\boldsymbol{\omega},\boldsymbol{t})}d\boldsymbol{\omega}\\ &= \overline{d(\gamma)}\int_{\mathbb{R}^2} \mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})e^{i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\boldsymbol{\omega}\right)}d\boldsymbol{\omega}.\end{aligned} \tag{16}$$

As a particular case, when $\alpha = \beta$, then one obtains

$$\begin{aligned}&\gamma = \alpha, \quad \delta = 0, \quad a(\alpha) = \frac{\cot\alpha}{2}, \quad b(\alpha) = \frac{1}{\sin\alpha},\\ &c = 0, \quad d(\alpha) = \frac{1 + i\cot\alpha}{2\pi}, \quad M = \begin{pmatrix} \frac{1}{\sin\alpha} & 0\\ 0 & \frac{1}{\sin\alpha}\end{pmatrix}.\end{aligned}$$

Substituting this into (9) yields

$$
\begin{aligned}
\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega}) &= \frac{1+i\cot\alpha}{2\pi}\int_{\mathbb{R}^2} f(\boldsymbol{t})e^{-i\left[\frac{\cot\alpha}{2}\left(t_1^2+t_2^2+\omega_1^2+\omega_2^2-\begin{pmatrix}t_1\\t_2\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sin\alpha} & 0\\ 0 & \frac{1}{\sin\alpha}\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right)\right]}d\boldsymbol{t}\\
&= \frac{1+i\cot\alpha}{2\pi}\int_{\mathbb{R}^2} f(\boldsymbol{t})e^{-i\left[(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)\frac{\cot\alpha}{2}-\begin{pmatrix}t_1\\t_2\end{pmatrix}\cdot\begin{pmatrix}\frac{\omega_1}{\sin\alpha}\\ \frac{\omega_2}{\sin\alpha}\end{pmatrix}\right]}d\boldsymbol{t}\\
&= \frac{1+i\cot\alpha}{2\pi}\int_{\mathbb{R}^2} f(\boldsymbol{t})e^{-i\left[\frac{\cot\alpha}{2}\left(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2\right)-\frac{t_1\omega_1}{\sin\alpha}-\frac{t_2\omega_2}{\sin\alpha}\right]}d\boldsymbol{t}\\
&= \frac{1+i\cot\alpha}{2\pi}\int_{\mathbb{R}^2} f(\boldsymbol{t})e^{-i\left[\frac{\cot\alpha}{2}\left(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2\right)-\frac{1}{\sin\alpha}(t_1\omega_1+t_2\omega_2)\right]}d\boldsymbol{t}\\
&= \frac{1+i\cot\alpha}{2\pi}\int_{\mathbb{R}^2} f(\boldsymbol{t})e^{-i\left[\frac{\cot\alpha}{2}\left(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2\right)-\csc\alpha(t_1\omega_1+t_2\omega_2)\right]}d\boldsymbol{t}\\
&= \frac{1+i\cot\alpha}{2\pi}\int_{\mathbb{R}^2} f(\boldsymbol{t})e^{-i\left[\frac{\cot\alpha}{2}\left(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2\right)-\csc\alpha(\boldsymbol{t}\cdot\boldsymbol{\omega})\right]}d\boldsymbol{t}\\
&= \frac{1+i\cot\alpha}{1-i\cot\alpha}\mathfrak{F}_{\alpha}\{f\}(\boldsymbol{\omega})\\
&= -e^{-2i\alpha}\mathfrak{F}_{\alpha}\{f\}(\boldsymbol{\omega}).
\end{aligned}
\tag{17}
$$

In this case, $\mathfrak{F}_{\alpha}\{f\}$ represents the 2-D FrFT described by identity (5). It is not difficult to verify that for $\beta=\alpha=\frac{\pi}{2}$, the CFrFT $\mathfrak{F}_{\alpha,\beta}\{f\}$ becomes the 2-D FT $\mathfrak{F}\{f\}$ defined in relation (15).

The next result will be needed to derive the uncertainty principle involving the CFrFT in the next section.

Theorem 2.6 (Parseval's formula)**.** *For all $f,g\in L^2(\mathbb{R}^2)$, the following relation holds true:*

$$\langle f,g\rangle_{L^2(\mathbb{R}^2)}=\langle\mathfrak{F}_{\alpha,\beta}\{f\},\mathfrak{F}_{\alpha,\beta}\{g\}\rangle_{L^2(\mathbb{R}^2)}, \tag{18}$$

and

$$\|f\|^2_{L^2(\mathbb{R}^2)}=\|\mathfrak{F}_{\alpha,\beta}\{f\}\|^2_{L^2(\mathbb{R}^2)}. \tag{19}$$

Proof. With the help of Parseval's relation concerning the 2-D Fourier transform, we find that

$$
\begin{aligned}
\int_{\mathbb{R}^2} f(\boldsymbol{t})\overline{g(\boldsymbol{t})}d\boldsymbol{t} &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathfrak{F}\{f\}(\boldsymbol{\omega})\overline{\mathfrak{F}\{g\}(\boldsymbol{\omega})}d\boldsymbol{\omega}\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\mathfrak{F}\{f\}(M\boldsymbol{\omega})\overline{\mathfrak{F}\{g\}(M\boldsymbol{\omega})}d(M\boldsymbol{\omega})\\
&= \frac{\det(M)}{4\pi^2}\int_{\mathbb{R}^2}\mathfrak{F}\{f\}(M\boldsymbol{\omega})\overline{\mathfrak{F}\{g\}(M\boldsymbol{\omega})}d\boldsymbol{\omega}.
\end{aligned}
\tag{20}
$$

Applying equation (14) to relation (20) results in

$$\begin{aligned}\int_{\mathbb{R}^2} f(\boldsymbol{t})\overline{g(\boldsymbol{t})}d\boldsymbol{t} &= \frac{1}{4\pi^2\sin^2\gamma}\int_{\mathbb{R}^2}\mathfrak{F}\{f(\boldsymbol{t})e^{-ia(\gamma)|\boldsymbol{t}|^2}\}(M\boldsymbol{\omega})\overline{\mathfrak{F}\{g(\boldsymbol{t})e^{-ia(\gamma)|\boldsymbol{t}|^2}\}(M\boldsymbol{\omega})}d\boldsymbol{\omega}\\ &= \frac{1}{4\pi^2\sin^2\gamma}\int_{\mathbb{R}^2} d^{-1}(\gamma)\overline{d^{-1}(\gamma)}\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})\overline{\mathfrak{F}_{\alpha,\beta}\{g\}(\boldsymbol{\omega})}d\boldsymbol{\omega}\\ &= \frac{|d^{-1}(\gamma)|^2}{4\pi^2\sin^2\gamma}\int_{\mathbb{R}^2}\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})\overline{\mathfrak{F}_{\alpha,\beta}\{g\}(\boldsymbol{\omega})}d\boldsymbol{\omega}\\ &= \int_{\mathbb{R}^2}\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})\overline{\mathfrak{F}_{\alpha,\beta}\{g\}(\boldsymbol{\omega})}d\boldsymbol{\omega},\end{aligned}$$

and the proof is complete. □

To verify the use of the CFrFT properties, we give an illustration with a simple example.

Example 1. *Consider the Gaussian function of the form*

$$f(\boldsymbol{t}) = e^{-k|\boldsymbol{t}|^2},\ k > 0. \tag{21}$$

Thus, with the help of equation (9), we get

$$\begin{aligned}&\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})\\ &= \int_{\mathbb{R}^2} e^{-k(t_1^2+t_2^2)d(\gamma)}e^{-i(\alpha(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\omega)}d\boldsymbol{t}\\ &= d(\gamma)\int_{\mathbb{R}^2} e^{-k(t_1^2+t_2^2)}e^{-i(\alpha(\gamma)(t_1^2+t_2^2+\omega_1^2+\omega_2^2)-i\begin{pmatrix}t_1\\t_2\end{pmatrix}\cdot\begin{pmatrix}b(\gamma,\delta) & c(\gamma,\delta)\\ -c(\gamma,\delta) & b_{(\gamma,\delta)}\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}}d\boldsymbol{t}\\ &= d(\gamma)e^{-i(\alpha(\gamma)(\omega_1^2+\omega_2^2)}\int_{\mathbb{R}^2} e^{-k(t_1^2+t_2^2)-i\begin{pmatrix}t_1\\t_2\end{pmatrix}\cdot\begin{pmatrix}b(\gamma,\delta)\omega_1+c(\gamma,\delta)\omega_2\\ -c(\gamma,\delta)\omega_1+b(\gamma,\delta)\omega_2\end{pmatrix}}d\boldsymbol{t}\\ &= d(\gamma)e^{-i\alpha(\gamma)(\omega_1^2+\omega_2^2)}\\ &\quad\times\int_{\mathbb{R}^2} e^{-k(t_1^2+t_2^2)}e^{-i\alpha(\gamma)(t_1^2+t_2^2)}e^{-i\left(t_1(b(\gamma,\delta)\omega_1+c(\gamma,\delta)\omega_2)+t_2(-c(\gamma,\delta)\omega_1+b(\gamma,\delta)\omega_2)\right)}d\boldsymbol{t}\\ &= d(\gamma)e^{-i\alpha(\gamma)+(\omega_1^2+\omega_2^2)}\\ &\quad\times\int_{\mathbb{R}^2} e^{-(k+i\alpha(\gamma))t_1^2}e^{-(k+i\alpha(\gamma))t_2^2}e^{-it_1(b(\gamma,\delta)\omega_1+c(\gamma,\delta)\omega_2)}e^{-it_2(-c(\gamma,\delta)\omega_1+b(\gamma,\delta)\omega_2)}d\boldsymbol{t}.\end{aligned} \tag{22}$$

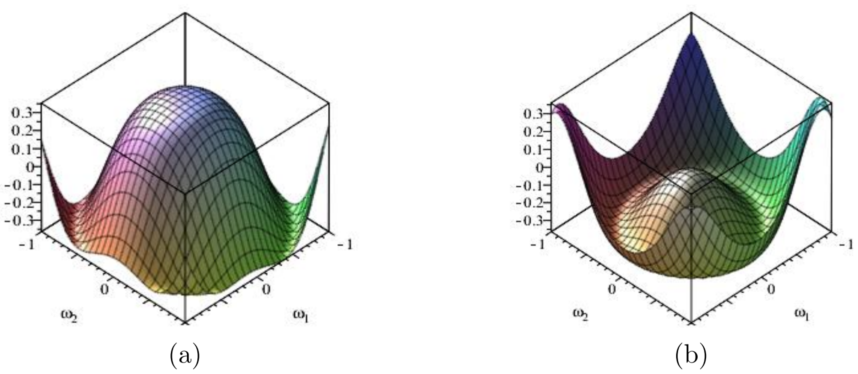
Further, we obtain

$$
\begin{aligned}
\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega}) &= d(\gamma)e^{-i\alpha(\gamma)(\omega_1^2+\omega_2^2)}\int_{\mathbb{R}} e^{-(k+i\alpha(\gamma)t_1^2-it_1(b(\gamma,\delta)\omega_1+b(\gamma,\delta)\omega_2)}dt_1\\
&\quad\times\int_{\mathbb{R}} e^{-(k+i\alpha(\gamma)t_1^2-it_2(-c(\gamma,\delta)\omega_1+b(\gamma,\delta)\omega_2)t_2}dt_2\\
&= d(\gamma)e^{-i\alpha(\gamma)(\omega_1^2+\omega_2^2)}\int_{\mathbb{R}} e^{-(k+i\alpha(\gamma))\left(t_1^2+\frac{i(b(\gamma,\delta)\omega_1+c(\gamma,\delta)\omega_2)t_1}{k+i\alpha(\gamma)}\right)}dt_1\\
&\quad\times\int_{\mathbb{R}} e^{-(k+i\alpha(\gamma))(t_2^2+\frac{i(-c(\gamma,\delta)\omega_1+b(\gamma,\delta)\omega_2)t_2}{k+i\alpha(\gamma)})}dt_2.
\end{aligned}
\tag{23}
$$

Hence,

$$
\begin{aligned}
&\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})\\
&= d(\gamma)e^{-i\alpha(\gamma)(\omega_1^2+\omega_2^2)}\int_{\mathbb{R}} e^{-(k+i\alpha(\gamma))\left(\left(t_1+\frac{i(b(\gamma,\delta)\omega_1+c(\gamma,\delta)\omega_2)}{2(k+i\alpha(\gamma))}\right)^2-\left(\frac{i(b(\gamma,\delta)\omega_1+c(\gamma,\delta)\omega_2)}{2(k+i\alpha(\gamma)}\right)^2\right)}dt_1\\
&\quad\times\int_{\mathbb{R}} e^{-(k+i\alpha(\gamma))\left(\left(t_2+\frac{i(-c(\gamma,\delta)\omega_1+b(\gamma,\delta)\omega_2)}{2(k+i\alpha(\gamma))}\right)^2-\left(\frac{i(-c(\gamma,\delta)\omega_1+b(\gamma,\delta)\omega_2)}{2(k+i\alpha(\gamma))}\right)^2\right)}dt_2\\
&= d(\gamma)e^{-i\alpha(\gamma)(\omega_1^2+\omega_2^2)-\frac{\left(b(\gamma,\delta)\omega_1)+c(\gamma,\delta)\omega_2\right)^2}{4(k+i\alpha(\gamma))}-\frac{\left(-c(\gamma,\delta)\omega_1+b(\gamma,\delta)\omega_2\right)^2}{4(k+i\alpha(\gamma))}}\\
&\quad\times\int_{\mathbb{R}} e^{-(k+i\alpha(\gamma))\left(t_1+\frac{i(b(\gamma,\delta)\omega_1+c(\gamma,\delta)\omega_2)}{4(k+i\alpha(\gamma))}\right)^2}dt_1\int_{\mathbb{R}} e^{-(k+i\alpha(\gamma))\left(t_2+\frac{i(-c(\gamma,\delta)\omega_1+b(\gamma,\delta)\omega_2)}{4(k+i\alpha(\gamma))}\right)^2}dt_2\\
&= d(\gamma)e^{-i\alpha(\gamma)(\omega_1^2+\omega_2^2)-\left(\frac{(b(\gamma,\delta)\omega_1+c(\gamma,\delta)\omega_2)^2-(b(\gamma,\delta)\omega_2-c(\gamma,\delta)\omega_1^2)}{4(k+i\alpha(\gamma))}\right)}\frac{\pi}{k+i\alpha(\gamma)},
\end{aligned}
\tag{24}
$$

as shown in Figure 1.



(a) (b)

FIGURE 1. (a) Real part, (b) imaginary part of Example 1 for $\gamma = \frac{\pi}{6}, \delta = \frac{\pi}{4}, k = 1, \alpha = 5, \omega_1 = -1...1, \omega_2 = -1..1.$

3. Properties of Coupled Fractional Fourier Transform

In this part, we investigate in detail several properties of the CFrFT. The results are modifications of the corresponding properties of the 2-D FrFT.

Theorem 3.1 (Linearity). *Let two functions $f, g \in L^2(\mathbb{R}^2)$ with $k_1, k_2 \in \mathbb{R}$, then we have*

$$\mathfrak{F}_{\alpha,\beta}\{k_1 f(\boldsymbol{t}) + k_2 g(\boldsymbol{t})\}(\boldsymbol{\omega}) = k_1 \mathfrak{F}_{\alpha,\beta}\{f(\boldsymbol{t})\}(\boldsymbol{\omega}) + k_2 \mathfrak{F}_{\alpha,\beta}\{g(\boldsymbol{t})\}(\boldsymbol{\omega}). \tag{25}$$

Proof. Due to the definition of the CFrFT (9), we get

$$\begin{aligned}\mathfrak{F}_{\alpha,\beta}\{k_1 f(\boldsymbol{t}) + k_2 g(\boldsymbol{t})\}(\boldsymbol{\omega}) &= d(\gamma) \int_{\mathbb{R}^2} \big(k_1 f(\boldsymbol{t}) + k_2 g(\boldsymbol{t})\big) e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot\boldsymbol{M}\boldsymbol{\omega}\right)} d\boldsymbol{t} \\ &= d(\gamma) \int_{\mathbb{R}^2} k_1 f(\boldsymbol{t}) e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot\boldsymbol{M}\boldsymbol{\omega}\right)} d\boldsymbol{t} \\ &\quad + d(\gamma) \int_{\mathbb{R}^2} k_2 g(\boldsymbol{t}) e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot\boldsymbol{M}\boldsymbol{\omega}\right)} d\boldsymbol{t}.\end{aligned} \tag{26}$$

The above equation can be rewritten in the form

$$\begin{aligned}\mathfrak{F}_{\alpha,\beta}\{k_1 f(\boldsymbol{t}) + k_2 g(\boldsymbol{t})\}(\boldsymbol{\omega}) &= k_1 d(\gamma) \int_{\mathbb{R}^2} f(\boldsymbol{t}) e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot\boldsymbol{M}\boldsymbol{\omega}\right)} d\boldsymbol{t} \\ &\quad + k_2 d(\gamma) \int_{\mathbb{R}^2} g(\boldsymbol{t}) e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot\boldsymbol{M}\boldsymbol{\omega}\right)} d\boldsymbol{t} \\ &= k_1 \mathfrak{F}_{\alpha,\beta}\{f(\boldsymbol{t})\}(\boldsymbol{\omega}) + k_2 \mathfrak{F}_{\alpha,\beta}\{g(\boldsymbol{t})\}(\boldsymbol{\omega}).\end{aligned} \tag{27}$$

Thus, the theorem was to be proved. □

Theorem 3.2 (Shifting). *Let f belongs to $L^2(\mathbb{R}^2)$ and $T_y f(\boldsymbol{t}) = f(\boldsymbol{t} - \boldsymbol{y})$, then we have*

$$\mathfrak{F}_{\alpha,\beta}\{T_y f(\boldsymbol{t})\}(\boldsymbol{\omega}) = e^{-ia(\gamma)|\boldsymbol{y}|^2 - \boldsymbol{y}\cdot M\boldsymbol{\omega}} \mathfrak{F}_{\alpha,\beta}\{\check{f}\}(\boldsymbol{\omega}), \tag{28}$$

where $\check{f}(\boldsymbol{t}) = f(\boldsymbol{t}) e^{-i2a(\gamma)\boldsymbol{t}\cdot\boldsymbol{y}}$.

Proof. With the help of equation (9), we immediately obtain

$$\mathfrak{F}_{\alpha,\beta}\{T_y f(\boldsymbol{t})\}(\boldsymbol{\omega}) = d(\gamma) \int_{\mathbb{R}^2} f(\boldsymbol{t} - \boldsymbol{y}) e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot\boldsymbol{M}\boldsymbol{\omega}\right)} d\boldsymbol{t}. \tag{29}$$

Substituting $\boldsymbol{u}=\boldsymbol{t}-\boldsymbol{y}$ in the above identity results in

$$
\begin{aligned}
&\mathfrak{F}_{\alpha,\beta}\{T_y f(\boldsymbol{t})\}(\boldsymbol{\omega})\\
&= d(\gamma)\int_{\mathbb{R}^2} f(\boldsymbol{u})e^{-i\left(a(\gamma)(|\boldsymbol{u}+\boldsymbol{y}|^2+|\boldsymbol{\omega}|^2)-(\boldsymbol{u}+\boldsymbol{y})\cdot \boldsymbol{M\omega}\right)}d\boldsymbol{u}\\
&= d(\gamma)\int_{\mathbb{R}^2} f(\boldsymbol{u})e^{-i\left(a(\gamma)(|\boldsymbol{u}|^2+2\boldsymbol{u}\cdot\boldsymbol{y}+|\boldsymbol{y}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{u}\cdot M\boldsymbol{\omega}-\boldsymbol{y}\cdot M\boldsymbol{\omega}\right)}d\boldsymbol{u}\\
&= d(\gamma)\int_{\mathbb{R}^2} f(\boldsymbol{u})e^{-i2a(\gamma)\boldsymbol{u}\cdot\boldsymbol{y}}e^{-i\left(a(\gamma)(|\boldsymbol{u}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{u}\cdot M\boldsymbol{\omega}\right)}e^{-ia(\gamma)|\boldsymbol{y}|^2-\boldsymbol{y}\cdot M\boldsymbol{\omega}}d\boldsymbol{u}\\
&= d(\gamma)\int_{\mathbb{R}^2} \check{f}(\boldsymbol{u})e^{-i\left(a(\gamma)(|\boldsymbol{u}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{u}\cdot M\boldsymbol{\omega}\right)}e^{-ia(\gamma)|\boldsymbol{y}|^2-\boldsymbol{y}\cdot M\boldsymbol{\omega}}d\boldsymbol{u}\\
&= e^{-ia(\gamma)|\boldsymbol{y}|^2-\boldsymbol{y}\cdot M\boldsymbol{\omega}}\mathfrak{F}_{\alpha,\beta}\{\check{f}\}(\boldsymbol{\omega}), \qquad (30)
\end{aligned}
$$

and the proof is complete. □

Theorem 3.3 (Modulation)**.** *Assume that $f\in L^2(\mathbb{R}^2)$. Let $\mathbb{M}_{\omega_0}f(\boldsymbol{t})=e^{i\boldsymbol{t}\cdot\boldsymbol{\omega}_0}$, then we have*

$$
\begin{aligned}
&\mathfrak{F}_{\alpha,\beta}\{\mathbb{M}_{\omega_0}f\}(\boldsymbol{\omega})\\
&= e^{-i\left(a(\gamma)\left(|M^{-1}\boldsymbol{\omega}_0|^2-2|(\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega}_0)\cdot M^{-1}\boldsymbol{\omega}_0|\right)\right)}\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega_0}). \qquad (31)
\end{aligned}
$$

Proof. It follows from equation (9) that

$$
\begin{aligned}
&\mathfrak{F}_{\alpha,\beta}\{\mathbb{M}_{\omega_0}f\}(\boldsymbol{\omega})\\
&= d(\gamma)\int_{\mathbb{R}^2}\mathbb{M}_{\omega_0}f(\boldsymbol{t})e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\boldsymbol{\omega}\right)}d\boldsymbol{t}\\
&= d(\gamma)\int_{\mathbb{R}^2}e^{i\boldsymbol{t}\cdot\boldsymbol{\omega}_0}f(\boldsymbol{t})e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\boldsymbol{\omega}\right)}d\boldsymbol{t}\\
&= d(\gamma)\int_{\mathbb{R}^2}f(\boldsymbol{t})e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot(M\boldsymbol{\omega}+\boldsymbol{\omega_0})\right)}d\boldsymbol{t}\\
&= d(\gamma)\int_{\mathbb{R}^2}f(\boldsymbol{t})e^{-i\left(a(\gamma)\left(|\boldsymbol{t}|^2+|\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega_0}-M^{-1}\boldsymbol{\omega}_0|^2\right)-\boldsymbol{t}\cdot M(\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega_0})\right)}d\boldsymbol{t}\\
&= d(\gamma)\int_{\mathbb{R}^2}f(\boldsymbol{t})\\
&\quad\times e^{-i\left(a(\gamma)\left(|\boldsymbol{t}|^2+|\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega_0}|^2-2|(\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega}_0)\cdot M^{-1}\boldsymbol{\omega}_0|+|M^{-1}\boldsymbol{\omega}_0|^2\right)-\boldsymbol{t}\cdot M(\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega}_0)\right)}d\boldsymbol{t}\\
&= e^{-i\left(a(\gamma)\left(|M^{-1}\boldsymbol{\omega}_0|^2-2|(\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega}_0)\cdot M^{-1}\boldsymbol{\omega}_0|\right)\right)}\\
&\quad\times d(\gamma)\int_{\mathbb{R}^2}f(\boldsymbol{t})e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega_0}|^2)-\boldsymbol{t}\cdot M(\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega_0})\right)}d\boldsymbol{t}\\
&= e^{-i\left(a(\gamma)\left(|M^{-1}\boldsymbol{\omega}_0|^2-2|(\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega}_0)\cdot M^{-1}\boldsymbol{\omega}_0|\right)\right)}\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega}+M^{-1}\boldsymbol{\omega_0}).
\end{aligned}
$$

This finishes the proof of the theorem. $\square$

Definition 3.4. *For any functions $f, g \in L^2(\mathbb{R}^2)$, the convolution operator for the CFrFT is defined by*

$$(f * g)(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{t}-\boldsymbol{x}) e^{i2a(\gamma)\boldsymbol{x}\cdot(\boldsymbol{t}-\boldsymbol{x})} d\boldsymbol{x}. \tag{32}$$

As an easy consequence we get the following theorem.

Theorem 3.5 (CFrFT Convolution)**.** *Let $f, g \in L^2(\mathbb{R}^2)$, one has*

$$\frac{e^{ia(\gamma)|\boldsymbol{\omega}|^2}}{d(\gamma)} \mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega}) \mathfrak{F}_{\alpha,\beta}\{g\}(\boldsymbol{\omega}) = \mathfrak{F}_{\alpha,\beta}\{f * g\}(\boldsymbol{\omega}). \tag{33}$$

Proof. In fact, we have

$$\begin{aligned}
&\mathfrak{F}_{\alpha,\beta}\{f * g\}(\boldsymbol{\omega}) \\
&= d(\gamma) \int_{\mathbb{R}^2} (f * g) e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{t} \\
&= d(\gamma) \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{t}-\boldsymbol{x}) e^{i2a(\gamma)\boldsymbol{x}\cdot(\boldsymbol{t}-\boldsymbol{x})} d\boldsymbol{x} \right) e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{t} \\
&= d(\gamma) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{t}-\boldsymbol{x}) e^{i2a(\gamma)\boldsymbol{x}\cdot(\boldsymbol{t}-\boldsymbol{x})} e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{x} d\boldsymbol{t}. \qquad (34)
\end{aligned}$$

The change of variables $\boldsymbol{z} = \boldsymbol{t} - \boldsymbol{x}$, yields

$$\begin{aligned}
\mathfrak{F}_{\alpha,\beta}\{f * g\}(\boldsymbol{\omega}) &= d(\gamma) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{z}) e^{i2a(\gamma)\boldsymbol{z}\cdot\boldsymbol{x}} \\
&\quad \times e^{-i\left(a(\gamma)(|\boldsymbol{z}|^2+|\boldsymbol{x}|^2+|\boldsymbol{\omega}|^2+2\boldsymbol{z}\cdot\boldsymbol{x})-\boldsymbol{z}\cdot M\boldsymbol{\omega}-\boldsymbol{x}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{z} d\boldsymbol{x}.
\end{aligned}$$

This equation implies that

$$\begin{aligned}
\mathfrak{F}_{\alpha,\beta}\{f * g\}(\boldsymbol{\omega}) &= d(\gamma) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{z}) e^{-i\left(a(\gamma)(|\boldsymbol{x}|^2+|\boldsymbol{z}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{z}\cdot M\boldsymbol{\omega}-\boldsymbol{x}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{z} d\boldsymbol{x} \\
&= \frac{d(\gamma) d(\gamma)}{d(\gamma)} e^{ia(\gamma)|\boldsymbol{\omega}|^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{z}) e^{-i\left(a(\gamma)(|\boldsymbol{z}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{z}\cdot M\boldsymbol{\omega}\right)} \\
&\quad \times e^{-i\left(a(\gamma)(|\boldsymbol{x}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{x}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{z} d\boldsymbol{x} \\
&= \frac{e^{ia(\gamma)|\boldsymbol{\omega}|^2}}{d(\gamma)} \left(d(\gamma) \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-i\left(a(\gamma)(|\boldsymbol{x}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{x}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{x} \right) \\
&\quad \times \left(d(\gamma) \int_{\mathbb{R}^2} g(\boldsymbol{z}) e^{-i\left(a(\gamma)(|\boldsymbol{z}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{z}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{z} \right) \\
&= \frac{e^{ia(\gamma)|\boldsymbol{\omega}|^2}}{d(\gamma)} \mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega}) \mathfrak{F}_{\alpha,\beta}\{g\}(\boldsymbol{\omega}).
\end{aligned}$$

This ends the proof of the theorem. $\square$

Definition 3.6. *For any functions $f, g \in L^2(\mathbb{R}^2)$. The correlation operator for the CFrFT is defined by*

$$(f \circ g)(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{t}+\boldsymbol{x}) e^{-i2a(\gamma)\boldsymbol{x}\cdot(\boldsymbol{t}+\boldsymbol{x})} d\boldsymbol{x}. \tag{35}$$

Theorem 3.7. *Let $f, g \in L^2(\mathbb{R}^2)$, we have*

$$\mathfrak{F}_{\alpha,\beta}\{f \circ g\}(\boldsymbol{\omega}) = \frac{e^{ia(\gamma)|\boldsymbol{\omega}|^2}}{d(\gamma)} \mathfrak{F}_{\alpha,\beta}\{f\}(-\boldsymbol{\omega}) \mathfrak{F}_{\alpha,\beta}\{g\}(\boldsymbol{\omega}). \tag{36}$$

Proof. Applying (35) results in

$$\begin{aligned}
&\mathfrak{F}_{\alpha,\beta}\{f \circ g\}(\boldsymbol{\omega}) \\
&= d(\gamma) \int_{\mathbb{R}^2} (f \circ g) e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{t} \\
&= d(\gamma) \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{t}+\boldsymbol{x}) e^{-i2a(\gamma)\boldsymbol{x}\cdot(\boldsymbol{t}+\boldsymbol{x})} d\boldsymbol{x} \right) e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{t} \\
&= d(\gamma) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{t}+\boldsymbol{x}) e^{-i2a(\gamma)\boldsymbol{x}\cdot(\boldsymbol{t}+\boldsymbol{x})} e^{-i\left(a(\gamma)(|\boldsymbol{t}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{t}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{x} d\boldsymbol{t}.
\end{aligned}$$

Setting $\boldsymbol{z} = \boldsymbol{t} + \boldsymbol{x}$, we find that

$$\begin{aligned}
&\mathfrak{F}_{\alpha,\beta}\{f \circ g\}(\boldsymbol{\omega}) \\
&= d(\gamma) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{z}) e^{-i2a(\gamma)\boldsymbol{z}\cdot\boldsymbol{x}} e^{-i\left(a(\gamma)(|\boldsymbol{z}-\boldsymbol{x}|^2+|\boldsymbol{\omega}|^2)-(\boldsymbol{z}-\boldsymbol{x})\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{z} d\boldsymbol{x} \\
&= d(\gamma) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{z}) e^{-i2a(\gamma)\boldsymbol{x}\cdot\boldsymbol{z}} \\
&\quad \times e^{-i\left(a(\gamma)(|\boldsymbol{z}|^2+|\boldsymbol{x}|^2-2\boldsymbol{x}\cdot\boldsymbol{z}+|\boldsymbol{\omega}|^2)-\boldsymbol{z}\cdot M\boldsymbol{\omega}+\boldsymbol{x}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{z} d\boldsymbol{x} \\
&= d(\gamma) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{z}) e^{-i\left(a(\gamma)(|\boldsymbol{z}|^2+|\boldsymbol{x}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{z}\cdot M\boldsymbol{\omega}+\boldsymbol{x}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{z} d\boldsymbol{x} \\
&= \frac{e^{ia(\gamma)|\boldsymbol{\omega}|^2}}{d(\gamma)} \Bigg(d(\gamma) \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-i\left(a(\gamma)(|\boldsymbol{x}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{x}\cdot M(-\boldsymbol{\omega})\right)} d\boldsymbol{x} \\
&\quad \times d(\gamma) \int_{\mathbb{R}^2} g(\boldsymbol{z}) e^{-i\left(a(\gamma)(|\boldsymbol{z}|^2+|\boldsymbol{\omega}|^2)-\boldsymbol{z}\cdot M\boldsymbol{\omega}\right)} d\boldsymbol{z} \Bigg) \\
&= \frac{e^{ia(\gamma)|\boldsymbol{\omega}|^2}}{d(\gamma)} \mathfrak{F}_{\alpha,\beta}\{f\}(-\boldsymbol{\omega}) \mathfrak{F}_{\alpha,\beta}\{g\}(\boldsymbol{\omega}),
\end{aligned}$$

which completes the proof. □

4. Uncertainty Principle for Coupled Fractional Fourier Transform

Recently, the authors of [13, 14, 19, 20] have proposed several uncertainty principles related to various transformations. In this part, we establish an uncertainty principle related to the CFrFT. The following theorem will be useful for deriving the main result in this section.

Theorem 4.1. *Let $f \in L^2(\mathbb{R}^2)$, one has* [10]

$$\left\{\int_{\mathbb{R}^2} |\boldsymbol{t}|^2 |f(\boldsymbol{t})|^2 d\boldsymbol{t}\right\}^{\frac{1}{2}} \left\{\int_{\mathbb{R}^2} |\boldsymbol{\omega}|^2 |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{2}} \geq \frac{\sin^4\gamma}{8\pi^2}\|f\|^2_{L^2(\mathbb{R}^2)}. \tag{37}$$

The extension of equation (37) mentioned above is given by the following result.

Theorem 4.2. *Let $f \in L^2(\mathbb{R}^2)$ with $\eta, \lambda \geq 1$, the following inequality is satisfied:*

$$\left(\int_{\mathbb{R}^2} |\boldsymbol{\omega}|^{2\eta} |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^{\eta} d\boldsymbol{\omega}\right)^{\frac{\lambda}{\eta+\lambda}} \left(\int_{\mathbb{R}^2} |\boldsymbol{t}|^{2\lambda} |f(\boldsymbol{t})|^2 d\boldsymbol{t}\right)^{\frac{\eta}{\eta+\lambda}} \geq \left(\frac{\sin^8\gamma}{64\pi^4}\right)^{\frac{\eta\lambda}{\eta+\lambda}} \|f\|^2_{L^2(\mathbb{R}^2)}. \tag{38}$$

Proof. It follows from Holder's inequality for the CFrFT that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\boldsymbol{\omega}|^2 |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} \\
&\quad= \int_{\mathbb{R}^2} |\boldsymbol{\omega}|^2 |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^{\frac{2}{\eta}+2-\frac{2}{\eta}} d\boldsymbol{\omega} \\
&\quad= \int_{\mathbb{R}^2} |\boldsymbol{\omega}|^2 |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^{\frac{2}{\eta}} |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^{2-\frac{2}{\eta}} d\boldsymbol{\omega} \\
&\quad\leq \left(\int_{\mathbb{R}^2} \left(|\boldsymbol{\omega}|^2 |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^{\frac{2}{\eta}}\right)^{\eta} d\boldsymbol{\omega}\right)^{\frac{1}{\eta}} \\
&\qquad\times \left(\int_{\mathbb{R}^2} \left(|\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^{2-\frac{2}{\eta}}\right)^{\frac{\eta}{\eta-1}} d\boldsymbol{\omega}\right)^{\frac{\eta-1}{\eta}} \\
&\quad= \left(\int_{\mathbb{R}^2} |\boldsymbol{\omega}|^{2\eta} |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right)^{\frac{1}{\eta}} \left(\int_{\mathbb{R}^2} |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right)^{\frac{\eta-1}{\eta}}
\end{aligned} \tag{39}$$

Applying Plancherel's formula (19) to equation (39), we get

$$\begin{aligned}
\int_{\mathbb{R}^2} |\boldsymbol{\omega}|^2 |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} &\leq \left(\int_{\mathbb{R}^2} |\boldsymbol{\omega}|^{2\eta} |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right)^{\frac{1}{\eta}} \left(\int_{\mathbb{R}^2} |f(\boldsymbol{t})|^2 d\boldsymbol{\omega}\right)^{\frac{\eta-1}{\eta}} \\
&= \left(\int_{\mathbb{R}^2} |\boldsymbol{\omega}|^{2\eta} |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right)^{\frac{1}{\eta}} \|f\|^{2\left(\frac{\eta-1}{\eta}\right)}_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{40}$$

Hence, assuming that $f \not\equiv 0$,

$$\left(\int_{\mathbb{R}^2} |\boldsymbol{\omega}|^{2\eta} |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right)^{\frac{1}{\eta}} \geq \frac{\int_{\mathbb{R}^2} |\boldsymbol{\omega}|^2 |\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}}{\|f\|^{2\left(\frac{\eta-1}{\eta}\right)}_{L^2(\mathbb{R}^2)}}. \tag{41}$$

In a similar way, we find that

$$\left(\int_{\mathbb{R}^2}|\boldsymbol{t}|^{2\lambda}|f(\boldsymbol{t})|^2d\boldsymbol{t}\right)^{\frac{1}{\lambda}}\geq\frac{\int_{\mathbb{R}^2}|\boldsymbol{t}|^2|f(\boldsymbol{t})|^2d\boldsymbol{t}}{\|f\|_{L^2(\mathbb{R}^2)}^{2\left(\frac{\lambda-1}{\lambda}\right)}}. \tag{42}$$

By combining equations (41) and (42), we easily obtain

$$\left(\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^{2\eta}|\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2d\boldsymbol{\omega}\right)^{\frac{1}{\eta}}\left(\int_{\mathbb{R}^2}|\boldsymbol{t}|^{2\lambda}|f(\boldsymbol{t})|^2d\boldsymbol{t}\right)^{\frac{1}{\lambda}}$$
$$\geq\frac{\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^2|\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2d\boldsymbol{\omega}\int_{\mathbb{R}^2}|\boldsymbol{t}|^2|f(\boldsymbol{t})|^2d\boldsymbol{t}}{\|f\|_{L^2(\mathbb{R}^2)}^{2\left(\frac{\lambda-1}{\lambda}+\frac{\eta-1}{\eta}\right)}}. \tag{43}$$

Applying equation (37) in Theorem 4.1 results in

$$\left(\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^{2\eta}|\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2d\boldsymbol{\omega}\right)^{\frac{1}{\eta}}\left(\int_{\mathbb{R}^2}|\boldsymbol{t}|^{2\lambda}|f(\boldsymbol{t})|^2d\boldsymbol{t}\right)^{\frac{1}{\lambda}}\geq\frac{\frac{\sin^8\gamma}{64\pi^4}\|f\|_{L^2(\mathbb{R}^2)}^4}{\|f\|_{L^2(\mathbb{R}^2)}^{2\left(\frac{\lambda-1}{\lambda}+\frac{\eta-1}{\eta}\right)}}. \tag{44}$$

This equation yields

$$\left(\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^{2\eta}|\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2d\boldsymbol{\omega}\right)^{\frac{1}{\eta}}\left(\int_{\mathbb{R}^2}|\boldsymbol{t}|^{2\lambda}|f(\boldsymbol{t})|^2d\boldsymbol{t}\right)^{\frac{1}{\lambda}}\geq\frac{\sin^8\gamma}{64\pi^4}\|f\|_{L^2(\mathbb{R}^2)}^{2\left(\frac{\eta+\lambda}{\eta\lambda}\right)}. \tag{45}$$

We finally arrive at

$$\left(\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^{2\eta}|\mathfrak{F}_{\alpha,\beta}\{f\}(\boldsymbol{\omega})|^2d\boldsymbol{\omega}\right)^{\frac{\lambda}{\eta+\lambda}}\left(\int_{\mathbb{R}^2}|\boldsymbol{t}|^{2\lambda}|f(\boldsymbol{t})|^2d\boldsymbol{t}\right)^{\frac{\eta}{\eta+\lambda}}\geq\left(\frac{\sin^8\gamma}{64\pi^4}\right)^{\frac{\eta\lambda}{\eta+\lambda}}\|f\|_{L^2(\mathbb{R}^2)}^2,$$

which completes the proof. □

Remark 1.

1 *Note that inequality in Theorem 4.2 also holds for* $f\equiv 0$.
2 *The forthcoming paper will investigate how the Gaussian signals minimize inequality* (38).

5. Conclusion

In this research paper, we have demonstrated several properties of the CFrFT including convolution and correlation theorems. We also established a generalized uncertainty inequality related to the proposed CFrFT. All these results are nontrivial extensions of the 2-D FT and 2-D FrFT properties.

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