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On Generalized Bourbaki Theorem in the Category $(\mathbb{S}, \mathrm{Q})$-Cat

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Abstract. Bourbaki developed the concept of a proper map in topological spaces and proved that a continuous map between topological spaces is proper if and only if it is perfect, known as Bourbaki theorem. Clementino and Tholen extended this concept to lax algebras, formulating a generalized Bourbaki theorem applicable to a special type of category called a $(\mathbb{S}, \mathrm{Q})$-category. Their theorem states that, under certain conditions, a $(\mathbb{S}, \mathrm{Q})$-functor is proper if and only if both pullbacks of the functor are closed and a specific transformation is closed. They also provide an equivalent characterization using compactness of fibers. Clementino and Tholen then posed a question: If we slightly modify the conditions in their generalized theorem, do the equivalences still hold? This paper aims to answer this question, investigating the impact of these modifications on the relationship between properness and closure properties.

Key words and Phrases: Bourbaki theorem, Lax Extension of Functor, Lax Extension of Monad, $(\mathbb{S}, \mathrm{Q})$-category, $(\mathbb{S}, \mathrm{Q})$-Functor.

1. INTRODUCTION

The notion of a $(\mathbb{S}, \mathrm{Q})$-category was first introduced by Clementino, Hofmann and Tholen (see [1] and [2]). A $(\mathbb{S}, \mathrm{Q})$-category is a lax Eilenberg-Moore algebra of $\mathbb{S}$ in the category of sets and Q-relations where $\mathbb{S}$ is a monad and Q is a unital quantale. Taking $\mathbb{S}$ as the identity monad, this notion captures ordered sets for $\mathrm{Q} = 2$. Further, when $\mathbb{S}$ is the ultrafilter monad, one obtains topological spaces for $\mathrm{Q} = 2$ (see [3]).

A $(\mathbb{S}, \mathrm{Q})$-functor is a lax homomorphism between $(\mathbb{S}, \mathrm{Q})$-categories and $(\mathbb{S}, \mathrm{Q})$-**Cat** is the category of $(\mathbb{S}, \mathrm{Q})$-categories and $(\mathbb{S}, \mathrm{Q})$-functors (see [3]). As an example if $\mathbb{I}$ is identity monad then $(\mathbb{I}, 2)$-**Cat** is the category **Prost** of preordered sets and monotone maps.

Morever, if $\mathbb{U}$ is an ultrafilter monad, then the category $(\mathbb{U}, 2)$-**Cat** is isomorphic to the category **Top** of topological spaces and continuous functions. Since **Top** is an example of $(\mathbb{S}, \mathrm{Q})$-**Cat**, we can say that $(\mathbb{S}, \mathrm{Q})$-**Cat** is a generalization of **Top** in category theoretical setting. Hence we can analyze what properties existing in the topological spaces and continuous functions can be generalized to $(\mathbb{S}, \mathrm{Q})$-categories and $(\mathbb{S}, \mathrm{Q})$-functors.

Clementino and Tholen generalized the Bourbaki theorem in the context of topological spaces to the category $(\mathbb{S}, \mathrm{Q})$-**Cat** (see [4]). The Bourbaki theorem states that a continuous map between topological spaces is perfect iff it is proper (see [5, Theorem 10.2.1] and [6, Theorem 10]).

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The generalized Bourbaki theorem then states that, under a certain condition, the statement that a $(\mathbb{S},\mathrm{Q})$-functor g is proper is equivalent to the statement that every pullback of g is closed and Sg is closed which in turn is equivalent to the statement that all fibres of g are compact and Sg is closed.

In Remark 6.2 (2) of their paper [4], a question is raised: is the statement that a $(\mathbb{S},\mathrm{Q})$-functor g is proper, also equivalent to the statement that g is closed? Furthermore, is the statement that a $(\mathbb{S},\mathrm{Q})$-functor g is proper still equivalent to the statement that all fibers of g are compact and g is closed? The aim of this paper is to answer that questions.

In this paper, the basic category theory, quantale and the category $(\mathbb{S},\mathrm{Q})$-**Cat** are taken from [7], [8], [9], [3], [10], [6], [11], [12] and [13]. The basic general topology is taken from [5], [6], [14] and [15].

2. PRELIMINARIES

2.1. **Eilenberg-Moore Category.**

This session kicks off with a discussion of monads. We will then leverage this concept to define the Eilenberg-Moore category. For foundational details on monads, please refer to [3] and [13]. In the following, we will recall specific details pertinent to this work.

Definition 2.1. A *monad* $\mathbb{S}$ on a category $\mathbf{C}$ is a triple (S,η,ε) where $S:\mathbf{C}\to\mathbf{C}$ is a functor, and $\eta: SS\to S$, $\varepsilon: 1_{\mathbf{C}}\to S$ are natural transformations, which make the following diagrams

$$\begin{tikzcd} SSS \arrow[r, "S\eta"] \arrow[d, "\eta S"] & SS \arrow[d, "\eta"] \\ SS \arrow[r, "\eta"] & S \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\varepsilon S"] \arrow[dr, "1_S"'] & SS \arrow[d, "\eta"] & S \arrow[l, "S\varepsilon"'] \arrow[dl, "1_S"] \\ & S & \end{tikzcd}$$

commute. Namely fulfilling

$$\eta\cdot\eta S=\eta\cdot S\eta \text{ and } 1_S=\eta\cdot\varepsilon S=\eta\cdot S\varepsilon.$$

Example 2.2. Given the category **Set** of sets and functions. Then there exists *the identity monad* $\mathbb{I}=(1_{\mathbf{Set}},1,1)$ on **Set** with $1_{\mathbf{Set}}:\mathbf{Set}\to\mathbf{Set}$ is a identity functor on **Set** and $1:1_{\mathbf{Set}}\to 1_{\mathbf{Set}}$ is a natural transformation given by

$$1_Y: Y\to Y$$
$$y\mapsto y$$

for every set Y.

To elucidate the theory of monads, we present an application involving ultrafilters.

Definition 2.3. Given a set Y, a *filter* on Y is a family $\mathcal{F}\subseteq 2^Y$ such that

(1) $Y\in\mathcal{F}$
(2) $A,B\in\mathcal{F}\Rightarrow A\cap B\in\mathcal{F}$
(3) $A\in\mathcal{F}, A\subseteq B\Rightarrow B\in\mathcal{F}$.

A filter $\mathcal{F}$ is called *proper* provided that $\emptyset\notin\mathcal{F}$. An *ultrafilter* on a set Y is a maximal element in the set of proper filters on Y, ordered by inclusion.

For the category **Set**, there exists the ultrafilter functor

$$U:\mathbf{Set}\to\mathbf{Set}$$
$$Y\mapsto UY$$
$$g:Y\to Z\mapsto Ug:UY\to UZ$$

where $UY=\left\{\mathcal{V}\subseteq 2^Y \mid \mathcal{V}\,\text{is an ultrafilter on}\,X\right\}$ and $Ug(\mathcal{W})=\left\{B\subseteq Z\mid g^{-1}(B)\in\mathcal{W}\right\}$, for every set Y, every map $g:Y\to Z$ and $\mathcal{W}\in UY$.

Example 2.4. There exists *the ultrafilter monad* $\mathbb{U} = (U, \eta, \varepsilon)$ on **Set**, with $U : \mathbf{Set} \to \mathbf{Set}$ is a ultrafilter functor and $\eta : UU \to U$, $\varepsilon : 1_{\mathbf{Set}} \to U$ are natural transformations given by

$$\begin{aligned} \eta_Y : UUY &\to UY & \qquad \varepsilon_Y : Y &\to UY \\ \mathscr{W} &\mapsto \left\{A \subseteq Y \mid A^{\mathbb{U}} \in \mathscr{W}\right\} & y &\mapsto \dot{y} \end{aligned}$$

where $A^{\mathbb{U}} = \{\mathcal{V} \in UY \mid A \in \mathcal{V}\}$ and $\dot{y} = \{A \subseteq Y \mid y \in A\}$ for every set Y.

We now formally define the Eilenberg-Moore category. From the Eilenberg-Moore category $\mathbf{Set}^{\mathbb{S}}$, we will later define the $(\mathbb{S}, \mathrm{Q})$-**Cat** category on **Set** to Q-**Rel**. We will start by defining an Eilenberg-Moore algebra.

Definition 2.5. Given a monad $\mathbb{S} = (S, \eta, \varepsilon)$ on a category **C**. A $\mathbb{S}$-*algebra* (or *Eilenberg-Moore algebra*) is a pair (Y, a), where Y is a **C**-object, and $a : SY \to Y$ is a **C**-morphism, which makes the diagrams

$$\begin{tikzcd} SSY \arrow[r, "\eta_Y"] \arrow[d, "Sa"] & SY \arrow[d, "a"] \\ SY \arrow[r, "a"] & Y \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\varepsilon_Y"] \arrow[dr, "1_Y"] & SY \arrow[d, "a"] \\ & Y \end{tikzcd}$$

commute. Namely fulfilling

$$a \cdot Sa = a \cdot \eta_Y \text{ and } 1_Y = a \cdot \varepsilon_Y .$$

A $\mathbb{S}$-*homomorphism* $g : (Y, a) \to (Z, b)$ is a **C**-morphism $g : Y \to Z$, which makes the following diagram

$$\begin{tikzcd} SY \arrow[r, "Sg"] \arrow[d, "a"] & SZ \arrow[d, "b"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

commutes, i.e., $g \cdot a = b \cdot Sg$.

The *Eilenberg-Moore category* of $\mathbb{S}$ (or $\mathbf{C}^{\mathbb{S}}$) is the category of $\mathbb{S}$-algebras and $\mathbb{S}$-homomorphisms.

2.2. Category $(\mathbb{S}, \mathrm{Q})$-CAT.

In the $(\mathbb{S}, \mathrm{Q})$-**Cat** category, $\mathbb{S}$ is a monad and Q is a unital quantale. So in this session, we will first discuss the concept of a quantale. For basic facts about quantale we refer to [7] and [9].

Definition 2.6. A *quantale* Q is a complete lattice, which is equipped with an associative binary operation $\odot : \mathrm{Q} \times \mathrm{Q} \to \mathrm{Q}$ (multiplication) such that

(1) $v \odot (\bigvee W) = \bigvee\limits_{w \in W} (v \odot w)$

(2) $(\bigvee W) \odot v = \bigvee\limits_{w \in W} (w \odot v)$

for every $v \in \mathrm{Q}$ and every $W \subseteq \mathrm{Q}$. Moreover a quantale Q is said to be *unital* provided that its multiplication has a unit $k \in \mathrm{Q}$.

Example 2.7. Consider the complete lattice two-element

$$(\bot, \top, \leqslant) = (\text{false}, \text{true}, \leqslant) ,$$

where $\bot \leqslant \top$. The complete lattice $(\bot, \top, \leqslant)$ becomes a unital quantale with $\odot = \wedge$ and $k = \top$, where

$$\wedge (v, w) = \inf_{\leq} \{v, w\}$$

for every $v, w \in \{\bot, \top\}$. We will denote this quantale by $2 = (\{\bot, \top\} , \wedge, \top)$.

In defining the $(\mathbb{S}, \mathrm{Q})$-**Cat** category on **Set** to Q-**Rel**, Q-**Rel** is a category of sets and Q-relations. So now we will discuss a Q-relation which is a generalization of a relation.

Definition 2.8. Given a unital quantale $2 = (\{\bot, \top\}, \wedge, \top)$. A *relation* r from a set Y to a set Z is a map

$$r : Y \times Z \to 2 \text{ (denoted } r : Y \nrightarrow Z \text{).}$$

Given $y \in Y$ and $z \in Z$, one uses $y\, r\, z$ as a shorthand for $r(y, z) = \top$. The *opposite* (or *dual*) of a relation $r : Y \nrightarrow Z$ is the relation $r^o : Z \nrightarrow Y$ defined by

$$z\, r^\circ\, y \Leftrightarrow y\, r\, z$$

2-**Rel** (or **Rel**) is the category, whose objects are sets, and whose morphisms are relations.
Definition 2.9. Let $(\mathrm{Q}, \odot, k)$ be a unital quantale. A Q-*relation* r from a set Y to a set Z is a map

$$r : Y \times Z \to \mathrm{Q} \text{ (denoted } r : Y \nrightarrow Z \text{).}$$

The *opposite* (or *dual*) of a Q-relation $r : Y \nrightarrow Z$ is the Q-relation $r^\circ : Z \nrightarrow Y$ defined by

$$r^\circ (z, y) = r (y, z).$$

Q-**Rel** is the category, whose objects are sets, and whose morphisms are Q-relations. Composition of Q-relations $r : Y \nrightarrow Z$ and $t : Z \nrightarrow W$ is defined by

$$(t \cdot r)(y, w) = \bigvee_{z \in Z} (r (y, z) \odot s (z, w)).$$

Given a set Y, the identity Q-relations $1_Y : Y \nrightarrow Y$ is defined by

$$1_Y (y, v) = \begin{cases} k & \text{, if } y = v \\ \bot_{\mathrm{Q}} & \text{, otherwise} \end{cases}$$

for every $y, v \in Y$.

As a stepping stone, we can construct a lax extension of any functor $S : \mathbf{Set} \to \mathbf{Set}$ into the category of Q-relations. This lax extension will be crucial later when we define the lax extension of monads on **Set** to the category of Q-relations.
Definition 2.10. Let $(\mathrm{Q}, \odot, k)$ be a unital quantale and $S : \mathbf{Set} \to \mathbf{Set}$ a functor. A *lax extension* $\hat{S} : \mathrm{Q}\text{-}\mathbf{Rel} \to \mathrm{Q}\text{-}\mathbf{Rel}$ of S to Q-**Rel** is a pair of maps $\hat{S}_{\mathcal{O}} : \mathcal{O}_{\mathrm{Q-}\mathbf{Rel}} \to \mathcal{O}_{\mathrm{Q-}\mathbf{Rel}}$ and $\hat{S}_{\mathcal{M}} : \mathcal{M}_{\mathrm{Q-}\mathbf{Rel}} \to \mathcal{M}_{\mathrm{Q-}\mathbf{Rel}}$ (both denoted $\hat{S}$), where $\mathcal{O}_{\mathrm{Q-}\mathbf{Rel}}$ is class of objects in Q-**Rel**, $\mathcal{M}_{\mathrm{Q-}\mathbf{Rel}}$ is class of morphisms in Q-**Rel**, $\hat{S}(Y) = SY$, $\hat{S}(r : Y \nrightarrow Z) = \hat{S}r : SY \nrightarrow SZ$ and satisfy the following axioms:

(1) $r \leqslant l \Rightarrow \hat{S}(r) \leqslant \hat{S}(l)$
(2) $\hat{S}t \cdot \hat{S}r \leqslant \hat{S}(t \cdot r)$
(3) $Sg \leqslant \hat{S}g$ and $(Sg)^\circ \leqslant \hat{S}(g^\circ)$

for every set Y, V-relations $r, l : Y \nrightarrow Z$, $t : Z \nrightarrow W$ and map $g : Y \to Z$.

In order to fully articulate this concept, we will now analyze the ensuing two examples.
Example 2.11. The identity functor

$$\begin{aligned} 1_{\mathrm{Q-}\mathbf{Rel}} : \mathrm{Q} - \mathbf{Rel} &\to \mathrm{Q} - \mathbf{Rel} \\ Y &\mapsto Y \\ r &\mapsto r \end{aligned}$$

is a lax extension of the identity functor $1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$.
Example 2.12. The ultrafilter functor $U : \mathbf{Set} \to \mathbf{Set}$ has a lax extension which is a functor

$$\begin{aligned} \hat{U} : \mathbf{Rel} &\to \mathbf{Rel} \\ Y &\mapsto UY \\ r : Y \nrightarrow Z &\mapsto \hat{U}r : UY \nrightarrow UZ \end{aligned}$$

given by

$$\mathcal{V}\ \hat{U}r\ \mathcal{W} \Leftrightarrow \forall A \in \mathcal{V}, \forall B \in \mathcal{W}, \exists y \in A, \exists z \in B : y\, r\, z,$$

for every relation $r : Y \nrightarrow Z$ and all $\mathcal{V} \in UY$, $\mathcal{W} \in UZ$.

Having established lax extensions for functors, we can now define lax extensions for monads on **Set** to Q-**Rel**. We will further solidify this concept with examples. The concept of lax extensions for monads is detailed in [8], [3] and [13].

Definition 2.13. Given a unital quantale Q and a monad $\mathbb{S}$ on **Set**. A *lax extension* $\hat{\mathbb{S}}$ of $\mathbb{S}$ to Q-**Rel** is a triple $(\hat{S}, \eta, \varepsilon)$, where $\hat{S}$ is a lax extension of S to Q$-$**Rel**, and $\eta : \hat{S}\hat{S} \to \hat{S}$, $\varepsilon : 1_{\mathrm{Q}-\mathbf{Rel}} \to \hat{S}$ are oplax natural transformations, which means that η and ε make the following diagrams

$$\begin{array}{ccc} SSY & \xrightarrow{\eta_Y} & SY \\ \big\downarrow{\scriptstyle \hat{S}\hat{S}r} & \leqslant & \big\downarrow{\scriptstyle \hat{S}r} \\ SSZ & \xrightarrow{\eta_Z} & SZ \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\varepsilon_Y} & SY \\ \big\downarrow{\scriptstyle r} & \leqslant & \big\downarrow{\scriptstyle \hat{S}r} \\ Z & \xrightarrow{\varepsilon_Z} & SZ \end{array}$$

lax commute, for every Q$-$relation $r : Y \nrightarrow Z$. Namely fulfilling

$$\eta_Z \cdot \hat{S}\hat{S}r = \hat{S}r \cdot \eta_Y \text{ and } \varepsilon_Z \cdot r \leq \hat{S}r \cdot \varepsilon_Y,$$

for every Q$-$relation $r : Y \nrightarrow Z$.

Example 2.14. The identity monad $\mathbb{I}$ on **Set** has a lax extension to Q-**Rel** given by the identity monad

$$\mathbb{I} = (1_{\mathrm{Q}-\mathbf{Rel}}, 1, 1)$$

where $1 : 1_{\mathrm{Q}-\mathbf{Rel}} \to 1_{\mathrm{Q}-\mathbf{Rel}}$ is a natural transformation with

$$\begin{aligned} 1_Y : Y &\to Y \\ y &\mapsto y \end{aligned}$$

for every set Y.

Example 2.15. The lax extension $\hat{U}$ of the ultrafilter functor U provide a lax extension $\hat{\mathbb{U}} = (\hat{U}, \eta, \varepsilon)$ of the ultrafilter monad $\mathbb{U}$ on **Set** to **Rel**, with $\eta : \hat{U}\hat{U} \to \hat{U}$ and $\varepsilon : 1_{\mathbf{Rel}} \to \hat{U}$ given by

$$\begin{aligned} \eta_X : UUY &\to UY \\ \mathscr{W} &\mapsto \left\{A \subseteq Y \mid A^{\mathbb{U}} \in \mathscr{W}\right\} \end{aligned} \qquad \begin{aligned} \varepsilon_Y : Y &\to UY \\ y &\mapsto \dot{y} \end{aligned}$$

where $A^{\mathbb{U}} = \{\mathcal{V} \in UY \mid A \in \mathcal{V}\}$ and $\dot{y} = \{A \subseteq Y \mid y \in A\}$ for every set Y.

To generalize Bourbaki's theorem, we will subsequently define a flat lax extension of a monad on **Set** to Q-**Rel**.

Definition 2.16. A lax extension $\hat{S}$ to Q-**Rel** of a functor S on **Set** is *flat* provided that $\hat{S}1_Y = S1_Y$, for every set Y.
A lax extension $\hat{\mathbb{S}}$ to Q-**Rel** of a monad $\mathbb{S}$ on **Set** is *flat* provided that the lax extension $\hat{S}$ of S is flat.

Example 2.17. The lax extension $\hat{\mathbb{U}}$ of the ultrafilter monad $\mathbb{U}$ on **Set** to 2-**Rel** (or **Rel**) is flat.

After knowing the concept of lax extension of a monad on **Set** to Q-**Rel**, we can finally define a $(\mathbb{S}, \mathrm{Q})$-category (which is a lax Eilenberg-Moore algebra of $\mathbb{S}$ in Q-**Rel**) and a $(\mathbb{S}, \mathrm{Q})$-functor (which is a lax $\mathbb{S}$-homomorphism in Q-**Rel**). A $(\mathbb{S}, \mathrm{Q})$-**Cat** is a category of $(\mathbb{S}, \mathrm{Q})$-categories and $(\mathbb{S}, \mathrm{Q})$-functors. A thorough discussion of $(\mathbb{S}, \mathrm{Q})$-**Cat** can be found in [8], [9], [3] and [12].

Definition 2.18. Suppose Q is a unital quantale, and $\hat{\mathbb{S}} = (\hat{S}, \eta, \varepsilon)$ is a lax extension of a monad $\mathbb{S}$ on **Set** to Q-**Rel**. A $(\mathbb{S}, \mathrm{Q})$-*category* (or *lax algebra*) is a pair (Y, a), which comprises a set Y

and a Q-relation $a : SY \nrightarrow Y$ such that the following diagrams

$$\begin{tikzcd} SSY \arrow[r, "\eta_Y"] \arrow[d, "\hat{S}a"'] \arrow[dr, phantom, "\leqslant"] & SY \arrow[d, "a"] \\ SY \arrow[r, "a"] & Y \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\varepsilon_Y"] \arrow[dr] & SY \arrow[d, "a"] \arrow[d, phantom, "1_Y \leqslant"'] \\ & Y \end{tikzcd}$$

lax commute. Namely fulfilling $a \cdot \hat{S}a \leq a \cdot \eta_Y$ and $1_Y \leq a \cdot \varepsilon_Y$.
A $(\mathbb{S}, \mathrm{Q})$-*functor* $g : (Y, a) \to (Z, b)$ is a map $g : Y \to Z$ such that the following diagram

$$\begin{tikzcd} SY \arrow[r, "Sg"] \arrow[d, "a"'] \arrow[dr, phantom, "\leqslant"] & SZ \arrow[d, "b"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

lax commute, i.e., $g \cdot a \leq b \cdot Sg$.
The $(\mathbb{S}, \mathrm{Q})$-**Cat** is the category of $(\mathbb{S}, \mathrm{Q})$-categories and $(\mathbb{S}, \mathrm{Q})$-functors. Moreover, the category $(\mathbb{I}, \mathrm{Q})$-**Cat** is denoted Q-**Cat**, whose objects (resp. morphisms) are called Q-categories (resp. Q-functors).
Example 2.19. The $(\mathbb{I}, 2)$-category (or 2-category) is a pair $(Y, \leqslant)$ with Y is a set and $\leqslant : Y \nrightarrow Y$ is a relation such that

(1) $v \leqslant v$ for every $v \in Y$
(2) $u \leqslant v, v \leqslant w$ imply $u \leqslant w$ for every $u, v, w \in Y$.

A 2-functor $g : (Y, \leqslant) \to (Z, \leqslant)$ is a map $g : Y \to Z$ such that

$$v \leqslant w \text{ imply } g(v) \leqslant g(w)$$

for every $v, w \in Y$. As a result, $(\mathbb{I}, 2)$-**Cat** (or 2-**Cat**) is the category **Prost** of preordered sets and monotone maps.

A cornerstone result in this area is the fundamental theorem of lax algebras. This theorem states that the category of topological spaces **Top** is equivalent to the category of lax algebras over a specific monad $(\mathbb{U}, 2)$-**Cat**. In simpler terms, this means we can understand topological spaces as a special kind of $(\mathbb{S}, \mathrm{Q})$-category, where $\mathbb{S}$ and Q are particular monad and quantale, respectively. As a consequence, continuous functions between topological spaces can be seen as morphisms in the $(\mathbb{S}, \mathrm{Q})$-category framework.

The key takeaway from the equivalence between **Top** and $(\mathbb{U}, 2)$-**Cat** is that properties of topological spaces and continuous functions naturally translate to $(\mathbb{U}, 2)$-categories and their morphisms. This paves the way for further generalization to arbitrary $(\mathbb{S}, \mathrm{Q})$-categories, as demonstrated by Clementino and Tholen [4]. Their work generalizes the Kuratowski-Mrówka theorem and Bourbaki's theorem, extending these well-known results from the realm of topological spaces to the more abstract framework of $(\mathbb{S}, \mathrm{Q})$-categories.
Theorem 2.20 (Fundamental example of a lax algebra in [10])**.** *The category* $(\mathbb{U}, 2)$-***Cat*** *is isomorphic to the category* ***Top****.*

Thereafter we can also generalize the concepts of indiscrete topology and discrete topology in topological spaces to $(\mathbb{S}, \mathrm{Q})$-categories into the concepts of indiscrete $(\mathbb{S}, \mathrm{Q})$-category structure and discrete $(\mathbb{S}, \mathrm{Q})$-category structure, as explained in the following theorem and example.
Theorem 2.21 (Theorem 33 in [10])**.** *Let Y be a set.*

(1) The discrete $(\mathbb{S}, \mathrm{Q})$-category structure on Y is given by $\varepsilon_Y^{\circ} \cdot \hat{S}1_Y$ (where ε_Y° is opposite of ε_Y) and denoted by $1_Y^{\#} = e_Y^{\circ}.\hat{S}1_Y$.
(2) The indiscrete $(\mathbb{S}, \mathrm{Q})$-category structure on Y is given by the constant map $\underline{\top_{\mathrm{Q}}} : SY \times Y \to \mathrm{Q}$ with value $\top_{\mathrm{Q}}$.

Example 2.22. In the category $(\mathbb{U}, 2)$-**Cat** $\cong$ **Top**, the discrete structure on a set Y is the discrete topology on Y (i.e. the power set of Y) and the indiscrete structure on Y is the indiscrete topology on Y (i.e. $\{\emptyset, Y\}$).

2.3. The Bourbaki Theorem.

N. Bourbaki's theorem for topological spaces says that a continuous map between topological spaces is perfect iff it is proper. So at the beginning of this session we will start by defining proper functions and perfect functions in topological spaces.

Definition 2.23. Let $g : (Y, \mathcal{T}) \to (Z, \mathcal{J})$ be a continuous map between topological spaces.

(1) g is *closed* provided that the image under g of every closed set in $(Y, \mathcal{T})$ is closed in $(Z, \mathcal{J})$.
(2) g is *proper* provided that for every topological space $(W, \mathcal{K})$, the map

$$f \times 1_W : Y \times W \to Z \times W$$

is closed.

We also define a compact set in a topological space, because we will use the concept of a compact set by defining a perfect function in a topological space.

Definition 2.24. Given a topological space $(Y, \mathcal{T})$, a subset $A \subseteq Y$ is said to be *compact* provided that for every family $\{O_i : i \in I\} \subseteq \mathcal{T}$ such that $A \subseteq \bigcup\limits_{i \in I} O_i$ there exists a finite subfamily $\{O_{i_1}, O_{i_2}, \ldots, O_{i_n}\} \subseteq \{O_i : i \in I\}$ such that $A \subseteq \bigcup\limits_{j=1}^{n} O_{i_j}$. A topological space $(Y, \mathcal{T})$ is said to be *compact* provided that its underlying set Y is compact.

Definition 2.25. A continuous map $g : (Y, \mathcal{T}) \to (Z, \mathcal{J})$ between topological spaces is called *perfect* provided that g is closed, and for every $z \in Z$, the fibre $g^{-1}(z)$ is a compact subset of Y.

We will first establish the concepts of proper functions and perfect functions within topological spaces. This foundation will be crucial when we present the Kuratowski-Mrówka theorem and the Bourbaki theorem. Subsequently, we will generalize these theorems from topological spaces to $(\mathbb{S}, \mathrm{Q})$-categories, resulting in the generalized Kuratowski-Mrówka theorem and the generalized Bourbaki theorem.

Theorem 2.26 (Kuratowski-Mrówka theorem in [6])**.** *Given a topological space $(Y, \mathcal{T})$. The following conditions are equivalent:*

(1) $(Y, \mathcal{T})$ is compact
(2) for every topological space $(Z, \mathcal{J})$, the projection $\pi_Z : Y \times Z \to Z$ is closed.

Theorem 2.27 (Bourbaki theorem in [6])**.** *A continuous map between topological spaces is proper iff it is perfect.*

2.4. The Generalized Bourbaki Theorem.

We will start this session by giving some properties that we will later use in generalizing N. Bourbaki's theorem for topological spaces to $(\mathbb{S}, \mathrm{Q})$-categories.

From now on, assume that the quantale Q and the lax extension $\hat{\mathbb{S}} = (\hat{S}, \eta, \varepsilon)$ of a monad $\mathbb{S}$ on **Set** to Q$-$**Rel** satisfy the following five conditions:

(1) Q is *strictly two-sided*, i.e., $(Q, \odot, \top_Q)$ is a monoid.
(2) Q is *cartesian closed*, i.e., $v \wedge (\bigvee W) = \bigvee\limits_{w \in W} (v \wedge w)$ for every $v \in \mathrm{Q}$, $W \subseteq \mathrm{Q}$.
(3) S is *taut*, i.e., S preserves pullbacks of monomorphisms along arbitrary maps.
(4) $\hat{S}$ is *left-whiskering*, i.e., $\hat{S}(g \cdot r) = Sg \cdot \hat{S}r$, for every Q-relation $r : Y \nrightarrow Z$ and every map $g : Z \to W$.

(5) $\eta^\circ : \hat{S} \to \hat{S}\hat{S}$ is *natural*, which means that the following diagram

$$\begin{tikzcd} SY \arrow[r, "\eta^\circ_Y"] \arrow[d, "\hat{S}r"] & SSY \arrow[d, "\hat{S}\hat{S}r"] \\ SZ \arrow[r, "\eta^\circ_Z"] & SSZ \end{tikzcd}$$

commutes for every Q-relation $r : Y \nrightarrow Z$. This means that

$$\eta^\circ_Y \cdot \hat{S}r = \hat{S}\hat{S}r \cdot \eta^\circ_Y,$$

for every Q-relation $r : Y \nrightarrow Z$.

Example 2.28. The lax extension $\hat{\mathbb{U}} = (\hat{U}, \eta, \varepsilon)$ of the ultrafilter monad $\mathbb{U}$ on **Set** to 2-**Rel** (or **Rel**), satisfes the following conditions:

(1) The quantale 2 is strictly two-sided.
(2) The quantale 2 is cartesian closed.
(3) The ultrafilter functor U on **Set** is taut.
(4) The lax extension $\hat{U}$ of the ultrafilter functor U on **Set** to 2-**Rel** is left-whiskering.
(5) $\eta^\circ : \hat{U} \to \hat{U}\hat{U}$ is natural.

In the following definition, we introduce a proper $(\mathbb{S}, \mathrm{Q})$-functor, drawing on concepts from [6].

Definition 2.29. A $(\mathbb{S}, \mathrm{Q})$-functor $g : (Y, a) \to (Z, b)$ is *proper* provided that the diagram

$$\begin{tikzcd} SY \arrow[r, "Sg"] \arrow[d, "a"] & SZ \arrow[d, "b"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

commutes, i.e., $g \cdot a = b \cdot Sg$.

Example 2.30. In the category $(\mathbb{U}, 2)$-**Cat** $\cong$ **Top** one gets that a $(\mathbb{U}, 2)$-functor $g : (Y, a) \to (Z, b)$ is proper provided that $g \cdot a = b \cdot Ug$. Since g is a $(\mathbb{U}, 2)$-functor this is equivalent to $b \cdot Ug \leq g \cdot a$ which in turn is also equivalent to

$$b \cdot Ug(\mathcal{W}, z) \leq g \cdot a(\mathcal{W}, z),$$

for every $\mathcal{W} \in UY$, $z \in Z$. This result is also equivalent to

$$b(Ug(\mathcal{W}), z) \leq \bigvee_{g(y)=z} a(\mathcal{W}, y),$$

for every $\mathcal{W} \in UY$, $z \in Z$. Therefore we get that for every ultrafilter $\mathcal{W} \in UY$ and every $z \in \lim Ug(\mathcal{W})$, there exists $y \in \lim \mathcal{W}$ such that $g(y) = z$. This is equivalent to the map $g \times 1_W : Y \times W \to Z \times W$ is closed, for every topological space $(W, \mathcal{K})$. In other words it means g is proper in the context of topological spaces.

In the next step, we introduce the concept of a closed $(\mathbb{S}, \mathrm{Q})$-functor. The following definition references [6].

Definition 2.31. A $(\mathbb{S}, \mathrm{Q})$-functor $g : (Y, a) \to (Z, b)$ is *closed* provided that for every $A \subseteq Y$,

$$g \cdot a \cdot Si_A \cdot !^\circ_{SA} = b \cdot Sg \cdot Si_A \cdot !^\circ_{SA}$$

where $i_A : A \hookrightarrow Y$ is the inclusion map and $!_{SA} : SA \to 1$ (where $1 = \{*\}$) is the unique map, i.e., $!_{SA}(z) = *$ for every $z \in SA$.

Example 2.32. In the category $(\mathbb{U}, 2)$-**Cat** $\cong$ **Top** one gets that $(\mathbb{U}, 2)$-functor $g : (Y, a) \to (Z, b)$ is closed provided that for every $A \subseteq Y$,

$$g \cdot a \cdot Ui_A \cdot !^\circ_{UA} = b \cdot Ug \cdot Ui_A \cdot !^\circ_{UA}.$$

This is equivalent (see [6, Lemma 56]) to

$$b \cdot Ug \cdot Ui_A \cdot !^\circ_{UA} \leqslant g \cdot a \cdot Ui_A \cdot !^\circ_{UA},$$

for every $A \subseteq Y$. This means that

$$\bigvee_{\mathcal{V} \in U(g(A))} b(Ui_{g(A)}(\mathcal{V}), z) \leq \bigvee_{\mathcal{W} \in UA} \bigvee_{f(y)=z} a(Ui_A(\mathcal{W}), y),$$

for every $z \in Z$. Therefore we get that the image under g of every closed set in Y is closed in Z. In other words it means g is closed in the context of topological spaces.

We will introduce the concept of a compact $(\mathbb{S}, \mathrm{Q})$-category. For a comprehensive treatment, refer to [6].

Definition 2.33. A $(\mathbb{S}, \mathrm{Q})$-category (Y, a) is said to be *compact* provided that the unique $(\mathbb{S}, \mathrm{Q})$-functor $!_Y : (Y, a) \to (1, \top)$ (where $1 = \{*\}$) is proper.

Example 2.34. In the category $(\mathbb{U}, 2)$-**Cat** $\cong$ **Top** one gets that (Y, a) is compact provided that the unique $(\mathbb{U}, 2)-$functor $!_Y : (Y, a) \to (1, \top)$ (where $1 = \{*\}$) is proper. This is equivalent to $!_Y \cdot a = \top \cdot U!_Y$. This also means that

$$\bigvee_{y \in Y} a(\mathcal{W}, y) = \top,$$

for every $\mathcal{W} \in UY$. Therefore we get that for every ultrafilter on the topological space Y has a limit point. In other words it means Y is compact in the context of topological spaces.

In our generalization of Bourbaki's theorem, we will employ the concepts of a finitely $(-)^\circ$-strict lax natural transformation and a fiber of a $(\mathbb{S}, \mathrm{Q})$-functor. These concepts are defined in the following two definitions and explained in Example 2.36 (see [6] for both).

Definition 2.35. A $\mathrm{Q}-$relation $r : Y \nrightarrow Z$ is said to *have finite fibres* provided that the set

$$r^\circ(z) = \{y \in Y : \bot_\mathrm{Q} < r(y, z)\} \text{ (where } r^\circ \text{ is opposite of } r)$$

is finite for every $z \in Z$.

A lax natural transformation $\varepsilon : 1_{\mathrm{Q-\mathbf{Rel}}} \to \hat{S}$ is said to be *finitely* $(-)^\circ$*-strict* provided that the following diagram

$$\begin{tikzcd} SY \arrow[r, "\varepsilon^\circ_Y"] \arrow[d, "\hat{S}r"'] & Y \arrow[d, "r"] \\ SZ \arrow[r, "\varepsilon^\circ_Z"] & Z \end{tikzcd}$$

commutes for every $\mathrm{Q}-$relation $r : Y \nrightarrow Z$ with finite fibres. That is satisfy

$$\varepsilon^\circ_Z \cdot \hat{S}r \leq r \cdot \varepsilon^\circ_Y,$$

for every Q-relation $r : Y \nrightarrow Z$ with finite fibres.

Example 2.36. Given the lax extension $\hat{\mathbb{U}} = (\hat{U}, \eta, \varepsilon)$ of the ultrafilter monad $\mathbb{U}$ on **Set** to **Rel**, one gets that the lax natural transformation $\varepsilon : 1_{\mathbf{Rel}} \to \hat{U}$ is finitely $(-)^\circ$-strict.

In the following definition, we present the concept of a fibre of a $(\mathbb{S}, \mathrm{Q})$-functor.

Definition 2.37. Let $\hat{\mathbb{S}} = (\hat{S}, \eta, \varepsilon)$ be a lax extension of monad $\mathbb{S}$ on **Set** to Q-**Rel**. Given a $(\mathbb{S}, \mathrm{Q})$-functor $g : (Y, a) \to (Z, b)$. *The fibre of* g *on* $z \in Z$ is the pullback

$$!_{g^{-1}(z)} : \left(g^{-1}(z), \tilde{a}\right) \to \left(1, 1^\#\right) \text{ (where } 1 = \{*\})$$

of g along the $(\mathbb{S}, \mathrm{Q})-$functor $z : \left(1, 1^\#\right) \to (Z, b)$, i.e.,

$$\begin{tikzcd} (g^{-1}(z), \tilde{a}) \arrow[r, "!_{g^{-1}(z)}"] \arrow[d, "i_{g^{-1}(z)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (1, 1^\#) \arrow[d, "z"] \\ (Y, a) \arrow[r, "g"] & (Z, b) \end{tikzcd}$$

where $1^{\#} = \varepsilon^{\circ} \cdot \hat{S}1_1$ is the discrete structure on 1 and

$$\tilde{a} = \left(i^{\circ}_{g^{-1}(z)} \,.\, a \,.\, Si_{g^{-1}(z)}\right) \wedge \left(!^{\circ}_{g^{-1}(z)} \,.\, \varepsilon^{\circ}_{1} \,.\, \hat{S}1_1 \,.\, S!_{g^{-1}(z)}\right).$$

This theorem introduces a functor from $(\mathbb{S}, \mathrm{Q})$-**Cat** to Q-**Cat**, which we will utilize in Bourbaki's generalization theorem.

Theorem 2.38 (Theorem 18 in [6]). *Let $\hat{\mathbb{S}} = (\hat{S}, \eta, \varepsilon)$ be a lax extension of monad $\mathbb{S}$ on **Set** to Q-**Rel**. Then there exists a functor $F : (\mathbb{S}, \mathrm{Q})$-**Cat** $\to$ Q-**Cat** which is given by $F(Y, a) = (SY, \hat{a})$ and $F(g : (Y, a) \to (Z, b)) = Sg : (SY, \hat{a}) \to \left(SZ, \hat{b}\right)$, for every $(\mathbb{S}, \mathrm{Q})$-category (Y, a) and $(\mathbb{S}, \mathrm{Q})$-functor $g : (Y, a) \to (Z, b)$, where $\hat{a} = \hat{S}a \cdot \eta^{\circ}_{Y}$.*

This section concludes by generalizing the Kuratowski-Mrówka and Bourbaki theorems, extending their applicability from topological spaces to $(\mathbb{S}, \mathrm{Q})$-categories. We will refer to [6] and [4] for concepts used in the following two generalization theorems.

Theorem 2.39 (The generalized Kuratowski-Mrówka theorem in [6] and Theorem 5.2 in [4]). *Let $\hat{\mathbb{S}} = (\hat{S}, \eta, \varepsilon)$ be a lax extension of monad $\mathbb{S}$ on **Set** to Q-**Rel**, where $\hat{S}$ is flat and $\varepsilon : 1_{\mathrm{Q}-\mathbf{Rel}} \to \hat{S}$ is finitely $(-)^{\circ}$-strict. The following are equivalent:*

1. *The $(\mathbb{S}, \mathrm{Q})$-category (Y, a) is compact*
2. *for every $(\mathbb{S}, \mathrm{Q})$-category (W, c), the projection $\pi_Z : (Y, a) \times (W, c) \to (W, c)$ is closed.*

Example 2.40. In the category $(\mathbb{U}, 2)$-**Cat** $\cong$ **Top**, one gets that the generalized Kuratowski-Mrówka theorem mentioned Kuratowski-Mrówka theorem.

Theorem 2.41 (The generalized Bourbaki theorem in [6] and Theorem 6.1 in [4]). *Let $\hat{\mathbb{S}} = (\hat{S}, \eta, \varepsilon)$ be a lax extension of monad $\mathbb{S}$ on **Set** to Q-**Rel**, where $\hat{S}$ is flat, $S1 \cong 1$ (where $1 = \{*\}$), and $\varepsilon : 1_{\mathrm{Q}-\mathbf{Rel}} \to \hat{S}$ be finitely $(-)^{\circ}$-strict. Given a $(\mathbb{S}, \mathrm{Q})-$functor $g : (Y, a) \to (Z, b)$, the following are equivalent:*

1. *g is proper*
2. *every pullback of g is closed, and $Sg : (SX, \hat{a}) \to \left(SZ, \hat{b}\right)$ is closed*
3. *all fibres of g are compact, and $Sg : (SY, \hat{a}) \to \left(SZ, \hat{b}\right)$ is closed.*

Example 2.42. In the category $(\mathbb{U}, 2)$-**Cat** $\cong$ **Top**, one gets that the generalized Bourbaki theorem mentioned Bourbaki theorem for topological spaces.

3. MAIN RESULTS

We will address the specific issue raised at the beginning. Let $\hat{\mathbb{S}} = (\hat{S}, \eta, \varepsilon)$ be a lax extension of monad $\mathbb{S}$ on **Set** to Q-**Rel**, where $\hat{S}$ is flat, $S1 \cong 1$ (where $1 = \{*\}$), and $\varepsilon : 1_{\mathrm{Q}-\mathbf{Rel}} \to \hat{S}$ be finitely $(-)^{\circ}$-strict. Given a $(\mathbb{S}, \mathrm{Q})-$functor $g : (Y, a) \to (Z, b)$, are the following are equivalent:

1. g is proper
2. every pullback of g is closed
3. all fibres of g are compact, and g is closed.

This section will establish that the provided statements are not mutually equivalent. In support of our proof, we will now examine the facts presented in two distinct subsections, namely Section 2.1: Examples and Section 7.9: Labeled graphs as $(\mathbb{H}, 2)$-categories from [4]. Let $(W, \cdot, e)$ be a monoid with identity element $e \in W$. We consider a functor $S = W \times - :$ **Set** $\to$ **Set** and monad $\mathbb{S} = (S, \eta, \varepsilon)$ on **Set**, where $SY = W \times Y$,

$$\begin{aligned} \eta_Y : W \times (W \times Y) &\to (W \times Y) \\ (v, (w, y)) &\mapsto (v \cdot w, y) \end{aligned}$$

$$\begin{aligned}\varepsilon_Y : Y &\to W \times Y\\ y &\mapsto (e, y)\end{aligned}$$

for every set Y, $y \in Y$ and $v, w \in W$. Then S possesses a lax extension functor $\hat{S}$ on **Set** to 2-**Rel** (or **Rel**) given by

$$(v, y)\left(\hat{S}r\right)(w, z) \Leftrightarrow v = w \text{ and } y\, r\, z\,,$$

for any $r : Y \nrightarrow Z$, $(v, y) \in W \times Y$ and $(w, z) \in W \times Z$.

We aim to prove that the functor $\hat{S}$: 2-**Rel** $\to$ 2-**Rel** is flat. Given $(v, y)\,, (w, j) \in W \times Y$. Then we get $(v, y)\left(\hat{S}1_Y\right)(w, j)$ iff $v = w$ and $y\,(1_Y)\,j$. Further we also get $y = j$. Thus $(v, y)\left(\hat{S}1_Y\right)(v, y)$. So it is obtained $\hat{S}1_Y = 1_{SY}$. Since $1_{SY} = S1_Y$, then we get $\hat{S}1_Y = S1_Y$. We have thus established that $\hat{S}$: 2-**Rel** $\to$ 2-**Rel** is flat.

We will demonstrate that the natural transformation $\varepsilon : 1_{\mathbf{Q-Rel}} \to \hat{S}$ is finitely $(-)^{\circ}$-strict. Consider a 2-relation $r : Y \nrightarrow Z$ with finite fibers. Let $(v, y) \in W \times Y$ and $z \in Z$. Then we get $(v, y)\left(\varepsilon_Z^{\circ} \cdot \hat{S}r\right) z$ iff there is $(w, j) \in W \times Z$ such that $(v, y)\left(\hat{S}r\right)(w, j)$ and $(w, j)\,(\varepsilon_Z^{\circ})\, z$. From $(v, y)\left(\hat{S}r\right)(w, j)$, we get $v = w$ and $y\, r\, j$. From $(w, j)\,(\varepsilon_Z^{\circ})\, z$, we get $z\,(\varepsilon_Z)\,(w, j)$, $w = e$ and $z = j$. Consequently $v = w = e$ and $y\, r\, j = y\, r\, z$. Thus we get $(v, y)\left(\varepsilon_Z^{\circ} \cdot \hat{S}r\right) z$ iff $v = e$ and $y\, r\, z$. On the other hand, we also get $(v, y)\,(r \cdot \varepsilon_Y^{\circ})\, z$ iff there is $q \in Y$ such that $(v, y)\,(\varepsilon_Y^{\circ})\, q$ and $q\, r\, z$. From $(v, y)\,(\varepsilon_Y^{\circ})\, q$, we get $q\,(\varepsilon_Y)\,(v, y)$, $v = e$ and $q = y$. Consequently $q\, r\, z = y\, r\, z$. Thus we get $(v, y)\,(r \cdot \varepsilon_Y^{\circ})\, z$ iff $v = e$ and $y\, r\, z$. In other words $(v, y)\left(\varepsilon_Z^{\circ} \cdot \hat{S}r\right) y = (v, y)\,(r \cdot \varepsilon_Y^{\circ})\, z$, for every $(v, y) \in W \times Y$ and $z \in Z$. This concludes that $\varepsilon : 1_{\mathbf{Q-Rel}} \to \hat{S}$ is finitely $(-)^{\circ}$-strict.

Let $r : W \times Y \nrightarrow Y$ be a 2-relation. Let $v \in W$ and $y, q \in Y$. We define the notation $y \xrightarrow[r]{v} q \Leftrightarrow (v, y)\, r\, q$. In this notation, a pair (Y, a) is considered a $(\mathbb{S}, 2)$-category if and only if Y is a set and $a : W \times Y \nrightarrow Y$ is a 2-relation satisfying conditions

(i) $y \xrightarrow[a]{e} y$,

(ii) $y \xrightarrow[a]{v} q \xrightarrow[a]{w} p \Rightarrow y \xrightarrow[a]{w \cdot v} p$,

for every $y, q, p \in Y$ and $u, v \in W$. We also get that a $(\mathbb{S}, 2)$-category (Y, a) is compact if and only if for every $v \in W$ and $y \in Y$, there is $q \in Y$ such that $y \xrightarrow[a]{v} q$. Moreover $g : (Y, a) \to (Z, b)$ is a $(\mathbb{S}, 2)$-functor if and only if it satisfies $y \xrightarrow[a]{e} q \Rightarrow g(y) \xrightarrow[b]{e} g(q)$, for every $y, q \in Y$ and $v \in W$. For a $(\mathbb{S}, 2)$-functor $g : (Y, a) \to (Z, b)$, we have that g is proper if and only if

$$\left(\forall v \in W, \forall y \in Y, \forall z \in Z : g(y) \xrightarrow[b]{v} z\right)\left(\exists\, q \in g^{-1}(z) : y \xrightarrow[a]{v} q\right).$$

A $(\mathbb{S}, 2)$-functor g is closed iff

$$\left(\forall v \in W, \forall y \in Y, \forall z \in Z : g(y) \xrightarrow[b]{v} z\right)\left(\exists q \in g^{-1}(q), \exists w \in W : y \xrightarrow[a]{w} q\right).$$

In what follows we will prove that statement (2) does not imply statement (1).

Proof for $(2) \nRightarrow (1)$**:**

Let $(W, \cdot, e)$ be a non trivial monoid with identity element $e \in W$ and $K = \{5, 6\}$. We define a $(\mathbb{S}, 2)$-functor $g : (K, a) \to (K, b)$ which is an identity map $g : K \to K$ and 2-relations $a : W \times K \nrightarrow K$ and $b : W \times K \nrightarrow K$ satisfy

$$a\left(\left(v,y\right),q\right)=\begin{cases}\top, \text{ if } \left(\left(v,y\right),q\right)\in\{\left(\left(e,5\right),5\right),\left(\left(e,5\right),6\right), \\ \qquad\qquad\qquad\qquad \left(\left(e,6\right),5\right),\left(\left(e,6\right),6\right)\} \\ \bot, \text{ otherwise}\end{cases}$$

and $b\left(\left(v,y\right),q\right)=\top$ for every $v\in W$ and $y,q\in K$.

(i) We will show that (K,a) and (K,b) are $(\mathbb{S},2)$-categories. For every $y\in K$ we have $y\xrightarrow[a]{e}y$ and $y\xrightarrow[b]{e}y$. Similarly, for every $y,q,p\in K$ and every $v,w,u,j\in M$ let us assume that

$$y\xrightarrow[a]{v}q\xrightarrow[a]{w}p \text{ and } y\xrightarrow[b]{u}q\xrightarrow[b]{j}p.$$

Using the definitions of a and b we conclude that $v=w=e$ and $w\cdot v=e\cdot e=e$. Further we also get

$$y\xrightarrow[a]{w\cdot v=e}p \text{ and } y\xrightarrow[b]{j\cdot u}p.$$

Therefore we have shown that (K,a) and (K,b) are $(\mathbb{S},2)$-categories.

(ii) We will show that $g:(K,a)\to(K,b)$ is a $(\mathbb{S},2)$-functor. Given $y,q\in K$ and $v\in W$ such that $y\xrightarrow[a]{v}q$. Then we get

$$v=e \text{ and } g(y)\xrightarrow[b]{v=e}f(z).$$

Therefore we have shown that $g:(K,a)\to(K,b)$ is a $(\mathbb{S},2)$-functor.

(iii) We will prove that g is closed. Let $v\in W$ and $y,q\in K$ such that $g(y)\xrightarrow[b]{v}q$. Since g is an identity map, we have $g^{-1}(q)=\{q\}$. Therefore, we can choose $q\in g^{-1}(q)$ and $e\in M$ such that $y\xrightarrow[a]{e}q$. Thus we have that g is closed.

(iv) We will prove that every pullback of g is closed. Since g is a identity map, we have that a pullback of g along $1_{(K,b)}$ is g, as in the following diagram

$$\begin{tikzcd}
(K,a) \arrow[r, "g"] \arrow[d, "1_{(K,a)}"] & (K,b) \arrow[d, "1_{(K,b)}"] \\
(K,a) \arrow[r, "g"] & (K,b)
\end{tikzcd}$$

Since g is closed, we have that the pullback of g along $1_{(K,b)}$ is closed.

(v) We will prove that g is not proper. Since there is $v\in W$ such that $v\neq e$ and there are $y,q\in K$ such that $g(y)\xrightarrow[b]{v}q$. But for every $p\in f^{-1}\left(q\right)$, we have that $y\not\xrightarrow[a]{\alpha}p$. Thus we have g is not proper.

Therefore we have proven that every pullback of g is closed but g is not proper. In what follows we will prove that statement (3) does not imply statement (1).

Proof for $(3)\not\Rightarrow(1)$:

By using the same $(\mathbb{S},2)$-functor $g:(K,a)\to(K,b)$ in the proof for $(2)\not\Rightarrow(1)$ above, we obtain that g is closed and not proper. Now will show that every fibre of g is compact. Let a fibre of g on $q\in K$, which is the pullback $!_{g^{-1}(q)}:\left(g^{-1}(q),\tilde{a}\right)\to\left(1,1^{\#}\right)$ of g along the $(\mathbb{S},2)$-functor $q:(1,\top)\to(K,b)$ as in the following diagram

$$\begin{tikzcd}
\left(g^{-1}(q),\tilde{a}\right) \arrow[r, "!_{g^{-1}(q)}"] \arrow[d, "i_{g^{-1}(q)}"] & (1,\top) \arrow[d, "q"] \\
(K,a) \arrow[r, "g"] & (K,b)
\end{tikzcd}$$

where $1 = \{*\}$, $\left(1, 1^{\#}\right) = (1, \top)$ and $\tilde{a} = \left(i^{\circ}_{g^{-1}(q)} \,.\, a \,.\, S i_{g^{-1}(q)}\right) \wedge \left(!^{\circ}_{g^{-1}(q)} \,.\, \varepsilon^{\circ}_{1} \,.\, \hat{S} 1_1 \,.\, S !_{g^{-1}(q)}\right)$.

We will show that that the fibre $\left(g^{-1}(q), \tilde{a}\right)$ of g is compact. Let (L, c) be a $(\mathbb{S}, 2)$-category. We have the following composition,

$$\begin{array}{ccc}
\left(g^{-1}(q), \tilde{a}\right) \times (L, c) & \xrightarrow{\pi_L} & (L, c) \\
\Big\downarrow{\scriptstyle \pi_{g^{-1}(q)}} & & \Big\downarrow{\scriptstyle !_L} \\
\left(g^{-1}(q), \tilde{a}\right) & \xrightarrow{!_{g^{-1}(q)}} & (1, \top) \\
\Big\downarrow{\scriptstyle i_{g^{-1}(q)}} & & \Big\downarrow{\scriptstyle q} \\
(K, a) & \xrightarrow{g} & (K, b)
\end{array}$$

where $\pi_L : \left(g^{-1}(q), \tilde{a}\right) \times (L, c) \to (L, c)$ and $\pi_{g^{-1}(q)} : \left(g^{-1}(q), \tilde{a}\right) \times (L, c) \to \left(g^{-1}(q), \tilde{a}\right)$ are projection functions. Since $\pi_L : \left(g^{-1}(q), \tilde{a}\right) \times (L, c) \to (L, c)$ is a pullback of g, then we have π_L is closed (from the proof for $(2) \not\Rightarrow (1)$, we have shown that every pullback of g is closed). Wherefore $\pi_L : \left(g^{-1}(q), \tilde{a}\right) \times (L, c) \to (L, c)$ is closed then according to the generalized Kuratowski-Mrówka theorem, we obtain that $\left(g^{-1}(q), \tilde{a}\right)$ is compact. Thus we have proven that every fibre of g are compact, but g is not proper.

4. CONCLUDING REMARKS

Our research demonstrates that, if we modify the conditions in the generalized Bourbaki theorem (which is the question in [4]), we can then provide a counterexample that there is a $(\mathbb{S}, \mathrm{Q})$-functor whose pullbacks are closed, but the $(\mathbb{S}, \mathrm{Q})$-functor is not proper. Furthermore, we can also give a refuting example which states that there is a $(\mathbb{S}, \mathrm{Q})$-functor which is closed and all its fibers are compact, but the $(\mathbb{S}, \mathrm{Q})$-functor is not proper. For further research, we intend to analyze the generalization of the characterization of proper mappings by local compactness properties in topological space described in [5] which can be developed in the category $(\mathbb{S}, \mathrm{Q})$-**Cat**.

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