J. Indones. Math. Soc.
Vol. 31, No. 03 (2025), pp. 1–18.

$\mathcal{R}$-Bounded Solution Operator for Navier-Stokes-Korteweg in Bent Half Space (Ω_+)

Suma Inna[1*] and Rahmi Purnomowati[2]

[1,2]Faculty of Science and Technology, Universitas Islam Negeri Syarif Hidayatullah Jakarta, Indonesia
[1]suma.inna@uinjkt.ac.id, [2]rahmi.purnomowati@uinjkt.ac.id

Abstract. This paper discusses the $\mathcal{R}$-bounded solution operator for a compressible fluid model of Korteweg type with slip boundary conditions in a bent half-space (Ω_+). This result provides a foundation for studying the Navier-Stokes-Korteweg system in the L_p in time and L_q in space maximal regularity class and contributes to the analysis of local and global well-posedness for the original nonlinear problem, which is a fundamental system equation to describe the motion of the viscous fluid.

Key words and Phrases: Bent-half space, Navier Stokes Korteweg, R-Boundedness, Slip boundary conditions.

1. INTRODUCTION

The theory of $\mathcal{R}$-boundedness has evolved significantly over the last decade. Analyzing $\mathcal{R}$-bounded solutions of systems of partial differential equations (PDEs) has become a crucial step in understanding their Maximal Regularity. Maximal Regularity theory is a powerful tool for solving nonlinear models. Recently, there has been considerable interest in the Maximal Regularity theory due to its effectiveness and reliability in handling nonlinear PDEs. Maximal Regularity provides an a priori estimate of the solution to a linear differential system, which is essential for solving the nonlinear system. This theory facilitates solving nonlinear equations through linearization techniques combined with the contraction mapping principle, where a priori estimates are crucial. This approach allows for deriving both local and global solutions for nonlinear systems.

Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2 (N \in \mathbb{N})$ where $\mathbb{N}$ denotes the set of natural numbers and

$$\mathbb{R}^N = \{x = (x_1, \cdots, x_N) \mid x_i \in \mathbb{R}, \forall i\}.$$

---

[*]Corresponding author

2020 Mathematics Subject Classification: 35Q30, 76N10
Received: 02-03-2025, accepted: 13-06-2025.

The Navier-Stokes-Korteweg (NSK) model captures the capillarity effects in fluids, and its corresponding resolvent equation is given by:

$$\begin{aligned} \lambda\rho + \alpha_1 \mathrm{div}\mathbf{u} &= d \text{ in } \Omega \\ \lambda\mathbf{u} - \alpha_4^{-1}\mathrm{Div}\left(\alpha_2\mathbf{D}(\mathbf{u}) + (\alpha_3-\alpha_2)\mathrm{div}\mathbf{u}\mathbf{I} - \alpha_1\Delta\rho\mathbf{I}\right) &= \mathbf{f} \text{ in } \Omega \end{aligned} \tag{1}$$

where $\rho = \rho(x)$ is a scalar function, $\mathbf{u} = \mathbf{u}(x) = (u_1(x), \ldots, u_N(x))^\top$ is a vector-valued function and the coefficients $\alpha_i = \alpha_i(x)$ $(i = 1, \ldots, 4)$ are real-valued uniformly Lipschitz continuous functions, i.e., there exists a positive constant c such that $|\alpha_i(x) - \alpha_i(y)| \leq c|x-y|$. The doubled stress tensor is denoted by $\mathbf{D}(\mathbf{u})$, i.e.,

$$\mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + (\nabla\mathbf{u})^\mathsf{T}; \text{ for } \quad \nabla\mathbf{u} = \begin{pmatrix} \partial_1 u_1 & \ldots & \partial_N u_1 \\ \vdots & \ddots & \vdots \\ \partial_1 u_N & \ldots & \partial_N u_N \end{pmatrix}, \tag{2}$$

while $\mathbf{I}$ is the $N\times N$ identity matrix. Here, λ is a resolvent parameter in $\Sigma_\sigma = \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| < \pi - \sigma\}$ for $\sigma \in (0, \pi/2)$ where $\mathbb{C}$ represents the set of complex numbers.

We set $\partial_j = \frac{\partial}{\partial x_j}$, then for a scalar function $u = u(x)$ and for a vector function $\mathbf{u} = (u_1(x), \ldots, u_N(x))^\top$ defined in $\mathbb{R}^N$,

$$\begin{aligned} \nabla u &= (\partial_1 u, \partial_2 u, \ldots, \partial_N u)^\top; \\ \nabla^2\mathbf{u} &= (\partial_J\partial_k u_l \mid \{J,k,l = 1, \ldots, N\}); \\ \mathrm{div}\,\mathbf{u} &= \sum_{J=1}^N \partial_J u_J; \\ \Delta u &= \sum_{J=1}^N \partial_J^2 u. \end{aligned} \tag{3}$$

For an $N\times N$ matrix-valued function $\mathbf{M} = (M_{ij}(x))_{1\leq i,j\leq N}$, we set

$$\mathrm{Div}\,\mathbf{M} = \left(\sum_{j=1}^N \partial_j M_{1j}, \ldots, \sum_{j=1}^N \partial_j M_{Nj}\right)^\mathsf{T}. \tag{4}$$

The development of the NSK model has a rich history rooted in the study of capillarity effects and two-phase fluid flows. In 1901, Korteweg formulated constitutive equations for the stress tensor, incorporating the fluid density gradient to model capillarity effects in fluids. Dunn and Serrin later explored the NSK model with Dirichlet boundary conditions within the framework of rational mechanics, introducing the concept of inertia work in thermomechanics [1]. The NSK model captures capillarity effects and two-phase liquid-vapor flows, with a transition phase representing a fluid diffusion interface, as discussed by Anderson et al. [2] and Liu et al.[3].

In 2003, Bresch, Desjardins, and Lin analyzed the weak solutions of the NSK model under specific boundary conditions [4]. Later, Kotschote discussed strong solutions in the exterior domain, introducing Maximal Regularity for the linear

NSK model in the L_p setting combined with the fixed-point theorem to obtain local solutions for the model (1) with Dirichlet boundary conditions [5]. Kotschote also examined non-isothermal cases for Newtonian and non-Newtonian fluids and proved the asymptotic stability of strong solutions to the dynamic NSK system [6].

Regarding $\mathcal{R}$-boundedness, Hirokazu Saito established $\mathcal{R}$-bounded solutions of the NSK with free boundary conditions in half-space and later in arbitrary bounded domains [7, 8]. Subsequently, Suma Inna et al. demonstrated the existence of $\mathcal{R}$-bounded solutions of the resolvent equations for system (1) with slip boundary conditions in half-space for the case where $(\frac{\mu+\nu}{2\kappa})^2 - \frac{1}{\kappa} \neq 0, \kappa \neq \mu\nu$ [9]. In 2023, Suma Inna analyzed the solution operator of the NSK for the coefficients $(\frac{\mu+\nu}{2\kappa})^2 - \frac{1}{\kappa} > 0, \kappa = \mu\nu$ [10], and in 2024, proved $\mathcal{R}$-bounded solutions of system (1) with slip boundary conditions in half-space for arbitrary coefficients ν, μ and κ [11]. Besides, Suma Inna and Saito also analyzed local solutions to the NSK model in 2023 [12].

Generally, to solve equation (1) with a boundary condition in $\Omega \subset \mathbb{R}^N$, the process typically involves: first, solving equation (1) in the whole space $\mathbb{R}^N$; second, solving equation (1) with a boundary condition in the half-space $\mathbb{R}^N_+$; and third, solving equation (1) in a bent-half space (Ω_+). The solution of system (1) in the whole space was achieved by Saito in [13], while Suma Inna solved system (1) with slip boundary conditions in [11].

This paper extends the previous work presented in [11], which addressed solving equation (1) in a bent-half space (Ω_+) with slip boundary conditions. The method to achieve the main result follows Saito's work in [13], which dealt with Dirichlet boundary conditions and proved the existence of an $\mathcal{R}$-bounded solution operator for the spectral parameter $\lambda \in \mathbb{C}_+$, where $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \Re\lambda > 0\}$. However, in this paper, we prove the existence of an $\mathcal{R}$-bounded solution operator with slip boundary conditions for spectral parameter $\lambda \in \Sigma_\sigma$ for $\sigma \in (0, \pi/2)$, which is a significant deviation from the Dirichlet boundary condition case treated by Saito.

Additionally, we provide notations that will be used throughout the paper. For any domain G, the Lebesgue space is represented by $L_q(G)$ while the Sobolev space of order $m, m \in \mathbb{N}$, is denoted by $H^m_q(G)$. When $m = 0, H^0_q(G) = L_q(G)$ and the norm in $H^m_q(G), m \in \mathbb{N}_0$, where $\mathbb{N}_0 = N \cup \{0\}$, is expressed as $\|\cdot\|_{H^m_q(G)}$. Let X and Y be Banach spaces, X^m, $m \in \mathbb{N}$, represents the multiplication of X m times, and the norm in X^m is abbreviated as $\|\cdot\|_X$. The notation $\mathcal{L}(X, Y)$ refers to the set of linear operators from X to Y, while $\mathcal{L}(X)$ denotes the set of linear operators from X to X. For any domain U in $\mathbb{C}$, we denote the set of all X-valued functions $f = f(\lambda)$ defined for $\lambda = \eta + i\tau \in U$, that are continuously differentiable with respect to τ when $\lambda \in U$, by $Hol(U, X)$.

In particular, we introduce specific notations for the function spaces in this paper. Let G be any domain in $\mathbb{R}^N$,

$$\begin{aligned}
&\mathcal{X}_q(G) = H_q^1(G) \times L_q(G)^N \times H_q^2(G) \times H_q^1(G)^N \times H_q^2(G),\\
&\mathfrak{X}_q(G) = L_q(G)^{\mathcal{M}}, \mathcal{M} = \ \big(N+1+N+N^2+N+1+N^2+N+N^2+N+1\big)\\
&\mathbf{H} = (d, \mathbf{f}, g, \mathbf{h}, h) \in \mathcal{X}_q(G),\\
&\mathcal{R}_\lambda \mathbf{H} = (\nabla d, \lambda^{1/2} d, \mathbf{f}, \nabla^2 g, \lambda^{1/2}\nabla g, \lambda g, \nabla \mathbf{h}, \lambda^{1/2}\mathbf{h}, \nabla^2 h, \lambda^{1/2}\nabla h, \lambda h) \in \mathfrak{X}_q(G),\\
&\mathcal{P}_q(G) = L_q(G)^{N^3+N^2+N+1}, \quad \mathcal{S}_\lambda \rho = (\nabla^3\rho, \lambda^{1/2}\nabla^2\rho, \lambda\nabla\rho, \lambda^{3/2}\rho) \in \mathcal{P}_q(G);\\
&\mathcal{Q}_q(G) = L_q(G)^{N^3+N^2+N}, \qquad \mathcal{T}_\lambda \mathbf{u} = (\nabla^2\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u}) \in \mathcal{Q}_q(G).
\end{aligned} \tag{5}$$

Before presenting the main result, this paper introduces the definition of $\mathcal{R}$-boundedness and some related theories.

Definition 1.1. *Let X and Y be two Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $p \in [1,\infty)$ such that for each natural number n, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, and $\{f_j\}_{j=1}^n \subset X$ there holds the inequality:*

$$\|\sum_{j=1}^n r_j(u) T_j f_j\|_{L_p((0,1),Y)} \leq C \|\sum_{j=1}^n r_j(u) f_j\|_{L_p((0,1),X)}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$. Here the Rademacher functions r_k, $k \in \mathbb{N}$, are given by $r_k\colon [0,1] \to \{-1,1\}, t \mapsto \operatorname{sign}(\sin(2^k\pi t))$.

Let us define the following sets:

$$\begin{aligned}
\mathbb{R}_+^N &= \{x = (x', x_N) \in \mathbb{R}^N \mid x' = (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1}, x_N > 0\},\\
\mathbb{R}_0^N &= \{x = (x', x_N) \in \mathbb{R}^N \mid x' = (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1}, x_N = 0\}.
\end{aligned}$$

Let Φ be a diffeomorphism from $\mathbb{R}_x^N$ to $\mathbb{R}_y^N$ of class C^3 (the class of 3-times continuously differentiable functions), where the subscripts x, y denote their variables, and let Φ^{-1} be the inverse map of Φ. We define:

$$\begin{aligned}
\Omega_+ &= \Phi(\mathbb{R}_+^N) = \{y = \Phi(x) \mid x = \Phi^{-1}(y), x \in \mathbb{R}_+^N\},\\
\Gamma_+ &= \Phi(\mathbb{R}_0^N) = \{y = \Phi(x) \mid x = \Phi^{-1}(y), x \in \mathbb{R}_0^N\}.
\end{aligned}$$

Let $\mathbf{n}_+$ be the outward normal vector to Γ_+. Define: $\Phi^{-1}(y) = (\phi_1^{-1}(y), \cdots, \phi_N^{-1}(y))$ Since $x = \Phi^{-1}(y)$, we have $x_N = \phi_N^{-1}(y) = 0$ on Γ_+, so Γ_+ can be written as: $\Gamma_+ = \Phi(\mathbb{R}_0^N) = \{x \in \mathbb{R}^N \mid x_N = \phi_N^{-1}(y) = 0\}$ and Γ_+ is the boundary of Ω_+. If $\alpha = (\alpha_1, \cdots, \alpha_N)$ is an N-tuple of non-negative integers α_j, we call α a multi-index and denote by x^α the monomial $x_1^{\alpha_1} \cdots x_N^{\alpha_N}$, which has degree $|\alpha| = \sum_{j=1}^N \alpha_j$. Similarly, if $\partial_j = \partial/\partial x_j$, then: $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$ denotes a differential operator of order $|\alpha|$. Note that $\partial^{(0,\cdots,0)} u = u$.

Now, we state the main result of this paper. This paper examines the resolvent system of compressible fluid equations of the Korteweg type with slip boundary conditions in a bent half-space Ω_+ presented below.

$$\begin{cases} \lambda\tilde{\rho} + \tilde{\alpha}_1 \mathrm{div}_y \tilde{\mathbf{u}} = \tilde{d} \text{ in } \Omega_+ \\ \lambda\tilde{\mathbf{u}} - \tilde{\alpha}_4^{-1} \mathrm{Div}\,(\tilde{\alpha}_2 \mathbf{D}(\tilde{\mathbf{u}}) + (\tilde{\alpha}_3 - \tilde{\alpha}_2)\mathrm{div}\tilde{\mathbf{u}}\mathbf{I} - \tilde{\alpha}_1 \Delta\tilde{\rho}\mathbf{I}) = \tilde{\mathbf{f}} \text{ in } \Omega_+ \\ \mathbf{n}_+ \cdot \nabla\tilde{\rho} = \tilde{g} \text{ on } \Gamma_+ \\ \mathbf{D}(\tilde{\mathbf{u}})\mathbf{n}_+ - \langle \mathbf{D}(\tilde{\mathbf{u}})\mathbf{n}_+, \mathbf{n}_+\rangle \mathbf{n}_+ = \tilde{\mathbf{h}} - \langle \tilde{\mathbf{h}}, \mathbf{n}_+\rangle \mathbf{n}_+ \text{ on } \Gamma_+ \\ \tilde{\mathbf{u}} \cdot \mathbf{n}_+ = \tilde{h} \text{ on } \Gamma_+. \end{cases} \tag{6}$$

To prove the existence of $\mathcal{R}$-bounded solution for System (6), we will demonstrate the following theorem.

Theorem 1.2. *Let $q \in (1, \infty)$, and let M_1 and M_2 be constants as in Equation (8). Assume there exist positive constants B_1 and B_2 such that $B_1 \le \tilde{\alpha}_i(y) \le B_2$ for every $y \in \Omega_+$ and $i = 1, 2, 3, 4$. Then, there exist positive constants $\delta \in (0, 1)$ and $M_1 \in (0, \frac{1}{2})$ that depend on N, q, B_1, and B_2 such that for any positive number $\epsilon > 0$ and for real-valued continuous Lipschitz functions $\tilde{\alpha}_i = \tilde{\alpha}_i(y)$ $(i = 1, 2, 3, 4)$ defined on Ω_+ and satisfying:*

(a) $\sup_{y\in\Omega_+} |\tilde{\alpha}_i(y) - \tilde{\alpha}_i^0| \le \delta$ *with positive constants* $\tilde{\alpha}_i^0 \in [B_1, B_2]$,

(b) $\|\nabla\tilde{\alpha}_j\|_{L_\infty(\Omega_+)} \le \epsilon$ *for* $j = 1, 2, 3, 4$,

there exists a constant $\eta \ge 1$, which depends on M_2, N, δ, q, B_1, and B_2, such that the following statements hold:

(1) *For any $\lambda \in \Sigma_\sigma$, there exist operators $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ with*

$$\mathcal{A}(\lambda) \in \mathrm{Hol}\big(\Sigma_\sigma,\ \mathcal{L}(\mathfrak{X}_q(\Omega_+), H_q^3(\Omega_+))\big),$$
$$\mathcal{B}(\lambda) \in \mathrm{Hol}\big(\Sigma_\sigma,\ \mathcal{L}(\mathfrak{X}_q(\Omega_+), H_q^2(\Omega_+)^N)\big),$$

such that for any $\tilde{\mathbf{H}} = (\tilde{d}, \tilde{f}, \tilde{g}, \tilde{\mathbf{h}}, \tilde{h}) \in \mathcal{X}_q(\Omega_+)$,

$$(\tilde{\rho}, \tilde{\mathbf{u}}) = \big(\mathcal{A}(\lambda)\, R_\lambda \tilde{\mathbf{H}},\ \mathcal{B}(\lambda)\, R_\lambda \tilde{\mathbf{H}}\big)$$

is a unique solution of System (6).

(2) *There exists a positive constant $C_{M_2,\eta}$, which depends on M_2, N, q, B_1, and B_2, such that for $n = 0, 1$,*

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q(\Omega_+), \mathcal{P}_q(\Omega_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{S}_\lambda \mathcal{A}(\lambda)) \middle| \lambda \in \Sigma_\sigma\right\}\right) \le C_{M_2,\eta},$$

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q(\Omega_+), \mathcal{Q}_q(\Omega_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda \mathcal{B}(\lambda)) \middle| \lambda \in \Sigma_\sigma\right\}\right) \le C_{M_2,\eta},$$

where $\mathcal{X}_q(\Omega_+), \mathfrak{X}_q(\Omega_+), \mathcal{P}_q(\Omega_+), \mathcal{Q}_q(\Omega_+), \mathcal{S}_\lambda$ and $\mathcal{T}_\lambda$ are defined in (5) with $G = \Omega_+$.

2. PRELIMINARIES

To solve System (6), we reduce the system in the bent half-space Ω_+ to a system in the half-space $\mathbb{R}^N_+$ by performing a diffeomorphism transformation on the domain Ω_+. Therefore, before proceeding to the proof of Theorem 1.2, we recall some results from [11] and[14].

Let $\mathbf{F}^0 = (d, \mathbf{f}, g, \mathbf{h}', h_N)$ with $\mathbf{h}' = (h_1, \ldots, h_{N-1})^\top$, we define the function space

$$\mathcal{X}^0_q(\mathbb{R}^N_+) = H^1_q(\mathbb{R}^N_+) \times L_q(\mathbb{R}^N_+)^N \times H^2_q(\mathbb{R}^N_+) \times H^1_q(\mathbb{R}^N_+)^{N-1} \times H^2_q(\mathbb{R}^N_+).$$

Then, we define $\mathcal{R}_\lambda \mathbf{F}^0$ and $\mathcal{X}^0_q(\mathbb{R}^N_+)$ as follows:

$$\mathfrak{X}^0_q(\mathbb{R}^N_+) = L_q(\mathbb{R}^N_+)^{\mathcal{N}}$$

with

$$\mathcal{N} = (N+1) + N + (N^2+N+1) + (N-1)(N+1) + (N^2+N+1),$$

and

$$\mathcal{R}_\lambda \mathbf{F}^0 = (\nabla d, \lambda^{1/2} d, f, \nabla^2 g, \lambda^{1/2}\nabla g, \lambda g,\\ \nabla \mathbf{h}', \lambda^{1/2}\mathbf{h}', \nabla^2 h_N, \lambda^{1/2}\nabla h_N, \lambda h_N) \in \mathfrak{X}^0_q(\mathbb{R}^N_+).$$

The system in the half-space is given by:

$$\begin{cases} \lambda\rho + \vartheta \mathrm{div}\, \mathbf{u} = d \text{ in } \mathbb{R}^N_+ \\ \lambda \mathbf{u} - \mu\Delta u - \nu\nabla \mathrm{div}\, \mathbf{u} - \kappa\Delta\nabla\rho = \mathbf{f} \text{ in } \mathbb{R}^N_+ \\ n\cdot\nabla\rho = g \text{ on } \mathbb{R}^N_0 \\ \partial_N v_s + \partial_s v_N = h_s \text{ on } \mathbb{R}^N_0, \quad s = 1, \ldots, N-1 \\ u_N = h \text{ on } \mathbb{R}^N_0 \end{cases} \tag{7}$$

Then, we obtain the following theorem discussed in [11].

Theorem 2.1 ([11]). *Let $q \in (1,\infty)$ and suppose that $\vartheta.\mu, \nu$, and κ are arbitrary positive constants. Then, for each $\lambda \in \Sigma_\sigma$ for $\sigma \in (0, \pi/2)$, there exist operators $\mathcal{A}^0(\lambda)$ and $\mathcal{B}^0(\lambda)$ with*

$$\mathcal{A}^0(\lambda) \in Hol(\Sigma_\sigma, \mathcal{L}(\mathfrak{X}^0_q(\mathbb{R}^N_+), H^3_q(\mathbb{R}^N_+))),$$

$$\mathcal{B}^0(\lambda) \in Hol(\Sigma_\sigma, \mathcal{L}(\mathfrak{X}^0_q(\mathbb{R}^N_+), H^2_q(\mathbb{R}^N_+)^N)),$$

such that for $\mathbf{F}^0 = (d, \mathbf{f}, g, \mathbf{h}', h_N) \in \mathcal{X}^0_q(\mathbb{R}^N_+)$, $(\rho, \mathbf{u}) = (A^0(\lambda)\mathcal{R}_\lambda\mathbf{F}^0, B^0(\lambda)\mathcal{R}_\lambda\mathbf{F}^0)$ is the solution operator of equation (7) satisfying the estimate

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}^0_q(\mathbb{R}^N_+), \mathcal{P}_q(\mathbb{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n \mathcal{S}_\lambda A^0(\lambda) \middle| \lambda \in \Sigma_\sigma\right\}\right) \le C,$$

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}^0_q(\mathbb{R}^N_+), \mathcal{Q}_q(\mathbb{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n \mathcal{T}_\lambda B^0(\lambda) \middle| \lambda \in \Sigma_\sigma\right\}\right) \le C$$

for $n = 0, 1$, where C is a positive constant depending at most on $N, q, \vartheta, \mu, \nu$, and κ. Here, $\mathcal{S}_\lambda$, $\mathcal{T}_\lambda$, $\mathcal{P}_q(\mathbb{R}^N_+)$, and $\mathcal{Q}_q(\mathbb{R}^N_+)$ are given by (5) with $G = \mathbb{R}^N_+$.

Moreover, we introduce a crucial theorem regarding the Neumann series expansion theorem proved in [14].

Theorem 2.2. *(Neumann series expansion [14]). Let X be a Banach space and $\mathcal{L}(X)$ be the set of linear operators from X to X. Let $A \in \mathcal{L}(X)$ with $\|A\|_{\mathcal{L}(X)} \leq 1$. Then the following statements are satisfied:*

(a) *The infinite series $\sum_{j=0}^{\infty} A_j$ is also in $\mathcal{L}(X)$. The series $\sum_{j=0}^{\infty} A_j$ is called the Neumann series,*

(b) *The operator $(I - A) \in \mathcal{L}(X)$ is bijective and $(I - A)^{-1} = \sum_{j=0}^{\infty} A_j$,*

(c) $\|(I - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{1 - \|A\|_{\mathcal{L}(X)}}$,

(d) *Let $g \in X$, then the equation $(I - A)u = g$, with the unknown $u \in X$, has a unique solution*

$$u = (I - A)^{-1} g = \sum_{j=0}^{\infty} A_j g.$$

3. MAIN RESULTS

3.1. Reducing the System.

This section discusses reducing the system in a bent half-space (6) to a system in a half-space. Let $\Phi(x) = x + \psi(x)$, where $\psi(x)$ is a function such that $\|\psi(x)\|_{L_\infty(\mathbb{R}^N)} \ll 1$. Then $\nabla\Phi(x) = \nabla x + \nabla\psi(x) =: \mathbf{A} + \mathbf{B}(x)$ and $\nabla\Phi^{-1}(y) =: \mathbf{A}_{-1} + \mathbf{B}_{-1}(y)$. Therefore, $\mathbf{A}$ and $\mathbf{A}_{-1}$ can be assumed to be orthonormal matrices, and $\mathbf{B}(x)$, $\mathbf{B}_{-1}(y)$ are matrix-valued functions in $H^2_\infty(\mathbb{R}^N)$ that satisfy:

$$\|(\mathbf{B}, \mathbf{B}_{-1})\|_{L_\infty(\mathbb{R}^N)} \leq M_1, \quad \|\nabla(\mathbf{B}, \mathbf{B}_{-1})\|_{H^1_\infty(\mathbb{R}^N)} \leq M_2. \tag{8}$$

We will eventually select M_1 to be sufficiently small, allowing us to assume $0 \leq M_1 \leq \frac{1}{2}$, and we assume $M_2 > 1$. Let the matrices $\mathbf{A}_{-1} = (a_{ij})$ and $\mathbf{B}_{-1}(y) = (b_{ij}(y))$. Then, the outward normal vector to Γ_+ can be written as:

$$\begin{aligned}
\mathbf{n}_+ = \mathbf{n}_+(y) &= -\frac{\nabla_y \phi_N^{-1}(y)}{|\nabla_y \phi_N^{-1}(y)|} = -\frac{\left(\frac{\partial\phi_N^{-1}(y)}{\partial y_1}, \cdots, \frac{\partial\phi_N^{-1}(y)}{\partial y_N}\right)^\top}{\sqrt{\sum_{i=1}^N \left(\frac{\partial\phi_N^{-1}(y)}{\partial y_i}\right)^2}} \\
&= -\frac{[a_{N1} + b_{N1}(y), \cdots, a_{NN} + b_{NN}(y)]^\top}{\sqrt{\sum_{i=1}^N (a_{Ni} + b_{Ni}(y))^2}} = -\frac{(\mathbf{A}_{-1} + \mathbf{B}_{-1}(y))^\top \mathbf{n}}{|(\mathbf{A}_{-1} + \mathbf{B}_{-1}(y))^\top \mathbf{n}|},
\end{aligned} \tag{9}$$

with $\mathbf{n} = (0, \cdots, -1)^\top$. Clearly, $\mathbf{n}_+$ is defined in $\mathbb{R}^N$. Moreover, the formula (9) implies that $|\mathbf{n}_+(y)| = 1$ for $y \in \mathbb{R}^N$ and by (8), we have:

$$|(\mathbf{A}_{-1} + \mathbf{B}_{-1}(y))^\top \mathbf{n}| \geq |A_{-1}^\top \mathbf{n}| - |\mathbf{B}_{-1}(y)^\top \mathbf{n}| \geq 1 - M_1 \geq \frac{1}{2}.$$

Let $D_j = \partial/\partial y_j$ and $\partial_j = \partial/\partial x_j$ for $j = 1, \dots, N$. By using the notations given by (2), (3) and (4) ,we have

$$\mathrm{Div}_y \mathbf{D}_y(\widetilde{\mathbf{u}}) \quad = \quad \Delta_y \widetilde{\mathbf{u}} + \nabla_y \mathrm{div}_y \widetilde{\mathbf{u}}.$$

Therefore

$$\begin{aligned}
\mathrm{Div}_y(\widetilde{\alpha}_2 \mathbf{D}_y(\widetilde{\mathbf{u}})) &= (\mathrm{Div}_y \alpha_2)\mathbf{D}_y(\widetilde{\mathbf{u}}) + \widetilde{\alpha}_2(\mathrm{Div}_y \mathbf{D}_y(\widetilde{\mathbf{u}})) \\
&= \nabla_y \widetilde{\alpha}_2 \mathbf{D}_y(\widetilde{\mathbf{u}}) + \widetilde{\alpha}_2(\Delta_y \widetilde{\mathbf{u}} + \nabla_y \mathrm{div}_y \widetilde{\mathbf{u}}) \\
&= \mathbf{D}_y(\widetilde{\mathbf{u}}) \nabla_y \widetilde{\alpha}_2 + \widetilde{\alpha}_2 \Delta_y \widetilde{\mathbf{u}} + \widetilde{\alpha}_2 \nabla_y \mathrm{div}_y \widetilde{\mathbf{u}}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathrm{Div}_y((\widetilde{\gamma}_3 - \widetilde{\gamma}_2)\mathrm{div}_y \widetilde{\mathbf{u}}\mathbf{I}) \quad = \quad & (\mathrm{Div}_y(\widetilde{\gamma}_3 - \widetilde{\gamma}_2)\mathbf{I}\mathrm{div}_y \widetilde{\mathbf{u}} + (\widetilde{\gamma}_3 - \widetilde{\gamma}_2) \\
& (\mathrm{Div}_y \mathrm{div}_y \widetilde{\mathbf{u}}\mathbf{I})
\end{aligned}$$

with

$$\begin{aligned}
\mathrm{Div}_y((\widetilde{\gamma}_3 - \widetilde{\gamma}_2)\mathbf{I}) \,\mathrm{div}\, \tilde{\mathbf{u}} &= \nabla_y(\widetilde{\gamma}_3 - \widetilde{\gamma}_2) \,\mathrm{div}\, \tilde{\mathbf{u}}, \\
\mathrm{Div}_y(\mathrm{div}_y \widetilde{\mathbf{c}}\mathbf{I}) &= \nabla_y \ \mathrm{div}_y \widetilde{\mathbf{c}}.
\end{aligned}$$

By using above equations, System (6) can be written as:

$$\begin{cases}
\lambda\tilde{\rho} + \alpha_1^0 \mathrm{div}_y(\tilde{\mathbf{u}}) + \mathcal{H}_1(\tilde{\mathbf{u}}) = \tilde{d} \text{ in } \Omega_+ \\
\lambda\tilde{\mathbf{u}} - (\alpha_1^0)^{-1}(\alpha_2^0 \Delta_y(\tilde{\mathbf{u}}) + \alpha_3^0 \nabla_y \mathrm{div}_y(\tilde{\mathbf{u}}) \\
\quad + \alpha_1^0 \nabla_y \Delta_y \tilde{\rho}) - \mathcal{H}_2(\tilde{\mathbf{u}}) - \mathcal{H}_3(\tilde{\rho}) = \tilde{\mathbf{f}} \text{ in } \Omega_+ \\
\mathbf{n}_+ \cdot \nabla_y \tilde{\rho} = \tilde{g} \text{ on } \Gamma_+ \\
\mathbf{D}(\tilde{\mathbf{u}})\mathbf{n}_+ - \langle \mathbf{D}(\tilde{\mathbf{u}})\mathbf{n}_+, \mathbf{n}_+ \rangle \mathbf{n}_+ = \tilde{\mathbf{h}} - \langle \tilde{\mathbf{h}}, \mathbf{n}_+ \rangle \mathbf{n}_+ \text{ on } \Gamma_+ \\
\tilde{\mathbf{u}} \cdot \mathbf{n}_+ = \tilde{h} \text{ on } \Gamma_+
\end{cases} \tag{10}$$

with

$$\begin{aligned}
\mathcal{H}_1(\tilde{\mathbf{u}}) &= (\tilde{\alpha}_1 - \alpha_1^0)\mathrm{div}_y(\tilde{\mathbf{u}}) \\
\mathcal{H}_2(\tilde{\mathbf{u}}) &= \left(\frac{\tilde{\alpha}_2}{\tilde{\alpha}_4} - \frac{\alpha_2^0}{\alpha_4^0} \right) \Delta_y \tilde{\mathbf{u}} + \left(\frac{\tilde{\alpha}_3}{\tilde{\alpha}_4} - \frac{\alpha_3^0}{\alpha_4^0} \right) \nabla_y \mathrm{div}_y(\tilde{\mathbf{u}}) \\
&\quad + \tilde{\alpha}_4^{-1} \left\{ \mathbf{D}_y(\tilde{\mathbf{u}}) \nabla_y \tilde{\alpha}_2 + (\mathrm{div}_y(\tilde{\mathbf{u}})) \nabla_y(\tilde{\alpha}_3 - \tilde{\alpha}_2) \right\} \\
\mathcal{H}_3(\tilde{\rho}) &= \left(\frac{\tilde{\alpha}_1}{\tilde{\alpha}_4} - \frac{\alpha_1^0}{\alpha_4^0} \right) \nabla_y \Delta_y \tilde{\rho} + (\Delta_y \tilde{\rho}) \nabla_y \tilde{\alpha}_1
\end{aligned} \tag{11}$$

Additionally, by changing variables $y = \Phi(x)$, we have the following fundamental properties:

$$D_j = \sum_{l=1}^{N} (a_{lj} + b_{lj}(x))\partial_l, \quad \nabla_y = (\mathbf{A}_{-1} + \mathbf{B}_{-1}(x))^\top \nabla_x,$$

$$D_j D_k = \sum_{l,m=1}^{N} a_{lj} a_{mk} \partial_l \partial_m + \sum_{l,m=1}^{N} (a_{lj} b_{mk}(x) + a_{mk} b_{lj}(x) + b_{lj}(x) b_{mk}(x)) \partial_l \partial_m$$

$$+\sum_{l,m=1}^{N}(a_{lj}+b_{lj}(x))(\partial_l b_{mk})(x)\partial_m$$

For $k, l, m = 1, \dots, N$, let us define:

$$C^1_{klm}(x) = (a_{lk} + b_{lk}(x))(\partial_l b_{mk})(x),$$
$$C^2_{klm}(x) = a_{lk}b_{mk}(x) + a_{mk}b_{lk}(x) + b_{lk}(x)b_{mk}(x)$$

Then we obtain:

$$\Delta_y = \Delta_x + \sum_{k,l,m=1}^{N}(C^1_{klm}(x)\partial_m + C^2_{klm}(x)\partial_m).$$

Let $\mathbf{u}(x) = \tilde{\mathbf{u}}(\Phi(x))$, then we get:

$$\begin{aligned} \mathrm{div}_y(\tilde{\mathbf{u}}) &= \mathrm{div}_x(\mathbf{A}_{-1}u) + \mathbf{B}_{-1}(x) : \nabla_x u \\ \nabla_y \mathrm{div}_y(\tilde{\mathbf{u}}) &= (\mathbf{A}_{-1} + \mathbf{B}_{-1}(x))^\top \nabla_x(\mathrm{div}_x(\mathbf{A}_{-1}u) + \mathbf{B}_{-1}(x) : \nabla_x u) \\ \mathbf{D}_y(\tilde{\mathbf{u}}) &= (\nabla_x \tilde{\mathbf{u}})(\mathbf{A}_{-1} + \mathbf{B}_{-1}(x)) + (\mathbf{A}_{-1} + \mathbf{B}_{-1}(x))^\top (\nabla_x u)^\top \end{aligned} \tag{12}$$

with $\mathbf{S} : \mathbf{T} = \sum_{i,j=1}^{N} S_{ij}T_{ij}$ for matrices $\mathbf{S} = (S_{ij}), \mathbf{T} = (T_{ij})$ and $\mathbf{A}^\top$ is the transpose of matrix $\mathbf{A}$. Let us define

$$\begin{aligned} \rho = \tilde{\rho}(\Phi(x)), \mathbf{v} = \mathbf{A}_{-1}\mathbf{u}(x) = \mathbf{A}_{-1}\tilde{\mathbf{u}}(\Phi(x)) \\ d = \tilde{d}(\Phi(x)), \mathbf{f} = \mathbf{A}_{-1}\tilde{\mathbf{f}}(\Phi(x)), g = \tilde{g}(\Phi(x)), \mathbf{h} = \tilde{\mathbf{h}}(\Phi(x)), h = \tilde{h}(\Phi(x)) \end{aligned} \tag{13}$$

Using the fact that $\mathbf{A}^\top_{-1} = (\mathbf{A}_{-1})^{-1}$ and the equations in (12), we obtain:

$$D_{ij}(\tilde{\mathbf{u}}) = \sum_{k,l=1}^{N} a_{ki}a_{lj}D_{kl}(v) + C_{ij} : \nabla \mathbf{v}$$

with $C_{ij} : \nabla \mathbf{v} = \sum_{k,l=1}^{N} a_{kj}b_{li}D_{kl}(v)$.

Let $\mathbf{n}_+ = -\mathbf{A}_N + \mathbf{B}_+(x) = -(A_{N1}, \dots, A_{NN})^\top + (B_{+1}, \dots, B_{+N})^\top$ with

$$A_{Ni} = \frac{a_{Ni}}{\sqrt{\sum_{i=1}^{N}(a_{Ni} + b_{Ni}(y))^2}}, B_{+i} = \frac{b_{Ni}}{\sqrt{\sum_{i=1}^{N}(a_{Ni} + b_{Ni}(y))^2}}, i = 1, \dots, N.$$

Then using the above equations and the third equation in (12), we obtain:

$$\begin{aligned} \mathbf{D}(\tilde{\mathbf{u}})\mathbf{n}_+ = \mathbf{D}(\mathbf{v})\mathbf{n} + R_1 : \nabla \mathbf{v} \\ \langle \mathbf{D}(\tilde{\mathbf{u}})\mathbf{n}_+, \mathbf{n}_+ \rangle \mathbf{n}_+ = \langle \mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n} \rangle \mathbf{n} + R_2 : \nabla \mathbf{v} \end{aligned} \tag{14}$$

with

$$R_m : \nabla \mathbf{v} = (R_m : \nabla \mathbf{v}|_1, \dots, R_m : \nabla \mathbf{v}|_N), \quad (m = 1, 2)$$

where

$$R_1 : \nabla \mathbf{v}|_s = \sum_{j=1}^{N} A_{Nj}C_{sj} : \nabla \mathbf{v} + \sum_{j=1}^{N}(A_{Nj} + B_{+j})C_{sj} : \nabla \mathbf{v}$$

and

$$
\begin{aligned}
R_2 : \nabla \mathbf{v}|_s =& 2\sum_{i,j,k,l}^{N} A_{Ns} a_{ji} B_{+j} D_{jN} - \sum_{i,j,k,l=1}^{N} A_{Ns} a_{ki} a_{lj} B_{+i} B_{+j} D_{kl}(\mathbf{v}) \\
&- \sum_{i,j=1}^{N} C_{ij} : \nabla \mathbf{v} A_{Ns}(A_{Nj} + B_{+i})(A_{Nj} + B_{+j}).
\end{aligned}
$$

with $s = 1, \dots N-1$. Let $h_s = \sum_{j=1}^{N} a_{sj}(h_j - \sum_{l=1}^{N} \langle h_l, (A_{Nj} + B_{+j}) \rangle (A_{Nj} + B_{+j})), (s = 1, \dots N-1)$. By using (12), (13) and (14), the system (10) is transformed into the following system in half-space:

$$
\begin{cases}
\lambda \rho + a_1^0(\mathrm{div}\mathbf{v} + \mathcal{K}_1(\mathbf{v})) + \mathcal{H}_1(\mathbf{A}_{-1}^\top \mathbf{v}) = d & \text{in } \mathbb{R}_+^N \\
\lambda \mathbf{v} - (a_4^0)^{-1}(a_2^0 \Delta \mathbf{v} + a_3^0 \nabla \mathrm{div}\mathbf{v} + a_1^0 \nabla \Delta \rho + \mathcal{K}_2(\mathbf{v}) + \mathcal{K}_3(\rho)) & \\
\qquad - \mathbf{A}_{-1}\mathcal{H}_2(\mathbf{A}_{-1}^\top v) - \mathbf{A}_{-1}\mathcal{H}_3(\rho) = f & \text{in } \mathbb{R}_+^N \\
\mathbf{n} \cdot \nabla \rho - \mathcal{K}_4(\rho) = g & \text{on } \mathbb{R}_0^N \\
\partial_N v_s + \partial_s v_N + \mathcal{K}_5(\mathbf{v}) = h_s & \text{on } \mathbb{R}_0^N \\
v_N + \mathcal{K}_6(\mathbf{v}) = h & \text{on } \mathbb{R}_0^N
\end{cases}
\tag{15}
$$

with $s = 1, \dots, N-1$ and

$$
\begin{aligned}
\mathcal{K}_1(\mathbf{v}) =& \mathbf{B}_{-1}(x) : (\mathbf{A}_{-1}^\top \nabla \mathbf{v}) \\
\mathcal{K}_2(\mathbf{v}) =& \alpha_2^0 \sum_{k,l,m=1}^{N} (C_{klm}^1(x)\partial_m + C_{klm}^2(x)\partial_l\partial_m)\mathbf{v} \\
&+ \alpha_3^0 \left\{ (\mathbf{I} + \mathbf{A}_{-1}\mathbf{B}_{-1}(x)^\top) \nabla(\mathrm{div}\mathbf{v} + \mathbf{B}_{-1}(x) : (\mathbf{A}_{-1}^\top \nabla \mathbf{v})) - \nabla \mathrm{div}\mathbf{v} \right\} \\
\mathcal{K}_3(\rho) =& \alpha_1^0 \left[(I + \mathbf{A}_{-1}\mathbf{B}_{-1}(x)^\top) \nabla \left\{ \Delta + \sum_{k,l,m=1}^{N} (C_{klm}^1(x)\partial_m + C_{klm}^2(x)\partial_l\partial_m) \right\} \rho \right. \\
&\left. - \Delta \nabla \rho \right] \\
\mathcal{K}_4(\rho) =& \frac{\sum_{j=1}^{N} b_{Nj}(x)(2a_{Nj} + b_{Nj}(x))}{|(\mathbf{A}_{-1}\mathbf{B}_{-1}(x))^\top \mathbf{n}|(|(\mathbf{A}_{-1}\mathbf{B}_{-1}(x))^\top \mathbf{n}| + 1)} \mathbf{n} \cdot \nabla \rho \\
&- \frac{(\mathbf{A}_{-1}\mathbf{B}_{-1}(x))^\top + \mathbf{B}_{-1}(x)\mathbf{A}_{-1}^\top + \mathbf{B}_{-1}\mathbf{B}_{-1}(x)^\top)\mathbf{n}}{|(\mathbf{A}_{-1} + \mathbf{B}_{-1}(x))^\top \mathbf{n}|} \cdot \nabla \rho \\
\mathcal{K}_5(\mathbf{v}) =& R_1 : \nabla \mathbf{v}|_s - R_2 : \nabla \mathbf{v}|_s \\
\mathcal{K}_6(\mathbf{v}) =& -\mathbf{v} \cdot (\mathbf{A}_{-1}\mathbf{B}_{+}(x)).
\end{aligned}
\tag{16}
$$

Then the System (15) is reduced to

$$
\begin{cases}
\lambda\rho + \alpha_1^0 \mathrm{div}\mathbf{v} - \mathcal{W}_1(\rho,\mathbf{v}) = d & \text{in } \mathbb{R}^N_+ \\
\lambda\mathbf{v} - (\alpha_4^0)^{-1}(\alpha_2^0\Delta\mathbf{v} + \alpha_3^0\nabla\mathrm{div}\mathbf{v} + \alpha_1^0\nabla\Delta\rho) - \mathcal{W}_2(\rho,\mathbf{v}) = \mathbf{f} & \text{in } \mathbb{R}^N_+ \\
\mathbf{n}\cdot\nabla\rho - \mathcal{W}_3(\rho,\mathbf{v}) = g & \text{on } \mathbb{R}^N_0 \\
\partial_N v_s + \partial_s v_N - \mathcal{W}_4(\rho,\mathbf{v}) = h_s & \text{on } \mathbb{R}^N_0 \\
v_N - \mathcal{W}_5(\rho,\mathbf{v}) = h & \text{on } \mathbb{R}^N_0
\end{cases} \tag{17}
$$

with $(s = 1, \cdots, N-1)$ and

$$
\begin{aligned}
&\mathcal{W}_1(\rho,\mathbf{v}) = -\alpha_1^0\mathcal{K}_1(\mathbf{v}) - \mathcal{H}_1(\mathbf{A}_{-1}^\top v),\\
&\mathcal{W}_2(\rho,\mathbf{v}) = (\alpha_4^0)^{-1}(\mathcal{K}_2(\mathbf{v}) + \mathcal{K}_3(\rho)) + \mathbf{A}_{-1}\mathcal{H}_2(\mathbf{A}_{-1}^\top v) + \mathbf{A}_{-1}\mathcal{H}_3(\rho),\\
&\mathcal{W}_3(\rho,\mathbf{v}) = \mathcal{K}_4(\rho),\\
&\mathcal{W}_4(\rho,\mathbf{v}) = \mathcal{K}_5(\rho),\\
&\mathcal{W}_5(\rho,\mathbf{v}) = \mathcal{K}_6(\mathbf{v}).
\end{aligned}
$$

Additionally, to demonstrate Theorem 1.2, we employ the solution for system (17) outlined above. This involves proving the existence and uniqueness of the $\mathcal{R}$-bounded solution operator for system (17). Thus, in the following subsection, we will establish the existence of the $\mathcal{R}$-bounded solution operator for system (17).

3.2. **Existence of $\mathcal{R}$-bounded Solutions Operator for System (17).**

In this subsection, we prove the existence of a unique solution to the System (17) and establish the $\mathcal{R}$-boundedness of the system. To this end, we start with the following lemma.

Lemma 3.1. *Let $q \in (1,\infty)$, and let M_1 and M_2 be constants appearing in equation* (8)*. Assume conditions (a) and (b) in Theorem 1.2 are satisfied for positive constants δ and ϵ such that $0 < \delta < \min(1, B_1/2)$. Then there exist positive constants α, β_{M_2} and $\alpha_{M_2,\epsilon}$ that do not depend on a_i^0 $(i = 1,2,3,4)$ such that for $\mathbf{v} \in H_q^2(\mathbb{R}^N_+)^N$ and $\rho \in H_q^3(\mathbb{R}^N_+)$, the following inequalities hold:*

$$
\begin{aligned}
\|\nabla(a_1^0\mathcal{K}_1(\mathbf{v}))\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla^2\mathbf{v}\|_{L_q(\mathbb{R}^N_+)} + \beta_{M_2}\|\nabla\mathbf{v}\|_{L_q(\mathbb{R}^N_+)},\\
\|a_1^0\mathcal{K}_1(\mathbf{v})\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla\mathbf{v}\|_{L_q(\mathbb{R}^N_+)},\\
\|(a_4^0)^{-1}\mathcal{K}_2(\mathbf{v})\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla^2\mathbf{v}\|_{L_q(\mathbb{R}^N_+)} + \beta_{M_2}\|\nabla\mathbf{v}\|_{L_q(\mathbb{R}^N_+)},\\
\|(a_4^0)^{-1}\mathcal{K}_3(\rho)\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla^3\rho\|_{L_q(\mathbb{R}^N_+)} + \beta_{M_2}\|(\nabla^2\rho,\nabla\rho)\|_{L_q(\mathbb{R}^N_+)},
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla^2\mathcal{K}_4(\rho)\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla^3\rho\|_{L_q(\mathbb{R}^N_+)} + \beta_{M_2}\|(\nabla^2\rho,\nabla\rho)\|_{L_q(\mathbb{R}^N_+)},\\
\|\nabla\mathcal{K}_4(\rho)\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla^2\rho\|_{L_q(\mathbb{R}^N_+)} + \beta_{M_2}\|\nabla\rho\|_{L_q(\mathbb{R}^N_+)},\\
\|\mathcal{K}_4(\rho)\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla\rho\|_{L_q(\mathbb{R}^N_+)},\\
\|\nabla\mathcal{K}_5(\mathbf{v})\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla^2\mathbf{v}\|_{L_q(\mathbb{R}^N_+)} + \beta_{M_2}\|(\nabla\mathbf{v})\|_{L_q(\mathbb{R}^N_+)},\\
\|\mathcal{K}_5(\mathbf{v})\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla\mathbf{v}\|_{L_q(\mathbb{R}^N_+)},\\
\|\nabla^2\mathcal{K}_6(\mathbf{v})\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla^2\mathbf{v}\|_{L_q(\mathbb{R}^N_+)} + \beta_{M_2}\|(\nabla\mathbf{v},\mathbf{v})\|_{L_q(\mathbb{R}^N_+)},\\
\|\nabla\mathcal{K}_6(\mathbf{v})\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla\mathbf{v}\|_{L_q(\mathbb{R}^N_+)} + \beta_{M_2}\|\mathbf{v}\|
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\|\nabla\mathcal{H}_1(\mathbf{A}_{-1}^\top\mathbf{v})\|_{L_q(\mathbb{R}^N_+)} &\le \alpha\delta\|\nabla^2\mathbf{v}\|_{L_q(\mathbb{R}^N_+)} + \alpha_{M_2,\epsilon}\|\nabla\mathbf{v}\|_{L_q(\mathbb{R}^N_+)},\\
\|\mathcal{H}_1(\mathbf{A}_{-1}^\top\mathbf{v})\|_{L_q(\mathbb{R}^N_+)} &\le \alpha\delta\|\nabla\mathbf{v}\|_{L_q(\mathbb{R}^N_+)},\\
\|\mathbf{A}_{-1}\mathcal{H}_2(\mathbf{A}_{-1}^\top\mathbf{v})\|_{L_q(\mathbb{R}^N_+)} &\le \alpha\delta\|\nabla^2\mathbf{v}\|_{L_q(\mathbb{R}^N_+)} + \alpha_{M_2,\epsilon}\|\nabla\mathbf{v}\|_{L_q(\mathbb{R}^N_+)},\\
\|\mathbf{A}_{-1}\mathcal{H}_2(a_1^0\rho)\|_{L_q(\mathbb{R}^N_+)} &\le \alpha\delta\|\nabla^3\rho\|_{L_q(\mathbb{R}^N_+)} + \alpha_{M_2,\epsilon}\|(\nabla^2\rho,\nabla\rho)\|_{L_q(\mathbb{R}^N_+)},\\
\|(a_4^0)^{-1}\mathcal{H}_3(\rho)\|_{L_q(\mathbb{R}^N_+)} &\le \alpha M_1\|\nabla^3\rho\|_{L_q(\mathbb{R}^N_+)} + \alpha_{M_2,\epsilon}\|(\nabla^2\rho,\nabla\rho)\|_{L_q(\mathbb{R}^N_+)}.
\end{aligned}
\tag{19}
$$

More precisely, α *depends on* N, q, B_1, B_2 *but do not depend on* M_1, M_2, δ *and* ϵ*;* β_{M_2} *depends on* N, M_2, q, B_1 *and* B_2 *but do not depend on* M_1, δ *and* ϵ*; and* $\alpha_{M_2,\epsilon}$ *depends on* $M-2, \delta, N, q, B_1$ *and* B_2 *but do not depend on* M_1, M_2, δ *and* ϵ*.*

Proof. First, we prove (18), especially $\mathcal{K}_2(\mathbf{v})$. Recall the formula $\mathcal{K}_2(\mathbf{v})$ in (16). Then by (8), we obtain

$$
\|C^1_{klm}\|_{H^1_\infty(\mathbb{R}^N_+)} + \|\nabla C^2_{klm}\|_{L_\infty(\mathbb{R}^N_+)} \le C_{N,M_2},
$$
$$
\|C^2_{klm}\|_{L_\infty(\mathbb{R}^N_+)} \le C_N M_1,
$$

for $k, l, m = 1, \ldots, N$ where the positive constants C_{N,M_2} and C_N do not depend on M_1. Then, by using the above inequalities, the estimates for $\mathcal{K}_2(\mathbf{v})$ is satisfied. We have proved the estimate of $\mathcal{K}_2(\mathbf{v})$ by using the same argument as for $\mathcal{K}_3(\mathbf{v})$. Similarly, by recalling the formula in (16) then using (8), then we immediately have the required estimates for $\mathcal{K}_1(\mathbf{v}), \mathcal{K}_4(\mathbf{v}), \mathcal{K}_5(\rho), \mathcal{K}_6(\rho)$. Therefore, we have proved the estimates for (18).

Next, we prove (19). Let $\alpha_1(x) = \tilde{\alpha}_1(\phi(x))$. Since

$$
\nabla_x\alpha_1(x) = (A + B(x))^\top(\nabla_y\tilde{\alpha}_1)(\phi(x)),
$$

it follows that

$$
\|\nabla_x\alpha_1\|_{L_\infty(\mathbb{R}^N_+)} \le C_N\epsilon
$$

with the positive constant C_N that only depends on N. Then it is known that

$$
\sup_{x\in\mathbb{R}^N_+}|\alpha_1(x) - \alpha_1^0| = \sup_{y\in\Omega_+}|\tilde{\alpha}_1(y) - \alpha_1^0| \le \delta.
$$

Recalling the formula of $\mathcal{H}_1$ in (11), the above inequalities result in the first estimate of (19). Since $0 < \delta < \min(1, B_1/2)$, for any $x \in \mathbb{R}$, we get

$$
\begin{aligned}
|\tilde{\alpha}_j(x)| &\leq B_2 + 1 \quad (j = 1, 2, 3), \\
B_1/2 \leq |\tilde{\alpha}_4(x)| &\leq B_2 + 1.
\end{aligned}
$$

Then, by using the above inequalities, we can prove the remaining estimates of (19). This completes the proof of the lemma. $\square$

Let $\mathcal{W}(\rho, \mathbf{v}) = (\mathcal{W}_1(\rho, \mathbf{v}), \mathcal{W}_2(\rho, \mathbf{v}), \mathcal{W}_3(\rho, \mathbf{v}), \mathcal{W}_4(\rho, \mathbf{v}), \mathcal{W}_5(\rho, \mathbf{v}))^\top$. Then, for any $(\rho, \mathbf{v}) \in H^3_q(\mathbb{R}^N_+) \times H^2_q(\mathbb{R}^N_+)^N$, $\eta \geq 1$, and $\lambda \in \Sigma_\sigma$, by Lemma 3.1 and Theorem 2.1, we have the following estimate:

$$
\begin{aligned}
\|\mathcal{R}_\lambda \mathcal{W}(\rho, \mathbf{v})\|_{\mathfrak{X}_q(\mathbb{R}^N_+)} \leq & \alpha(12M_1 + 4\epsilon_2) \\
& + (8\beta_{M_2} + 3\alpha_{M_2,\eta_2})\lambda_2^{1/2} \|\mathcal{S}_\lambda \rho, \mathcal{T}_\lambda \mathbf{v}\|_{\mathcal{P}_q(\mathbb{R}^N_+) \times \mathcal{Q}_q(\mathbb{R}^N_+)}.
\end{aligned} \tag{20}
$$

Let $\vartheta = \alpha^0_1$, $\mu = \alpha^0_2/\alpha^0_4$, $\nu = \alpha^0_3/\alpha^0_4$, and $\kappa = \alpha^0_1/\alpha^0_4$ in (7). By Theorem 2.1, there exist operators $\mathcal{A}^0(\lambda)$ and $\mathcal{B}^0(\lambda)$ with

$$
\begin{aligned}
\mathcal{A}^0(\lambda) &\in Hol(\Sigma_\sigma, \mathcal{L}(\mathfrak{X}_q(\mathbb{R}^N_+), H^3_q(\mathbb{R}^N_+))), \\
\mathcal{B}^0(\lambda) &\in Hol(\Sigma_\sigma, \mathcal{L}(\mathfrak{X}_q(\mathbb{R}^N_+), H^2_q(\mathbb{R}^N_+)),
\end{aligned}
$$

such that for $\mathbf{F}^0 = (d, \mathbf{f}, g, \mathbf{h}', h_N) \in \mathcal{X}^0_q(\mathbb{R}^N_+)$, $(r, \mathbf{w}) = (\mathcal{A}^0(\lambda)\mathcal{R}_\lambda \mathbf{F}^0, \mathcal{B}^0(\lambda)\mathcal{R}_\lambda \mathbf{F}^0)$ is the unique solution for system

$$
\begin{cases}
\lambda r + \alpha^0_1 \mathrm{div}\, \mathbf{w} = d & \text{in}\, \mathbb{R}^N_+, \\
\lambda \mathbf{w} - \dfrac{\alpha^0_2}{\alpha^0_4} \Delta \mathbf{w} - \dfrac{\alpha^0_3}{\alpha^0_4} \Delta \,\mathrm{div}\, \mathbf{w} - \dfrac{\alpha^0_1}{\alpha^0_4} \Delta \nabla r = \mathbf{f} & \text{in}\, \mathbb{R}^N_+, \\
\mathbf{n} \cdot \nabla r = g & \text{on}\, \mathbb{R}^N_0, \\
\partial_N w_s + \partial_s w_N = h_s & \text{on}\, \mathbb{R}^N_0,\ (s = 1, ..., N-1), \\
\mathbf{n} \cdot \mathbf{w} = h & \text{on}\, \mathbb{R}^N_0,
\end{cases} \tag{21}
$$

satisfying the estimates

$$
\begin{aligned}
\mathcal{R}_{((\mathfrak{X}^0_q(\mathbb{R}^N_+), \mathcal{P}_q(\mathbb{R}^N_+)))} \left\{ \left(\lambda \frac{d}{d\lambda} \right)^n \mathcal{S}_\lambda \mathcal{A}^0(\lambda) \mid \lambda \in \Sigma_\sigma \right\} &\leq M, \\
\mathcal{R}_{(\mathfrak{X}^0_q(\mathbb{R}^N_+), \mathcal{Q}_q(\mathbb{R}^N_+))} \left\{ \left(\lambda \frac{d}{d\lambda} \right)^n \mathcal{T}_\lambda \mathcal{B}^0(\lambda) \mid \lambda \in \Sigma_\sigma \right\} &\leq M,
\end{aligned}
$$

for $n = 0, 1$, where M is a positive constant that depends on N, q, B_1, and B_2, but does not depend on $\alpha^0_i (i = 1, 2, 3, 4)$.

Next, we will solve the system (17) using the approach developed for solving the system (21). Let we define

$$
\mathcal{V}(\lambda)\mathbf{F} = (\mathcal{V}_1(\lambda)\mathbf{F}, \mathcal{V}_2(\lambda)\mathbf{F}, \mathcal{V}_3(\lambda)\mathbf{F}, \mathcal{V}_4(\lambda)\mathbf{F}, \mathcal{V}_5(\lambda)\mathbf{F}),
$$

where

$$
\mathcal{V}_j(\lambda)\mathbf{F} = \mathcal{W}_j(\mathcal{A}^0(\lambda)\mathbf{F}, \mathcal{B}^0(\lambda)\mathbf{F}) \quad (j = 1, 2, 3, 4, 5),
$$

for $\mathbf{F} \in \mathfrak{X}_q^0(\mathbb{R}_+^N)$ and $\lambda \in \Sigma_\sigma$. For the estimate $\mathcal{V}(\lambda)\mathbf{F}$ mentioned above, we have the following lemma, with a proof that parallels the proof of Lemma 3.7 in [13].

Lemma 3.2. *Let $q \in (1, \infty)$ and let M_1 and M_2 be positive constants in (8). Assume that (a) and (b) in Theorem 1.2 are satisfied for positive constants δ and ϵ where δ satisfies $0 < \delta < \min(1, B_1/2)$. Then there exists $\gamma > 12$ that depends only on q such that for $n = 0, 1$ and for any $\eta \geq 1$,*

$$\begin{aligned}\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^0(\mathbb{R}_+^N))} &\left(\left\{ (\lambda \frac{d}{d\lambda})^n (\mathcal{R}_\lambda \mathcal{V}(\lambda)) \mid \lambda \in \Sigma_\sigma \right\} \right) \\ &\leq \gamma M \left(\alpha(M_1 + \delta) + (\beta_{M_2} + \alpha_{M_2,\epsilon})\eta^{-\frac{1}{2}} \right).\end{aligned} \tag{22}$$

Furthermore, from Lemma 22, we can choose M_1 and δ sufficiently small such that

$$\gamma M(\alpha M_1 + \delta) \leq 1/4, \quad \text{and} \quad M(\beta_{M_2} + \alpha_{M_2,\epsilon})\eta^{-1/2} \leq 1/4, \tag{23}$$

and choose η to be very large such that

$$\gamma M(\alpha M_1 + \delta) \leq 1/4, \quad \text{and} \quad M(\beta_{M_2} + \alpha_{M_2,\epsilon})\eta^{-1/2} \leq 1/4, \tag{24}$$

resulting

$$\mathcal{R}_{\mathcal{L}}(\mathfrak{X}_q^0(\mathbb{R}_+^N)) \left(\left\{ (\lambda \frac{d}{d\lambda})^n (\mathcal{R}_\lambda \mathcal{V}(\lambda)) \mid \lambda \in \Sigma_\sigma \right\} \right) \leq 1/2 \tag{25}$$

for $n = 0, 1$.

By (25), we see that $\|\mathcal{V}(\lambda)\|_{\mathfrak{X}_q^0(\mathbb{R}_+^N)} \leq 1/2$. Therefore, based on the Neumann series expansion theorem 2.2 (b) and (c), for every $\lambda \in \Sigma_\sigma$, there exists an inverse operator $(I - \mathcal{R}_\lambda \mathcal{V}(\lambda))^{-1}$ of $I - \mathcal{R}_\lambda \mathcal{V}(\lambda)$ in $\mathcal{L}(\mathfrak{X}_q^0(\mathbb{R}_+^N))$ such that

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^0(\mathbb{R}_+^N))} \left(\left\{ (\lambda \frac{d}{d\lambda})^n (I - \mathcal{R}_\lambda \mathcal{V}(\lambda))^{-1} \mid \lambda \in \Sigma_\sigma \right\} \right) \leq 4.$$

Let, for $\mathbf{F}^0 \in \mathfrak{X}_q^0(\mathbb{R}_+^N)$ and for $\lambda \in \Sigma_\sigma$, we define the following operators:

$$\Theta(\lambda)\mathbf{F}^0 = \mathcal{A}^0(\lambda)(I - \mathcal{R}_\lambda \mathcal{V}(\lambda))^{-1}\mathbf{F}^0, \quad \Xi(\lambda)\mathbf{F}^0 = \mathcal{B}^0(\lambda)(I - \mathcal{R}_\lambda \mathcal{V}(\lambda))^{-1}\mathbf{F}^0.$$

Then, based on the Neumann series expansion theorem (d), we obtain that $(\rho, \mathbf{v}) = (\Theta(\lambda)\mathcal{R}_\lambda \mathbf{F}^0, \Xi(\lambda)\mathcal{R}_\lambda \mathbf{F}^0)$ is the solution to (17) satisfying the estimate

$$\begin{aligned}\mathcal{R}_{\mathcal{L}}(\mathfrak{X}_q^0(\mathbb{R}_+^N), \mathcal{P}_q(\mathbb{R}_+^N)) \left(\left\{ (\lambda \frac{d}{d\lambda})^n (\mathcal{S}_\lambda \Theta(\lambda)) \mid \lambda \in \Sigma_\sigma \right\} \right) \leq 12M, \\ \mathcal{R}_{\mathcal{L}}(\mathfrak{X}_q^0(\mathbb{R}_+^N), \mathcal{Q}_q(\mathbb{R}_+^N)) \left(\left\{ (\lambda \frac{d}{d\lambda})^n (\mathcal{T}_\lambda \Xi(\lambda)) \mid \lambda \in \Sigma_\sigma \right\} \right) \leq 12M.\end{aligned} \tag{26}$$

for $n = 0, 1$. Thus, we have obtained an $\mathcal{R}$-bounded solution operator to System (17). The uniqueness of the solution to System (17) is obtained based on the prior estimate of the solution. Suppose $(\rho, \mathbf{v})$ satisfies System (17) for $(d, \mathbf{f}, g, \mathbf{h}', h) =$

$(0,0,0,0,0)$, then by (20), (23) and (24), we have

$$
\begin{aligned}
\|(\mathcal{S}_\lambda\rho, \mathcal{T}_\lambda\mathbf{v})\|_{\mathcal{P}_q(\mathbb{R}^N_+)\times\mathcal{Q}_q(\mathbb{R}^N_+)} &\le M\|\mathcal{R}_\lambda\mathcal{W}(\rho,\mathbf{v})\|_{\mathfrak{X}^0_q(\mathbb{R}^N_+)} \\
&\le 12M\left(\alpha(M_1+\delta)+(\beta_{M_2}+\alpha_{M_2,\epsilon})\eta^{-1/2}\right)\|(\mathcal{S}_\lambda\rho, \mathcal{T}_\lambda\mathbf{v})\|_{\mathcal{P}_q(\mathbb{R}^N_+)\times\mathcal{Q}_q(\mathbb{R}^N_+)} \\
&\le \gamma M\left(\alpha(M_1+\delta)+(\beta_{M_2}+\gamma_{M_2,\epsilon})\eta^{-1/2}\right)|(\mathcal{S}_\lambda\rho, \mathcal{T}_\lambda\mathbf{v})\|_{\mathcal{P}_q(\mathbb{R}^N_+)\times\mathcal{Q}_q(\mathbb{R}^N_+)} \\
&\le 1/2\|(\mathcal{S}_\lambda\rho, \mathcal{T}_\lambda\mathbf{v})\|_{\mathcal{P}_q(\mathbb{R}^N_+)\times\mathcal{Q}_q(\mathbb{R}^N_+)}.
\end{aligned}
$$

The above inequality results in $(\rho,\mathbf{v})=(0,0)$, and this shows the uniqueness of the solution to System (17), which completes the proof of the uniqueness.

3.3. **Proof of Theorem 1.2.**

In this subsection, we will prove Theorem 1.2, demonstrating the existence of an $\mathcal{R}$-bounded solution to System (6). This proof is based on the solution obtained from the previous reduction, where we reconstruct $\Theta(\lambda)$ and $\Xi(\lambda)$ as derived in the earlier subsection.

Using the definition given by (13), we can show that

$$
\begin{aligned}
\mathcal{R}_\lambda(d,\mathbf{f},g,\mathbf{h},h) =&(\nabla d, \lambda^{1/2}d, \mathbf{f}, \nabla^2 g, \lambda^{1/2}\nabla g, \lambda g, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h}, \nabla^2 h, \lambda^{1/2}\nabla h, \lambda h) \\
=&\Big((\nabla\Phi)^\top(\nabla\tilde{d})\circ\Phi, \lambda^{1/2}\tilde{d}\circ\Phi, \mathbf{A}_{-1}\tilde{\mathbf{f}}\circ\Phi, \mathcal{G}_1(\nabla^2\tilde{g})\circ\Phi \\
&+\mathcal{G}_2(\nabla\tilde{g})\circ\Phi, \lambda^{1/2}(\nabla\Phi)^\top(\nabla\tilde{g})\circ\Phi, \lambda\tilde{g}\circ\Phi, \\
&(\nabla\Phi)^\top(\nabla\tilde{h})\circ\Phi, \lambda^{1/2}\tilde{h}\circ\Phi, \mathcal{G}_1(\nabla^2\tilde{h})\circ\Phi \\
&+\mathcal{G}_2(\nabla\tilde{h})\circ\Phi, \lambda^{1/2}(\nabla\Phi)^\top(\nabla\tilde{h})\circ\Phi, \lambda\tilde{h}\circ\Phi\Big).
\end{aligned} \tag{27}
$$

Here, $\mathcal{G}_1(\nabla^2\tilde{f})\circ\Phi$ and $\mathcal{G}_2(\nabla\tilde{f})\circ\Phi$ are $N\times N$ matrices whose (i,j)-th components $(\mathcal{G}_1(\nabla^2\tilde{f})\circ\Phi)_{ij}$ and $(\mathcal{G}_2(\nabla\tilde{f})\circ\Phi)_{ij}$ are respectively given by

$$
(\mathcal{G}_1(\nabla^2\tilde{f})\circ\Phi)_{ij} = \sum_{k,l=1}^N\left(\frac{\partial^2\tilde{g}}{\partial y_k\partial y_l}(\Phi(x))\frac{\partial\Phi_k(x)}{\partial x_i}\frac{\partial\Phi_l(x)}{\partial x_j}\right),
$$

$$
(\mathcal{G}_2(\nabla\tilde{f})\circ\Phi)_{ij} = \sum_{k,l=1}^N\left(\frac{\partial\tilde{g}}{\partial y_k}(\Phi(x))\frac{\partial^2\Phi_k}{\partial x_i\partial x_j}\right),
$$

for $\Phi=(\Phi_1,\dots,\Phi_N)$, with $\tilde{f}\in\{\tilde{g},\tilde{h}\}$.

Let $\tilde{G}_1,\dots,\tilde{G}_{11}$ be the variables corresponding to

$$
\nabla\tilde{d}, \lambda^{1/2}\tilde{d}, \tilde{f}, \nabla^2\tilde{g}, \lambda^{1/2}\nabla\tilde{g}, \lambda\tilde{g}, \nabla\tilde{h}, \lambda^{1/2}\tilde{h}, \nabla^2\tilde{h}, \lambda^{1/2}\nabla\tilde{h}, \lambda\tilde{h}
$$

respectively. Note again that $\tilde{\rho} = \rho \circ \Phi^{-1}$ and $\tilde{\mathbf{u}} = \mathbf{A}_{-1}^{\top}\mathbf{v} \circ \Phi^{-1}$. Then, by using Equation (27), we define

$$\begin{aligned}\mathcal{A}(\lambda)\tilde{G} =&\Big[\Theta(\lambda)\Big((\nabla\Phi)^{\top}\tilde{G}_1 \circ \Phi, \tilde{G}_2 \circ \Phi, \mathbf{A}_{-1}\tilde{G}_3 \circ \Phi, \mathcal{G}_1\tilde{G}_4 \circ \Phi \\ &+ \lambda^{-1/2}\mathcal{G}_2\tilde{G}_5 \circ \Phi, (\nabla\Phi)^{\top}\tilde{G}_5 \circ \Phi, \tilde{G}_6 \circ \Phi, (\nabla\Phi)^{\top}\tilde{G}_8 \circ \Phi, \mathcal{G}_1\tilde{G}_9 \circ \Phi \\ &+ \lambda^{-1/2}\mathcal{G}_2\tilde{G}_{10} \circ \Phi, (\nabla\Phi)^{\top}\tilde{G}_{10} \circ \Phi, \tilde{G}_{11} \circ \Phi\Big)\Big] \circ \Phi^{-1}\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}(\lambda)\tilde{G} =&\mathbf{A}_{-1}^{\top}[\Xi(\lambda)((\nabla\Phi)^{\top}\tilde{G}_1 \circ \Phi, \tilde{G}_2 \circ \Phi, \mathbf{A}_{-1}\tilde{G}_3 \circ \Phi, C_2\tilde{G}_4 \circ \Phi \\ &+ \lambda^{-1/2}C_1\tilde{G}_5 \circ \Phi, (\nabla\Phi)^{\top}\tilde{G}_5 \circ \Phi, \tilde{G}_6 \circ \Phi, (\nabla\Phi)^{\top}\tilde{G}_7 \circ \Phi, \tilde{G}_8 \circ \Phi, C_2\tilde{G}_9 \circ \Phi \\ &+ \lambda^{-1/2}C_1\tilde{G}_{10} \circ \Phi, (\nabla\Phi)^{\top}\tilde{G}_{10} \circ \Phi, \tilde{G}_{11} \circ \Phi)] \circ \Phi^{-1}.\end{aligned}$$

for $\tilde{G} = (\tilde{G}_1, \ldots, \tilde{G}_{11}) \in \mathfrak{X}_q(\Omega_+)$.

Let $(\tilde{\rho}, \tilde{\mathbf{u}}) = (\mathcal{A}(\lambda)\mathcal{R}_\lambda\tilde{\mathbf{H}}, \mathcal{B}(\lambda)\mathcal{R}_\lambda\tilde{\mathbf{H}})$ with $\tilde{\mathbf{H}} = (\tilde{d}, \tilde{\mathbf{f}}, \tilde{g}, \tilde{\mathbf{h}}, \tilde{h}) \in \mathfrak{X}_q(\Omega_+)$, then $(\tilde{\rho}, \tilde{\mathbf{u}})$ is a solution to System (6). Additionally, the uniqueness of the solution to System (6) follows from that of System (17). The required estimate in Theorem 1.2 is obtained from the $\mathcal{R}$-boundedness of the solution to System (6), as given by (26). Therefore, we have established the existence of an $\mathcal{R}$-bounded solution operator for System (6), which completes the proof of Theorem 1.2. .

4. CONCLUDING REMARKS

This study is part of a series of investigations aimed at analyzing the Navier-Stokes-Korteweg (NSK) system in general and complex domains. As an initial step, it is essential to gain a deep understanding of the system's behavior in simpler domains, namely the whole space, half-space, and bent half-space. Solving the system in these domains serves as an important foundation, as the results will be used to formulate and establish the concept of maximal regularity for the NSK system.

In particular, the findings of this study will serve as one of the key foundations for proving maximal regularity, which will subsequently be used to establish the local and global well-posedness of the system using the maximal regularity approach. Maximal regularity plays a crucial role in proving the existence, uniqueness, and stability of solutions for systems with more general domains. Through this step-by-step approach, it is expected that the mathematical analysis of the NSK system can be systematically developed, starting from simpler domains and progressing to more complex and realistic settings.

Acknowledgement. The author gratefully acknowledges the financial support from the Research and Publishing Center of Universitas Islam Negeri Syarif Hidayatullah Jakarta, as well as the insightful discussions with Dr. Hirokazu Saito.

REFERENCES

[1] J. E. Dunn and J. Serrin, "On the thermomechanics of interstitial working," *Arch. Rational Mech. Anal.*, vol. 88, no. 2, pp. 95–133, 1985. https://doi.org/10.1007/BF00250907.

[2] D. M. Anderson, A. A. Wheeler, and G. B. McFadden, "Diffuse-interface methods in fluid mechanics,"

[3] J. Liu, C. M. Landis, H. Gomez, and T. J. R. Hughes, "Liquid–vapor phase transition: Thermomechanical theory, entropy stable numerical formulation, and boiling simulations," *Computer Methods in Applied Mechanics and Engineering*, vol. 297, no. 1, pp. 476–553, 2015. https://doi.org/10.1016/j.cma.2015.09.007.

[4] D. Bresch, B. Desjardins, and C. K. Lin, "On some compressible fluid models: Korteweg, lubrication, and shallow water systems," *Comm. Partial Differ. Equ.*, vol. 28, no. 3–4, pp. 843–868, 2003. https://doi.org/10.1081/PDE-120020499.

[5] M. Kotschote, "Strong solutions for a compressible fluid model of korteweg type," *Annales de l'Institut Henri Poincare ́ C, Analyse non line ́aire*, vol. 25, no. 4, pp. 679–696, 2008. https://doi.org/10.1016/j.anihpc.2007.03.005.

[6] M. Kotschote, "Strong well-posedness for a korteweg-type model for the dynamics of a compressible non-isothermal fluid," *J. Math. Fluid Mech.*, vol. 12, no. 4, pp. 473–484, 2010. https://doi.org/10.1007/s00021-009-0298-1.

[7] H. Saito, "Compressible fluid model of korteweg type with free boundary condition: model problem," *Funkcial. Ekvac*, vol. 62, no. 3, pp. 337–386, 2019. https://doi.org/10.1619/fesi.62.337.

[8] H. Saito, "Existence of -bounded solution operator families for a compressible fluid model of korteweg type on the half-space," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 2, pp. 1744–1787, 2020. https://doi.org/10.1002/mma.6875.

[9] S. Inna, S. Maryani, and H. Saito, "Half-space model problem for a compressible fluid model of korteweg type with slip boundary condition," in *Journal of Physics: Conference Series*, vol. 1494, pp. 1–11, IOP Publishing Ltd, 2020. https://doi.org/10.1088/1742-6596/1494/1/012014.

[10] S. Inna, I. Fauziah, M. Manaqib, and P. M. Puri, "Operator solusi model fluida termampatkan tipe korteweg dengan kondisi batas slip di half-space kasus koefisien $((\mu+\nu)/(2\kappa))^2 - (1/\kappa) \neq 0$, dengan $\kappa = \mu\nu$ dan $\mu \neq \nu$," *Limits: Journal of Mathematics and its Applications*, vol. 20, no. 2, pp. 191–208, 2023. http://dx.doi.org/10.12962/limits.v20i2.12954.

[11] S. Inna, "The existence of r-bounded solution operator of navier stokes korteweg model with slip boundary conditions in half-space," *Math. Meth. Appl. Sci.*, vol. 47, no. 11, pp. 8581–8610, 2024. https://doi.org/10.1002/mma.10033.

[12] S. Inna and H. Saito, "Local solvability for a compressible fluid model of korteweg type on general domains," *Mathematics*, vol. 11, no. 2368, pp. 1–41, 2023. https://doi.org/10.3390/math11102368.

[13] H. Saito, "On the maximal lp-lq regularity for a compressible fluid model of korteweg type on general domains," *Journal of Differential Equations*, vol. 268, no. 6, pp. 2802–2851, 2020. https://doi.org/10.1016/j.jde.2019.09.040.

[14] V. Lebedev, *An Introduction to Functional Analysis in Computational Mathematics.* Birkhauser, Boston, 1997. https://doi.org/10.1007/978-1-4612-4128-7.