J. Indones. Math. Soc.
Vol. 32, No. 03 (2026), pp. 1–26.

SG-Lightlike Submanifolds of a Locally Bronze Semi-Riemannian Manifold Equipped With (l, m)-Type Connection

Rajinder Kaur[1], Jasleen Kaur[1*]

[1]Department of Mathematics, Punjabi University, India

Abstract. This research work introduces the geometry of the SG (Screen Generic)-lightlike submanifolds of a locally bronze semi-Riemannian manifold endowed with an (l,m)-type connection. The characterization theorems on geodesicity of such submanifolds with respect to the integrability and parallelism of the distributions are established. It is shown that there exists no coisotropic, isotropic or totally proper SG-lightlike submanifold of a locally bronze semi-Riemannian manifold. Assertions for the smooth transversal vector fields in totally umbilical proper SG-lightlike submanifold are obtained. Furthermore, the structure of a minimal SG-lightlike submanifold of a locally bronze semi-Riemannian manifold is detailed with an example.

Key words and Phrases: Locally bronze semi-Riemannian manifold, SG-lightlike submanifold, Totally umbilical lightlike submanifold.

1. INTRODUCTION

De Spinadel [1] developed the theory of metallic means family, in which bronze ratio has an important location in studying topics such as dynamical system and quasicrystals. The geometry of bronze structures has an important relation with pure Riemannian metrics according to the corresponding structure. Pandey [2] introduced the concept of bronze structure on Riemannian manifold which was further studied by Aknipar [3]. Moreover, Duggal and Bejancu [4] initiated the theory of lightlike submanifolds of semi-Riemannian manifolds which provides an outstanding framework for geometric characteristics and has versatile applications, particularly in general relativity.

In view of this, Jin [5] developed generic lightlike submanifolds for an indefinite cosymplectic manifold which was further explored by [6, 5] for indefinite

*Corresponding author : jasleen_math@pbi.ac.in

2020 Mathematics Subject Classification: Primary: 53C15; Secondary:53C40, 53C50
Received: 02-02-2025, accepted: 05-04-2026.

Sasakian manifold and indefinite Kaehler manifold, respectively. Several new connections such that semi-symmetric non-metric connection [7], non-metric ϕ- symmetric connection [8] etc. applied to the SG-lightlike submanifolds came into existence. As the generalization of screen Cauchy Riemann(SCR) and generic lightlike submanifolds, Dogan et al. [9] introduced the notion of SG-lightlike submanifold for indefinite Kaehler manifold. Then, a few more similar classes namely, contact totally umbilical SG-lightlike and minimal SG-lightlike submanifolds of indefinite Sasakian manifolds were researched by Gupta [10]. Recently, the theory of SG-lightlike submanifolds was investigated for golden semi-Riemannian manifold [11], cosympletic manifold [12] and semi-Riemannian product manifold [13].

Jin[14] introduced the notion of non-symmetric and non-metric (l, m)-type connection on semi-Riemannian manifold as follows:
A linear connection $\bar{\Omega}$ on a semi-Riemannian manifold $(\bar{M}^{\circ}, \bar{g}^{'})$ is called an (l, m)-type connection if its torsion tensor $\bar{T}^{\circ}$ satisfies

$$\bar{T}^{\circ}(X^{\circ}, Y^{\circ}) = l\{\theta(Y^{\circ})X^{\circ} - \theta(X^{\circ})Y^{\circ}\} + m\{\theta(Y^{\circ})\bar{J}'X^{\circ} - \theta(X^{\circ})\bar{J}'Y^{\circ}\}$$

where l and m are smooth functions, $\bar{J}'$ is a tensor field of $(1, 1)$-type and θ is a 1-form associated with a smooth unit spacelike vector field η, which is called the characteristic vector field, by $\theta(X^{\circ}) = \bar{g}^{'}(X^{\circ}, \eta)$. We set $(l, m) \neq (0, 0)$ and denote by X°, Y° and Z°, the smooth vector fields on $\bar{M}^{\circ}$. Jin et al. [15] further developed the geometry of generic lightlike submanifolds for an indefinite Kaehler manifold equipped with an (l, m)-type connection. However, the concept of this connection for a locally bronze semi-Riemannian manifold having SG-lightlike submanifolds is yet to be investigated.

In this paper, we introduce the bronze structure for semi-Riemannian manifold and analyze the geometry of SG-lightlike submanifolds of a locally bronze semi-Riemannian manifolds equipped with an (l, m)-type connection. The integrability, parallelism and geodesicity of the distributions are characterized. We prove the non-existence of coisotropic, isotropic or totally proper SG-lightlike submanifold of a locally bronze semi-Riemannian manifold. Totally umbilical proper SG-lightlike submanifolds for locally bronze semi-Riemannian manifolds are also worked upon. Additionally, the structure of a minimal SG-lightlike submanifolds has been substantiated with an example.

2. PRELIMINARIES

Let $(\bar{M}^{\circ}, \bar{g}^{'})$ be an $(m+n)$-dimensional semi-Riemannian manifold with semi-Riemannian metric $\bar{g}^{'}$ of constant index q such that $m, n \geq 1$, $1 \leq q \leq m + n - 1$. Let $(M^{\circ}, g^{'})$ be an m-dimensional lightlike submanifold of $\bar{M}^{\circ}$. There exists a smooth distribution $RadTM^{\circ}$ on M° of rank $r > 0$, known as radical distribution on M° such that $RadTM^{\circ}_p = TM^{\circ}{}_p \cap TM^{\circ}{}^{\perp}_p, \forall\ p \in M^{\circ}$ where TM°_p and $TM^{\circ}{}^{\perp}_p$ are degenerate orthogonal spaces but not complementary. Then M° is called an r-lightlike submanifold of $\bar{M}^{\circ}$.

Consider complementary distribution of radical distribution in TM°, called screen distribution, denoted by $S(TM^\circ)$

$$i.e.\, TM^\circ = RadTM^\circ \perp S(TM^\circ),$$

and $S(TM^{\circ\perp})$, called screen transversal vector bundle which is a complementary vector subbundle to $Rad(TM^\circ)$ in $TM^{\circ\perp}$

$$TM^{\circ\perp} = RadTM^\circ \perp S(TM^{\circ\perp}).$$

Since $S(TM^\circ)$ is a non degenerate vector subbundle of $T\bar{M}^\circ|_{M^\circ}$, we have

$$T\bar{M}^\circ|_{M^\circ} = S(TM^\circ) \perp S(TM^\circ)^\perp,$$

where $S(TM^\circ)^\perp$ is the complementary orthogonal vector subbundle of $S(TM^\circ)$ in $T\bar{M}^\circ|_{M^\circ}$.
Let $tr(TM^\circ)$ and $ltr(TM^\circ)$ be complementary vector bundles to TM° in $T\bar{M}^\circ|_{M^\circ}$ and to $RadTM^\circ$ in $S(TM^{\circ\perp})^\perp$. Then

$$\begin{aligned} tr(TM^\circ) &= ltr(TM^\circ) \perp S(TM^{\circ\perp}), \\ T\bar{M}^\circ|_{M^\circ} &= TM^\circ \oplus tr(TM^\circ), \\ &= (RadTM^\circ \oplus ltr(TM^\circ)) \perp S(TM^\circ) \perp S(TM^{\circ\perp}). \end{aligned}$$

Following four cases exist for the lightlike submanifold $(M^\circ, g^{'}, S(TM^\circ), S(TM^{\circ\perp}))$:
(1) M° is r-lightlike if $r < min(m, n)$,
(2) M° is co-isotropic if $r = n < m$, i.e. $S(TM^{\circ\perp}) = \{0\}$,
(3) M° is isotropic if $r = m < n$, i.e. $S(TM^\circ) = \{0\}$ and
(4) M° is totally lightlike if $r = n = m$, i.e. $S(TM^{\circ\perp}) = \{0\} = S(TM^\circ)$.

Theorem 2.1. [4] *"Let $(M^\circ, g^{'}, S(TM^\circ), S(TM^{\circ\perp}))$ be a r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}^\circ, \bar{g}^{'})$. Then there exists a complementary vector bundle $ltr(TM^\circ)$, called a lightlike transversal bundle of $RadTM^\circ$ in $S(TM^{\circ\perp})^\perp$, and a basis of $\Gamma(ltr(TM^\circ)|_{U^\circ})$ consisting of smooth sections $\{N_1^\circ, \cdots, N_r^\circ\}$ of $S(TM^{\circ\perp})^\perp|_{U^\circ}$, where U° is a coordinate neighborhood of M°, such that*

$$\bar{g}^{'}(N_i^\circ, \xi_j^\circ) = \delta_{ij}, \quad \bar{g}^{'}(N_i^\circ, N_j^\circ) = 0, \quad i, j = 0, 1, \cdots, r$$

where $\{\xi_1^\circ, \cdots, \xi_r^\circ\}$ is a lightlike basis of $\Gamma(RadTM^\circ)|_{U^\circ}$."

Let $\bar{\nabla}^\circ$ be the Levi-Civita connection on $\bar{M}^\circ$. Then, the corresponding Gauss and Weingarten formulae are as follows:

$$\bar{\nabla}^\circ_{X^\circ} Y^\circ = \nabla^\circ_{X^\circ} Y^\circ + h^\circ(X^\circ, Y^\circ), \quad \forall\ X^\circ, Y^\circ \in \Gamma(TM^\circ), \tag{1}$$

$$\bar{\nabla}^\circ_{X^\circ} V^\circ = -A_{V^\circ} X^\circ + \nabla^{\circ t}_{X^\circ} V^\circ, \quad \forall\ X^\circ \in \Gamma(TM^\circ), V^\circ \in \Gamma(tr(TM^\circ)), \tag{2}$$

where $\{\nabla^\circ_{X^\circ} Y^\circ, A_{N^\circ} X^\circ\}$ and $\{h^\circ(X^\circ, Y^\circ), \nabla^{\circ t}_{X^\circ} N^\circ\}$ belong to $\Gamma(TM^\circ)$ and $\Gamma(tr(TM^\circ))$, respectively. ∇° and $\nabla^{\circ t}$ are linear connections on M° and on the vector bundle $tr(TM^\circ)$, respectively.

Considering the projection morphisms L and S of $tr(TM^\circ)$ on $ltr(TM^\circ)$ and $S(TM^{\circ\perp})$ respectively, we have

$$\bar{\nabla}^\circ_{X^\circ}Y^\circ = \nabla^\circ_{X^\circ}Y^\circ + h^{\circ l}(X^\circ, Y^\circ) + h^{\circ s}(X^\circ, Y^\circ), \tag{3}$$

$$\bar{\nabla}^\circ_{X^\circ}N^\circ = -A_{N^\circ}X^\circ + \nabla^{\circ l}_{X^\circ}N^\circ + D^{\circ s}(X^\circ, N^\circ), \tag{4}$$

$$\bar{\nabla}^\circ_{X^\circ}W^\circ = -A_{W^\circ}X^\circ + \nabla^{\circ s}_{X^\circ}W^\circ + D^{\circ l}(X^\circ, W^\circ), \tag{5}$$

where $h^{\circ l}(X^\circ, Y^\circ) = Lh^\circ(X^\circ, Y^\circ), h^{\circ s}(X^\circ, Y^\circ) = Sh^\circ(X^\circ, Y^\circ)$, $\{\nabla^\circ{}_{X^\circ}Y^\circ, A_{N^\circ}X^\circ, A_{W^\circ}X^\circ\} \in \Gamma(TM^\circ)$, $\{\nabla^{\circ l}_{X^\circ}N^\circ, D^{\circ l}(X^\circ, W^\circ)\} \in \Gamma(ltr(TM^\circ))$ and $\{\nabla^{\circ s}_{X^\circ}W^\circ, D^{\circ s}(X^\circ, N^\circ)\} \in \Gamma(S(TM^{\circ\perp}))$. Then, considering $(3)-(5)$ and the fact that $\bar{\nabla}^\circ$ is a metric connection, the following holds:

$$\bar{g}'(h^{\circ s}(X^\circ, Y^\circ), W^\circ) + \bar{g}'(Y^\circ, D^{\circ l}(X^\circ, W^\circ)) = \bar{g}'(A_{W^\circ}X^\circ, Y^\circ), \tag{6}$$

$$\bar{g}'(D^{\circ s}(X^\circ, N^\circ), W^\circ) = \bar{g}'(A_{W^\circ}X^\circ, N^\circ). \tag{7}$$

Let J' be a projection of TM° on $S(TM^\circ)$. Then, we have

$$\nabla^\circ{}_{X^\circ}J'Y^\circ = \nabla^{\circ *}_{X^\circ}J'Y^\circ + h^{\circ *}(X^\circ, J'Y^\circ), \tag{8}$$

$$\nabla^\circ{}_{X^\circ}\xi^\circ = -A^*_{\xi^\circ}X^\circ + \nabla^{\circ *t}_{X^\circ}\xi^\circ, \tag{9}$$

for any $X^\circ, Y^\circ \in \Gamma(TM^\circ)$ and $\xi^\circ \in \Gamma(Rad(TM^\circ))$, where$\{\nabla^{\circ *}_{X^\circ}J'Y^\circ, A^*_{\xi^\circ}X^\circ\}$ and $\{(h^{\circ *}(X^\circ, J'Y^\circ), \nabla^{\circ *t}_{X^\circ}\xi^\circ\}$ belong to $\Gamma(S(TM^\circ))$ and $\Gamma(RadTM^\circ)$, respectively. By using the above equations, we obtain

$$\bar{g}'(h^{\circ l}(X^\circ, J'Y^\circ), \xi^\circ) = g'(A^*_{\xi^\circ}X^\circ, J'Y^\circ), \tag{10}$$

$$\bar{g}'(h^{\circ *}(X^\circ, J'Y^\circ), N^\circ) = g'(A_{N^\circ}X^\circ, J'Y^\circ), \tag{11}$$

$$\bar{g}'(h^{\circ l}(X^\circ, \xi^\circ), \xi^\circ) = 0, A^*_{\xi^\circ}\xi^\circ = 0. \tag{12}$$

In general, the connection ∇° on M° is not metric connection. Since $\bar{\nabla}^\circ$ is a metric connection, from (3), we derive

$$(\nabla^\circ_{X^\circ}g')(Y^\circ, Z^\circ) = \bar{g}'(h^{\circ l}(X^\circ, Y^\circ), Z^\circ) + \bar{g}'(h^{\circ l}(X^\circ, Z^\circ), Y^\circ),$$

for any $X^\circ, Y^\circ, Z^\circ \in \Gamma(TM^\circ)$. Here, $\nabla^{\circ *}$ is a metric connection on $S(TM^\circ)$.

Definition 2.2. *A polynomial structure on a semi-Riemannian manifold $\bar{M}^\circ$ is known as bronze structure if it is determined by $\bar{J}'$ such that*

$$\bar{J}'^2 = 3\bar{J}' + I. \tag{13}$$

where $\bar{J}'$ is a tensor field of type $(1,1)$ on $\bar{M}^\circ$.

Definition 2.3. *If a semi-Riemannian metric $\bar{g}'$ satisfies the equation*

$$\bar{g}'(X^\circ, \bar{J}'Y^\circ) = \bar{g}'(\bar{J}'X^\circ, Y^\circ), \tag{14}$$

which yields

$$\bar{g}'(\bar{J}'X^\circ, \bar{J}'Y^\circ) = 3\bar{g}'(X^\circ, \bar{J}'Y^\circ) + \bar{g}'(X^\circ, Y^\circ), \ \forall\ X^\circ\ Y^\circ \in \Gamma(T\bar{M}^\circ), \tag{15}$$

then $\bar{g}'$ is called $\bar{J}'$-compatible.

Definition 2.4. *"If semi-Riemannian metric $\bar{g}^{'}$ is compatible with the bronze structure $\bar{J}^{'}$, then the pair $(\bar{g}^{'}, \bar{J}')$ is called a bronze semi-Riemannian structure and $(\bar{M}^{\circ}, \bar{g}^{'}, \bar{J}')$ a bronze semi-Riemannian manifold."*

Definition 2.5. *If $(\bar{M}^{\circ}, \bar{g}^{'}, \bar{J}^{'})$ is a bronze semi-Riemannian manifold and*

$$\bar{\nabla}^{\circ}\bar{J}^{'} = 0, \tag{16}$$

then $(\bar{M}^{\circ}, \bar{g}^{'}, \bar{J}^{'})$ is termed as a locally bronze semi- Riemannian manifold.

Let $(\bar{M}^{\circ}, \bar{g}^{'}, \bar{J}^{'})$ be a locally bronze semi-Riemannian manifold, where $\bar{g}^{'}$ is a semi-Riemannian metric and $\bar{J}^{'}$ is a bronze structure. Then,
For each X° tangent to M°, $\bar{J}^{'}X^{\circ}$ can be written as follows

$$\bar{J}^{'}X^{\circ} = f^{'}X^{\circ} + w^{'}X^{\circ} = f^{'}X^{\circ} + w_{l}^{'}X^{\circ} + w_{s}^{'}X^{\circ}, \tag{17}$$

where $f^{'}X^{\circ}$ and $w^{'}X^{\circ}$ are the tangential and the transversal parts of $\bar{J}^{'}X^{\circ}$, $w_{l}^{'}$ and $w_{s}^{'}$ are the projections on $ltr(TM^{\circ})$ and $S(TM^{\circ\perp})$, respectively. In addition, for any $V^{\circ} \in \Gamma(tr(TM^{\circ}))$, $\bar{J}^{'}V^{\circ}$ can be written as

$$\bar{J}^{'}V^{\circ} = B^{'}V^{\circ} + C^{'}V^{\circ}, \tag{18}$$

where $B^{'}V^{\circ}$ and $C^{'}V^{\circ}$ are the tangential and the transversal parts of $\bar{J}^{'}V^{\circ}$, respectively.

3. SG-LIGHTLIKE SUBMANIFOLDS

Definition 3.1. [9] *A real r-lightlike submanifold M° of locally bronze semi-Riemannian manifold $\bar{M}^{\circ}$ is said to be a SG-lightlike submanifold of $\bar{M}^{\circ}$ if the following conditions are satisfied:*
(a) $Rad(TM^{\circ})$ is invariant with respect to $\bar{J}^{'}$, that is

$$\bar{J}^{'}(RadTM^{\circ}) = RadTM^{\circ}, \tag{19}$$

(b) There exist a subbundle B_o of $S(TM^{\circ})$ such that

$$B_o = \bar{J}^{'}(S(TM^{\circ})) \cap S(TM^{\circ}), \tag{20}$$

where B_o is a non-degenerate distribution on M°.
The above definition implies that there exist a complementary non-degenerate distribution $B^{'}$ in $S(TM^{\circ})$ such that

$$S(TM^{\circ}) = B_o \oplus B^{'},\ \bar{J}^{'}(B^{'}) \not\subseteq S(TM^{\circ}),\ \bar{J}^{'}(B^{'}) \not\subseteq S(TM^{\circ\perp}). \tag{21}$$

Let J_o, J_1 and Q be the projection morphisms on B_o, $RadTM^{\circ}$ and $B^{'}$ respectively. Then for all $X^{\circ} \in \Gamma(TM^{\circ})$,

$$X^{\circ} = J_oX^{\circ} + J_1X^{\circ} + QX^{\circ} = J^{'}X^{\circ} + QX^{\circ},$$

where $B = B_o \perp RadTM^\circ$, B *is invariant and* $J^{'}X^\circ \in \Gamma(B)$, $QX^\circ \in \Gamma(B^{'})$.
It is clear that $\bar{J}^{'}(B^{'}) \neq B^{'}$.
Now for a vector field $Y^\circ \in \Gamma(B^{'})$ *and using* (17), *we have*

$$\bar{J}^{'}Y^\circ = f^{'}Y^\circ + w^{'}Y^\circ, \tag{22}$$

where $f^{'}Y^\circ \in \Gamma(B^{'})$ *and* $w^{'}Y^\circ \in \Gamma(S(TM^{\circ\perp}))$.

M° is said to be a proper SG-lightlike submanifold of a locally bronze semi-Riemannian manifold if $B_o \neq \{0\}$ *and* $B^{'} \neq \{0\}$.

Proposition 3.2. *Let M° be an SG-lightlike submanifold of a locally bronze semi-Riemannian manifold* $(\bar{M}^\circ, \bar{g}^{'}, \bar{J}^{'})$. *Then, the distribution $ltr(TM^\circ)$ is invariant with respect to* $\bar{J}^{'}$.

4. (l, m)-TYPE CONNECTION

For Levi-Civita connection $\bar{\nabla}^\circ$ on the locally bronze semi-Riemannian manifold $(\bar{M}^\circ, \bar{g}^{'}, \bar{J}^{'})$, we set

$$\bar{\Omega}_{X^\circ}Y^\circ = \bar{\nabla}^\circ_{X^\circ}Y^\circ + \theta(Y^\circ)\{lX^\circ + m\bar{J}^{'}X^\circ\}, \tag{23}$$

for any $X^\circ, Y^\circ \in \Gamma(TM^\circ)$. Since $\bar{\nabla}^\circ$ is torsion free and metric connection, therefore we obtain

$$\begin{aligned}(\bar{\Omega}_{X^\circ}\bar{g}^{'})(Y^\circ, Z^\circ) = &- l\{\theta(Y^\circ)\bar{g}^{'}(X^\circ, Z^\circ) + \theta(Z^\circ)\bar{g}^{'}(X^\circ, Y^\circ)\} \\ &- m\theta(Y^\circ)\bar{g}^{'}(\bar{J}^{'}X^\circ, Z^\circ) - m\theta(Z^\circ)\bar{g}^{'}(\bar{J}^{'}X^\circ, Y^\circ),\end{aligned} \tag{24}$$

$$\bar{T^\circ}(X^\circ, Y^\circ) = l\{\theta(Y^\circ)X^\circ - \theta(X^\circ)Y^\circ\} + m\{\theta(Y^\circ)\bar{J}^{'}X^\circ - \theta(X^\circ)\bar{J}^{'}Y^\circ\},$$

for any $X^\circ, Y^\circ, Z^\circ \in \Gamma(TM^\circ)$, where $\bar{T^\circ}^{\bar{\Omega}}$ is a torsion tensor of the connection $\bar{\Omega}$, where l and m are smooth functions and θ is a 1-form associated with a smooth unit spacelike vector field η, termed as the characteristic vector field, by $\theta(X^\circ) = \bar{g}^{'}(X^\circ, \eta)$. Setting $(l, m) \neq (0, 0)$, $\bar{\Omega}$ is called an (l, m)-type connection. Since $\bar{M}^\circ$ admits a tensor field $\bar{J}^{'}$ of type $(1, 1)$, therefore for any $X^\circ, Y^\circ \in \Gamma(TM^\circ)$, we have

$$\begin{aligned}(\bar{\Omega}_{X^\circ}\bar{J}^{'})(Y^\circ) = l\{&\theta(\bar{J}^{'}Y^\circ)X^\circ - \theta(Y^\circ)\bar{J}^{'}X^\circ\} + m\{\theta(\bar{J}^{'}Y^\circ)\bar{J}^{'}X^\circ \\ &- 3\theta(Y^\circ)\bar{J}^{'}X^\circ - \theta(Y^\circ)X^\circ\},\end{aligned}$$

$$\begin{aligned}\bar{\Omega}_{X^\circ}\bar{J}^{'}Y^\circ = \bar{J}^{'}(\bar{\Omega}_{X^\circ}Y^\circ) &+ l\{\theta(\bar{J}^{'}Y^\circ)X^\circ - \theta(Y^\circ)\bar{J}^{'}X^\circ\} + m\{\theta(\bar{J}^{'}Y^\circ)\bar{J}^{'}X^\circ \\ &- 3\theta(Y^\circ)\bar{J}^{'}X^\circ - \theta(Y^\circ)X^\circ\}.\end{aligned} \tag{25}$$

Let $(M^\circ, g^{'}, S(TM^\circ), S(TM^{\circ\perp}))$ be a SG-lightlike submanifold of a locally bronze semi-Riemannian manifold $(\bar{M}^\circ, \bar{g}^{'})$ with an (l, m) type connection $\bar{\Omega}$. Let Ω be the induced linear connection on M° from $\bar{\Omega}$. Therefore the Gauss formula inherent in its structure is as follows:

$$\bar{\Omega}_{X^\circ}Y^\circ = \Omega_{X^\circ}Y^\circ + \bar{h^\circ}^l(X^\circ, Y^\circ) + \bar{h^\circ}^s(X^\circ, Y^\circ), \tag{26}$$

for any $X^\circ, Y^\circ \in \Gamma(TM^\circ)$, where $\Omega_{X^\circ}Y^\circ \in \Gamma(TM^\circ)$, and $\bar{h^\circ}^l$, $\bar{h^\circ}^s$ are lightlike second fundamental form and the screen second fundamental form of M°, respectively. Now from (3), (17), (23) and (26), we obtain

$$\Omega_{X^\circ}Y^\circ = \nabla^\circ{}_{X^\circ}Y^\circ + l\theta(Y^\circ)X^\circ + m\theta(Y^\circ)f^{'}X^\circ, \tag{27}$$

$$\bar{h^\circ}^l(X^\circ, Y^\circ) = h^{\circ l}(X^\circ, Y^\circ) + m\theta(Y^\circ)w_l^{'}X^\circ, \tag{28}$$

$$\bar{h^\circ}^s(X^\circ, Y^\circ) = h^{\circ s}(X^\circ, Y^\circ) + m\theta(Y^\circ)w_s^{'}X^\circ, \tag{29}$$

Moreover using, (17), (22), (24) and (26), we get,

$$\begin{aligned}(\Omega_{X^\circ}g^{'})(Y^\circ, Z^\circ) = {} & g^{'}(h^{\circ l}(X^\circ, Y^\circ), Z^\circ) + g^{'}(Y^\circ, h^{\circ l}(X^\circ, Z^\circ)) \\ & - l(\theta(Y^\circ)g^{'}(X^\circ, Z^\circ) + \theta(Z^\circ)g^{'}(Y^\circ, X^\circ)) \\ & - m(\theta(Y^\circ)g^{'}(f^{'}X^\circ, Z^\circ) + \theta(Z^\circ)g^{'}(Y^\circ, f^{'}Z^\circ)),\end{aligned} \tag{30}$$

$$T^{\circ\Omega}(X^\circ, Y^\circ) = l\{\theta(Y^\circ)X^\circ - \theta(X^\circ)Y^\circ\} + m\{\theta(Y^\circ)f^{'}X^\circ - \theta(X^\circ)f^{'}Y^\circ\},$$

for any $X^\circ, Y^\circ, Z^\circ \in \Gamma(TM^\circ)$, where $T^{\circ\Omega}$ is torsion tensor of the induced connection Ω on M°. Hence, the following result holds:

Theorem 4.1. *The induced connection Ω on the SG-lightlike submanifold M° of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$ with an (l,m)-type connection $\bar{\Omega}$, is also an (l,m)-type connection.*

Suppose that $\bar{h^\circ}^l$ vanishes identically on M°. Therefore

$$\begin{aligned}(\Omega_{X^\circ}g^{'})(Y^\circ, Z^\circ) = & - l(\theta(Y^\circ)g^{'}(X^\circ, Z^\circ) + \theta(Z^\circ)g^{'}(Y^\circ, X^\circ)) \\ & - m(\theta(Y^\circ)g^{'}(f^{'}X^\circ, Z^\circ) + \theta(Z^\circ)g^{'}(Y^\circ, f^{'}Z^\circ)),\end{aligned} \tag{31}$$

follows from (30).
Consequently, we attain the following result:

Theorem 4.2. *For a SG-lightlike submanifold M° of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$ with an (l,m)-type connection $\bar{\Omega}$, the induced connection Ω on M° is also an (l,m)-type connection if and only if $\bar{h^\circ}^l$ vanishes identically on M° and the characterstic vector field $\zeta \in \Gamma(S(TM^{\circ\perp}))$ such that $\theta(X^\circ) = \bar{g}^{'}(X^\circ, \zeta)$.*

Corresponding to an (l,m)-type connection $\bar{\Omega}$, the Weingarten formulae are given by:

$$\bar{\Omega}_{X^\circ}N^\circ = -\bar{A}_{N^\circ}X^\circ + \bar{\Omega}^l_{X^\circ}N^\circ + \bar{D^\circ}^s(X^\circ, N^\circ), \tag{32}$$

$$\bar{\Omega}_{X^\circ}W^\circ = -\bar{A}_{W^\circ}X^\circ + \bar{\Omega}^s_{X^\circ}W^\circ + \bar{D^\circ}^l(X^\circ, W^\circ), \tag{33}$$

$N^\circ \in \Gamma(ltr(TM^\circ))$ and $W^\circ \in \Gamma(S(TM^{\circ\perp}))$, where $\{\bar{A}_{N^\circ}X^\circ, \bar{A}_{W^\circ}X^\circ\} \in \Gamma(TM^\circ)$, $\{\bar{\Omega}^l_{X^\circ}N^\circ, \bar{D^\circ}^l(X^\circ, W^\circ)\} \in \Gamma(ltr(TM^\circ))$ and $\{\bar{\Omega}^s_{X^\circ}W^\circ, \bar{D^\circ}^s(X^\circ, N^\circ)\} \in \Gamma(S(TM^{\circ\perp}))$. $\bar{\Omega}^l$ and $\bar{\Omega}^s$ are linear connections on $ltr(TM^\circ)$ and $S(TM^{\circ\perp})$, respectively. Both $\bar{A}_{N^\circ}$ and $\bar{A}_{W^\circ}$ are linear operators on $\Gamma(TM^\circ)$.

Now, employing equations (4), (5), (23), (32) and (33), we attain

$$\bar{A}_{N^\circ}X^\circ = A_{N^\circ}X^\circ - l\theta(N^\circ)X^\circ - m\theta(N^\circ)f^{'}X^\circ, \tag{34}$$

$$\bar{\Omega}^l_{X^\circ}N^\circ = \nabla^{\circ l}_{X^\circ}N^\circ + m\theta(N^\circ)w^{'}_l X^\circ,$$

$$\bar{D^\circ}^s(X^\circ, N^\circ) = D^{\circ s}(X^\circ, N^\circ) + m\theta(N^\circ)w^{'}_s X^\circ, \tag{35}$$

$$\bar{A}_{W^\circ}X^\circ = A_{W^\circ}X^\circ - l\theta(W^\circ)X^\circ - m\theta(W^\circ)f^{'}X^\circ, \tag{36}$$

$$\bar{\Omega}^s_{X^\circ}W^\circ = \nabla^s_{X^\circ}W^\circ + m\theta(W^\circ)w^{'}_s X^\circ,$$

$$\bar{D^\circ}^l(X^\circ, W^\circ) = D^{\circ l}(X^\circ, W^\circ) + m\theta(W^\circ)w^{'}_l X^\circ, \tag{37}$$

Now, using equations (6), (7), (29), (35) and (37), we obtain

$$\begin{aligned} \bar{g}^{'}(\bar{h^\circ}^s(X^\circ, Y^\circ), W^\circ) + \bar{g}^{'}(Y^\circ, \bar{D^\circ}^l(X^\circ, W^\circ)) &= \bar{g}^{'}(\bar{A}_{W^\circ}X^\circ, Y^\circ) \\ &+ l\theta(W^\circ)\bar{g}^{'}(X^\circ, Y^\circ) + m\theta(W^\circ)\bar{g}^{'}(f^{'}X^\circ, Y^\circ) \\ &+ m\theta(Y^\circ)\bar{g}^{'}(w^{'}_s X^\circ, W^\circ) + m\theta(W^\circ)\bar{g}^{'}(X^\circ, w^{'}_l X^\circ), \end{aligned} \tag{38}$$

$$\begin{aligned} \bar{g}^{'}(\bar{D^\circ}^s(X^\circ, N^\circ), W^\circ) &= \bar{g}^{'}(\bar{A}_{W^\circ}X^\circ, N^\circ) + l\theta(W^\circ)\bar{g}^{'}(X^\circ, N^\circ) \\ &+ m\theta(W^\circ)\bar{g}^{'}(f^{'}X^\circ, N^\circ) + m\theta(N^\circ)\bar{g}^{'}(w^{'}_s X^\circ, W^\circ), \end{aligned}$$

Let $J^{'}$ be the projection of TM° on $S(TM^\circ)$, then any $X^\circ \in \Gamma(TM^\circ)$ can be written as $X^\circ = J^{'}X^\circ + \Sigma^r_{i=1}\eta_i(X^\circ)\xi^\circ_i$,

$$\eta_i(X^\circ) = g(X^\circ, N^\circ_i),$$

where $\{\xi^\circ_i\}^r_{i=1}$ is a basis for $Rad\, TM^\circ$. The subsequent decomposition w.r.t Ω is

$$\Omega_{X^\circ}J^{'}Y^\circ = \Omega^*_{X^\circ}J^{'}Y^\circ + \bar{h^\circ}^*(X^\circ, J^{'}Y^\circ), \tag{39}$$

$$\Omega_{X^\circ}\xi^\circ = -\bar{A}^*_{\xi^\circ}X^\circ + \bar{\Omega}^{*t}_{X^\circ}\xi^\circ, \tag{40}$$

for any $X^\circ, Y^\circ \in \Gamma(TM^\circ)$, where $\{\Omega^*_{X^\circ}J^{'}Y^\circ, \bar{A}^*_{\xi^\circ}X^\circ\} \in \Gamma(S(TM^\circ))$ and $\{\bar{h^\circ}^*(X^\circ, J^{'}Y^\circ), \bar{\Omega}^{*t}_{X^\circ}\xi^\circ\} \in \Gamma(RadTM^\circ)$.

From (8), (9), (17), (39) and (40), we obtain

$$\Omega^*_{X^\circ}J^{'}Y^\circ = \nabla^{\circ *}_{X^\circ}J^{'}Y^\circ + m\theta(J^{'}Y^\circ)J^{'}f^{'}X^\circ + l\theta(J^{'}Y^\circ)J^{'}X^\circ,$$

$$\begin{aligned} \bar{h^\circ}^*(X^\circ, J^{'}Y^\circ) &= h^{\circ *}(X^\circ, J^{'}Y^\circ) + l\theta(J^{'}Y^\circ)\Sigma^r_{i=1}\eta_i(X^\circ)\xi^\circ_i \\ &+ m\theta(J^{'}Y^\circ)\Sigma^r_{i=1}\eta_i(f^{'}X^\circ)\xi^\circ_i, \end{aligned} \tag{41}$$

$$\bar{A}^*_{\xi^\circ}X^\circ = A^*_{\xi^\circ}X^\circ - l\theta(\xi^\circ)J^{'}X^\circ - m\theta(\xi^\circ)J^{'}f^{'}X^\circ, \tag{42}$$

$$\bar{\Omega}^{*t}_{X^\circ}\xi^\circ = \nabla^{\circ *t}_{X^\circ}\xi^\circ + l\theta(\xi^\circ)\eta(X^\circ)\xi^\circ + m\theta(\xi^\circ)\eta(f^{'}X^\circ)\xi^\circ,$$

Further, using (10), (11), (12), (41) and (42), we derive

$$\begin{aligned} g^{'}(\bar{h^\circ}^l(X^\circ, J^{'}Y^\circ), \xi^\circ) &= g^{'}(\bar{A}^*_{\xi^\circ}X^\circ, J^{'}Y^\circ) + l\theta(\xi^\circ)g^{'}(J^{'}X^\circ, J^{'}Y^\circ) + \\ &m\theta(\xi^\circ)g^{'}(J^{'}f^{'}X^\circ, J^{'}Y^\circ) + m\theta(J^{'}Y^\circ)g^{'}(w^{'}_l X^\circ, \xi^\circ), \end{aligned}$$

$$g^{'}(\bar{h^\circ}^*(X^\circ, J^{'}Y^\circ), N^\circ) = g^{'}(\bar{A}_{N^\circ}X^\circ, J^{'}Y^\circ) + l\theta(N^\circ)g^{'}(X^\circ, J^{'}Y^\circ)$$
$$+ m\theta(N^\circ)g^{'}(f^{'}X^\circ, J^{'}Y^\circ) + m\theta(J^{'}Y^\circ)\eta(f^{'}X^\circ),$$
$$g^{'}(\bar{h^\circ}^l(X^\circ, \xi^\circ), \xi^\circ) = m\theta(\xi^\circ)g^{'}(w_l^{'}X^\circ, \xi^\circ),$$
$$\bar{A}^*_{\xi^\circ}\xi^\circ = -l\theta(\xi^\circ)J^{'}\xi^\circ - m\theta(\xi^\circ)J^{'}f^{'}\xi^\circ,$$

5. SG-LIGHTLIKE SUBMANIFOLDS WITH AN (l, m)-TYPE CONNECTION

Definition 5.1. *"Let M° be a lightlike submanifold of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$. If*

$$\bar{J}'Rad(TM^\circ) = Rad(TM^\circ), \quad \bar{J}'S(TM^\circ) = S(TM^\circ), \tag{43}$$

then M° is an invariant lightlike submanifold of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$."

Proposition 5.2. *A screen Cauchy Riemann (SCR) lightlike submanifold of a locally bronze semi-Riemannian manifold is a SG-lightlike submanifold such that distribution $B^{'}$ is totally anti-invariant , that is,*

$$S(TM^{\circ\perp}) = \omega^{'}B^{'} \oplus \mu^\circ, \tag{44}$$

where $\omega^{'}$ is the normal component of bronze semi-Riemannian structure and μ° is a non-degenerate invariant distribution.

Proposition 5.3. *There is a non-existence of coisotropic , isotropic or totally proper SG-lightlike submanifold M° of a locally bronze semi-Riemannian manifold. If M° is SG-isotropic, coisotropic or totally lightlike submanifold, then it is invariant lightlike submanifold.*

Proof. Let M° be a proper SG-lightlike submanifold of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$. Let M° is isotropic, $S(TM^\circ) = \{0\}$ which implies that $B_o = \{0\}$ and $B^{'} = \{0\}$. Therefore, $\bar{J}'Rad(TM^\circ) = Rad(TM^\circ)$. Hence, $TM^\circ = Rad(TM^\circ) = \bar{J}'Rad(TM^\circ)$ which shows that M° is invariant submanifold with respect to $\bar{J}'$. If M° is coisotropic, then $S(TM^{\circ\perp}) = \{0\}$. Therefore, using (44), we obtain $\mu^\circ = 0$ and $\bar{J}'(B^{'}) = 0$. Also, $TM^\circ = B_o \oplus \bar{J}'(B^{'}) \oplus Rad(TM^\circ)$ and M° is invariant with respect to $\bar{J}'$. Further, if M° is totally lightlike, then $S(TM^\circ) = \{0\}$ and $S(TM^{\circ\perp}) = \{0\}$. Thus, we obtain $TM^\circ = Rad(TM^\circ)$ which proves M° is invariant.
This proves the assertion. □

Example 5.4. *Consider a locally bronze semi-Riemannian manifold $\bar{M}^\circ = (R^{16}_2, \bar{g}^{'})$ of signature $(-, -, +, +, +, +, +, +, +, , +, +, +, +, +, +, +)$ with respect to the basis $\{\partial z^\circ_1, \partial z^\circ_2, \partial z^\circ_3, \partial z^\circ_4, \partial z^\circ_5, \partial z^\circ_6, \partial z^\circ_7, \partial z^\circ_8, \partial z^\circ_9, \partial z^\circ_{10}, \partial z^\circ_{11}, \partial z^\circ_{12}, \partial z^\circ_{13}, \partial z^\circ_{14}, \partial z^\circ_{15}, \partial z^\circ_{16}\}$, where the bronze structure $\bar{J}^{'}$ is defined by*
$\bar{J}'(z^\circ_1, z^\circ_2, z^\circ_3, z^\circ_4, z^\circ_5, z^\circ_6, z^\circ_7, z^\circ_8, z^\circ_9, z^\circ_{10}, z^\circ_{11}, z^\circ_{12}, z^\circ_{13}, z^\circ_{14}, z^\circ_{15}, z^\circ_{16}) = (\sigma z^\circ_1, \sigma z^\circ_2, \sigma z^\circ_3, \sigma z^\circ_4, (3-\sigma)z^\circ_5, \sigma z^\circ_6, \sigma z^\circ_7, \sigma z^\circ_8, (3-\sigma)z^\circ_9, \sigma z^\circ_{10}, (3-\sigma)z^\circ_{11}, \sigma z^\circ_{12}, 3z^\circ_{13} + z^\circ_{14}, z^\circ_{13}, 3z^\circ_{15} + z^\circ_{16}, z^\circ_{15})$.

Let M° be a submanifold $\mathbb{R}^{16}_2$ defined as

$$z^\circ_1 = y^{'}_1 - y^{'}_2,\ z^\circ_2 = \sigma y^{'}_4,$$
$$z^\circ_3 = y^{'}_3 + y^{'}_5,\ z^\circ_4 = y^{'}_2 + y^{'}_3,$$
$$z^\circ_5 = y^{'}_4,\ z^\circ_6 = y^{'}_3 - y^{'}_5,$$
$$z^\circ_7 = y^{'}_2 + y^{'}_1,\ z^\circ_8 = -y^{'}_2 + y^{'}_3,\ z^\circ_9 = -cos\alpha\, y^{'}_6,$$
$$z^\circ_{10} = -cos\alpha\, y^{'}_7,\ z^\circ_{11} = sin\alpha\, y^{'}_6,\ z^\circ_{12} = sin\alpha\, y^{'}_7,$$
$$z^\circ_{13} = -sin y^{'}_7\, cosh y^{'}_8,\ z^\circ_{14} = 0,\ z^\circ_{15} = cos y^{'}_7\, sinh y^{'}_8,\ z^\circ_{16} = 0.$$

Here TM° is spanned by $\{B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9\}$, where

$$B_1 = -\partial z^\circ_1 + \partial z^\circ_4 + \partial z^\circ_7 - \partial z^\circ_8,$$
$$B_2 = \partial z^\circ_3 + \partial z^\circ_4 + \partial z^\circ_6 + \partial z^\circ_8,$$
$$B_3 = \partial z^\circ_1 + \partial z^\circ_7,$$
$$B_4 = \partial z^\circ_3 - \partial z^\circ_6,$$
$$B_5 = \sigma \partial z^\circ_2 + \partial z^\circ_5,$$
$$B_6 = -cos\alpha\, \partial z^\circ_9 + sin\alpha\, \partial z^\circ_{11},$$
$$B_7 = -cos\alpha\, \partial z^\circ_{10} + sin\alpha\, \partial z^\circ_{12},$$
$$B_8 = -cos y^{'}_7\, cosh y^{'}_8\, \partial z^\circ_{13} + sin y^{'}_7\, sinh y^{'}_8\, \partial z^\circ_{15},$$
$$B_9 = cos y^{'}_7\, cosh y^{'}_8\, \partial z^\circ_{15} + sin y^{'}_7\, sinh y^{'}_8\, \partial z^\circ_{13},$$

Therefore, M° is a 2-lightlike submanifold with $Rad(TM^\circ) = Span\{B_1, B_2\}$ such that $\bar{J}' B_1 = \sigma B_1$ and $\bar{J}'(B_2) = \sigma B_2$. This shows that $Rad(TM^\circ)$ is invariant with respect to $\bar{J}'$. Since $\bar{J}' B_3 = \sigma B_3$, $\bar{J}' B_4 = \sigma B_4$, $\bar{J}' B_6 = (3 - \sigma) B_6$ and $\bar{J}' B_7 = \sigma B_7$. Therefore, $B_o = Span\{B_3, B_4, B_6, B_7\}$ is invariant with respect to $\bar{J}'$.

Furthermore, $S(TM^{\circ\perp})$ is spanned by $\{W_1, W_2, W_3, W_4, W_5\}$, where

$$W_1 = sin\alpha\, \partial z^\circ_9 + cos\alpha\, \partial z^\circ_{11},$$
$$W_2 = -\partial z^\circ_2 + \sigma \partial z^\circ_5,$$
$$W_3 = sin\alpha\, \partial z^\circ_{10} + cos\alpha\, \partial z^\circ_{12},$$
$$W_4 = sin y^{'}_7\, sinh y^{'}_8\, \partial z^\circ_{14} + cos y^{'}_7\, cosh y^{'}_8\, \partial z^\circ_{16},$$
$$W_5 = -cos y^{'}_7\, cosh y^{'}_8\, \partial z^\circ_{14} + sin y^{'}_7\, sinh y^{'}_8\, \partial z^\circ_{16},$$

We derive, $\bar{J}' B_5 = 3B_5 - W_2$, $\bar{J}' B_8 = 3B_8 + W_5$ and $\bar{J}' B_9 = 3B_9 + W_4$, which means $B^{'} = Span\{B_5, B_8, B_9\}$, is not invariant with respect to $\bar{J}'$ and $\bar{J}'(B^{'}) \not\subset S(TM^\circ)$ and $\bar{J}'(B^{'}) \not\subset S(TM^{\circ\perp})$.

Therefore, $ltr(TM^\circ)$ is spanned by

$$N_1 = \frac{1}{4}(-\partial z^\circ_1 - \partial z^\circ_4 + \partial z^\circ_7 + \partial z^\circ_8),\ N_2 = \frac{1}{4}(\partial z^\circ_3 - \partial z^\circ_4 + \partial z^\circ_6 - \partial z^\circ_8),$$

which is invariant with respect to $\bar{J}'$. Further, $\bar{J}'W_1 = (3-\sigma)W_1$, $\bar{J}'W_3 = \sigma W_3$, which shows that $\bar{J}'\mu^\circ = \mu^\circ$, where $\mu^\circ = Span\{W_1, W_3\}$ i.e., μ° is invariant with respect to $\bar{J}'$.

Hence, M° become a proper SG-2-lightlike submanifold of R_2^{16}.

Definition 5.5. *Let M° be a SG-lightlike submanifold of a locally bronze semi-Riemannian manifold $(\bar{M}^\circ, \bar{J}', \bar{g}^{'})$ with an (l,m)-type connection $\bar{\Omega}$. Then the distribution B is said to be integrable if and only if $[X^\circ, Y^\circ] \in \Gamma(B)$ for any $X^\circ, Y^\circ \in \Gamma(B)$.*

Remark 5.6. *The above definition implies that the distribution B is said to be integrable if and only if $g^{'}([X^\circ, Y^\circ], Z^\circ) = 0$ for any $X^\circ, Y^\circ \in \Gamma(B)$ and $Z^\circ \in \Gamma(B^\perp)$.*

Theorem 5.7. *Let M° be a SG-lightlike submanifold of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$ with an (l,m)- type connection $\bar{\Omega}$. Then, B_o is integrable if and only if the following conditions hold:*

(i) $g^{'}(\bar{h^\circ}^*(X^\circ, \bar{J}'Y^\circ), \bar{J}'N^\circ) + 3g^{'}(\bar{h^\circ}^*(Y^\circ, \bar{J}'X^\circ), N^\circ) = g^{'}(\bar{h^\circ}^*(Y^\circ, \bar{J}'X^\circ), \bar{J}'N^\circ) + 3g^{'}(\bar{h^\circ}^*(X^\circ, \bar{J}'Y^\circ), N^\circ)$.

(ii) $g^{'}(\Omega^*_{X^\circ}\bar{J}'Y^\circ, f^{'}Z^\circ) + g^{'}(\bar{h^\circ}^s(X^\circ, \bar{J}'Y^\circ), \bar{J}'Z^\circ) + 3g^{'}(\Omega^*_{Y^\circ}\bar{J}'X^\circ, Z^\circ) = g^{'}(\Omega^*_{Y^\circ}\bar{J}'X^\circ, f^{'}Z^\circ) + g^{'}(\bar{h^\circ}^s(Y^\circ, \bar{J}'X^\circ), \bar{J}'Z^\circ) + 3g^{'}(\Omega^*_{X^\circ}\bar{J}'Y^\circ, Z^\circ)$,

for any $X^\circ, Y^\circ \in \Gamma(B_o)$, $Z^\circ \in \Gamma(B^{'})$ and $N^\circ \in \Gamma(ltr(TM^\circ))$.

Proof. The distribution B_o is integrable if and only if
$\bar{g}^{'}([X^\circ, Y^\circ], N^\circ) = 0$, $\quad \bar{g}^{'}([X^\circ, Y^\circ], Z^\circ) = 0$.
Therefore, from equations (23), (13), (21),(14), (25), (26) and (39), we obtain

$$\begin{aligned}
\bar{g}^{'}([X^\circ, Y^\circ], N^\circ) &= \bar{g}^{'}(\bar{\Omega}_{X^\circ}Y^\circ, N^\circ) - l\theta(Y^\circ)\bar{g}^{'}(X^\circ, N^\circ) \\
&\quad - m\theta(Y^\circ)\bar{g}^{'}(\bar{J}'X^\circ, N^\circ) - \bar{g}^{'}(\bar{\Omega}_{Y^\circ}X^\circ, N^\circ) \\
&\quad + l\theta(X^\circ)\bar{g}^{'}(Y^\circ, N^\circ) + m\theta(X^\circ)\bar{g}^{'}(\bar{J}'Y^\circ, N^\circ) \\
&= \bar{g}^{'}(\bar{\Omega}_{X^\circ}\bar{J}^{'}(\bar{J}'Y^\circ), N^\circ) - 3\bar{g}^{'}(\bar{\Omega}_{X^\circ}\bar{J}'Y^\circ, N^\circ) \\
&\quad - \bar{g}^{'}(\bar{\Omega}_{Y^\circ}\bar{J}^{'}(\bar{J}'X^\circ), N^\circ) + 3\bar{g}^{'}(\bar{\Omega}_{Y^\circ}\bar{J}'X^\circ, N^\circ) \\
&= g^{'}(\bar{h^\circ}^*(X^\circ, \bar{J}'Y^\circ) - \bar{h^\circ}^*(Y^\circ, \bar{J}'X^\circ), \bar{J}'N^\circ) \\
&\quad - 3\Big\{g^{'}(\bar{h^\circ}^*(X^\circ, \bar{J}'Y^\circ) - \bar{h^\circ}^*(Y^\circ, \bar{J}'X^\circ), N^\circ)\Big\} \\
&= 0.
\end{aligned}$$

Similarly, on applying (13), (14), (20), (22), (26) and (39), we obtain

$$\begin{aligned}\bar{g}^{'}([X^{\circ},Y^{\circ}],Z^{\circ}) &= \bar{g}^{'}(\bar{\Omega}_{X^{\circ}}Y^{\circ}-\theta(Y^{\circ})\{lX^{\circ}+m\bar{J}'X^{\circ}\},Z^{\circ})\\ &\quad -\bar{g}^{'}(\bar{\Omega}_{Y^{\circ}}X^{\circ}-\theta(X^{\circ})\{lY^{\circ}+m\bar{J}'Y^{\circ}\},Z^{\circ})\\ &= \bar{g}^{'}(\bar{\Omega}_{X^{\circ}}\bar{J}'Y^{\circ},\bar{J}'Z^{\circ})-3\bar{g}^{'}(\bar{\Omega}_{X^{\circ}}\bar{J}'Y^{\circ},Z^{\circ})\\ &\quad +3l\theta(\bar{J}'Y^{\circ})\bar{g}^{'}(X^{\circ},Z^{\circ})-l\theta(\bar{J}'Y^{\circ})\bar{g}^{'}(\bar{J}'X^{\circ},Z^{\circ})\\ &\quad -m\theta(\bar{J}'Y^{\circ})\bar{g}^{'}(X^{\circ},Z^{\circ})-\bar{g}^{'}(\bar{\Omega}_{Y^{\circ}}\bar{J}'X^{\circ},\bar{J}'Z^{\circ})\\ &\quad +3\bar{g}^{'}(\bar{\Omega}_{Y^{\circ}}\bar{J}'X^{\circ},Z^{\circ})-3l\theta(\bar{J}'X^{\circ})\bar{g}^{'}(Y^{\circ},Z^{\circ})\\ &\quad +l\theta(\bar{J}'X^{\circ})\bar{g}^{'}(\bar{J}'Y^{\circ},Z^{\circ})+m\theta(\bar{J}'X^{\circ})\bar{g}^{'}(Y^{\circ},Z^{\circ})\\ &= g^{'}(\Omega^{*}_{X^{\circ}}\bar{J}'Y^{\circ},f^{'}Z^{\circ})+g^{'}(\bar{h^{\circ}}^{s}(X^{\circ},\bar{J}'Y^{\circ}),\bar{J}'Z^{\circ})\\ &\quad +3g^{'}(\Omega^{*}_{Y^{\circ}}\bar{J}'X^{\circ},Z^{\circ})-g^{'}(\Omega^{*}_{Y^{\circ}}\bar{J}'X^{\circ},f^{'}Z^{\circ})\\ &\quad -g^{'}(\bar{h^{\circ}}^{s}(Y^{\circ},\bar{J}'X^{\circ}),\bar{J}'Z^{\circ})-3g^{'}(\Omega^{*}_{X^{\circ}}\bar{J}'Y^{\circ},Z^{\circ})\\ &= 0.\end{aligned}$$

The required assertion is obtained. □

Theorem 5.8. *The distribution $B^{'}$ of a SG-lightlike submanifold M° in a locally bronze semi-Riemannian manifold $\bar{M^{\circ}}$ with an (l,m)-type connection $\bar{\Omega}$ is integrable if and only if*

(i) $\Omega^{*}_{Y^{\circ}}f^{'}Z^{\circ}+\bar{A}_{w'Y^{\circ}}Z^{\circ}+3\Omega^{*}_{Z^{\circ}}Y^{\circ}-\Omega^{*}_{Z^{\circ}}f^{'}Y^{\circ}-\bar{A}_{w'Z^{\circ}}Y^{\circ}-3\Omega^{*}_{Y^{\circ}}Z^{\circ}$ *has no component in* $\Gamma(B_o)$,

(ii) $\bar{A}_{w'Y^{\circ}}Z^{\circ}+\bar{h^{\circ}}^{*}(Y^{\circ},f^{'}Z^{\circ})+3\bar{h^{\circ}}^{*}(Z^{\circ},Y^{\circ}) = \bar{A}_{w'Z^{\circ}}Y^{\circ}+\bar{h^{\circ}}^{*}(Z^{\circ},f^{'}Y^{\circ})+3\bar{h^{\circ}}^{*}(Y^{\circ},Z^{\circ})$,

for any $Y^{\circ},Z^{\circ}\in\Gamma(B^{'})$, $X^{\circ}\in\Gamma(B_o)$, $N^{\circ}\in\Gamma(ltr(TM^{\circ}))$.

Proof. $B^{'}$ is integrable if and only if $\bar{g}^{'}([Y^{\circ},Z^{\circ}],N^{\circ})=0$, $\bar{g}^{'}([Y^{\circ},Z^{\circ}],X^{\circ})=0$. Since, $\bar{\Omega}$ is an (l,m)-type connection, therefore applying equations (14), (15), (25), (26), (32) and (22), we obtain

$$\begin{aligned}\bar{g}^{'}([Y^{\circ},Z^{\circ}],N^{\circ}) &= \bar{g}^{'}(\bar{\Omega}_{Y^{\circ}}Z^{\circ}-l\theta(Z^{\circ})Y^{\circ}-m\theta(Z^{\circ})\bar{J}'Y^{\circ},N^{\circ})\\ &\quad -\bar{g}^{'}(\bar{\Omega}_{Z^{\circ}}Y^{\circ}-l\theta(Y^{\circ})Z^{\circ}-m\theta(Y^{\circ})\bar{J}'Z^{\circ},N^{\circ})\\ &= g^{'}(\Omega_{Y^{\circ}}f^{'}Z^{\circ},\bar{J}'N^{\circ})+g^{'}(-\bar{A}_{w'Z^{\circ}}Y^{\circ},\bar{J}'N^{\circ})\\ &\quad -3g^{'}(\Omega_{Y^{\circ}}Z^{\circ},\bar{J}'N^{\circ})-g^{'}(\Omega_{Z^{\circ}}f^{'}Y^{\circ},\bar{J}'N^{\circ})\\ &\quad -g^{'}(-\bar{A}_{w'Y^{\circ}}Z^{\circ},\bar{J}'N^{\circ})+3g^{'}(\Omega_{Z^{\circ}}Y^{\circ},\bar{J}'N^{\circ})\\ &= 0.\end{aligned}$$

Now, using equation (39), we get

$$g^{'}(\bar{h^{\circ}}^{*}(Y^{\circ},f^{'}Z^{\circ}),\bar{J}'N^{\circ})-g^{'}(\bar{A}_{w'Z^{\circ}}Y^{\circ},\bar{J}'N^{\circ})-3g^{'}(\bar{h^{\circ}}^{*}(Y^{\circ},Z^{\circ}),\bar{J}'N^{\circ})$$

$$- g^{'}(\bar{h^{\circ}}^{*}(Z^{\circ}, f^{'}Y^{\circ}), \bar{J}^{'}N^{\circ}) + g^{'}(\bar{A}_{w^{'}Y^{\circ}}Z^{\circ}, \bar{J}^{'}N^{\circ}) + 3g^{'}(\bar{h^{\circ}}^{*}(Z^{\circ}, Y^{\circ}), \bar{J}^{'}N^{\circ}) = 0.$$

Similarly,

$$\begin{aligned}
\bar{g}^{'}([Y^{\circ}, Z^{\circ}], X^{\circ}) = {} & g^{'}(\Omega^{*}_{Y^{\circ}}f^{'}Z^{\circ}, \bar{J}^{'}X^{\circ}) + g^{'}(\bar{A}_{w^{'}Y^{\circ}}Z^{\circ}, \bar{J}^{'}X^{\circ}) + 3g^{'}(\Omega^{*}_{Z^{\circ}}Y^{\circ}, \bar{J}^{'}X^{\circ}) \\
& + (9m + 3l)\{\theta(Z^{\circ})g^{'}(f^{'}Y^{\circ}, X^{\circ}) - \theta(Y^{\circ})g^{'}(f^{'}Z^{\circ}, X^{\circ})\} \\
& + 3m\{\theta(Z^{\circ})g^{'}(Y^{\circ}, X^{\circ}) - \theta(Y^{\circ})g^{'}(Z^{\circ}, X^{\circ})\} - g^{'}(\Omega^{*}_{Z^{\circ}}f^{'}Y^{\circ}, \bar{J}^{'}X^{\circ}) \\
& - g^{'}(\bar{A}_{w^{'}Z^{\circ}}Y^{\circ}, \bar{J}^{'}X^{\circ}) - 3g^{'}(\Omega^{*}_{Y^{\circ}}Z^{\circ}, \bar{J}^{'}X^{\circ}) \\
& - (l + 3m)\{\theta(\bar{J}^{'}Z^{\circ})g^{'}(f^{'}Y^{\circ}, X^{\circ}) - \theta(\bar{J}^{'}Y^{\circ})g^{'}(f^{'}Z^{\circ}, X^{\circ})\} \\
& - m\{\theta(\bar{J}^{'}Z^{\circ})g^{'}(Y^{\circ}, X^{\circ}) - \theta(\bar{J}^{'}Y^{\circ})g^{'}(Z^{\circ}, X^{\circ})\} \\
= {} & 0.
\end{aligned}$$

Then, from the equations (20) and (21), we get the desired result. □

Theorem 5.9. *For a SG-lightlike submanifold M° of a locally bronze semi-Riemannian manifold with an (l, m)-type connection $\bar{\Omega}$, the distribution B is integrable if and only if*

$$\begin{aligned}
g^{'}(\Omega_{X^{\circ}}\bar{J}^{'}Y^{\circ}, f^{'}Z^{\circ}) + g^{'}(\bar{h^{\circ}}^{s}(X^{\circ}, \bar{J}^{'}Y^{\circ}), \bar{J}^{'}Z^{\circ}) + 3g^{'}(\Omega_{Y^{\circ}}\bar{J}^{'}X^{\circ}, Z^{\circ}) \\
= g^{'}(\Omega_{Y^{\circ}}\bar{J}^{'}X^{\circ}, f^{'}Z^{\circ}) + g^{'}(\bar{h^{\circ}}^{s}(Y^{\circ}, \bar{J}^{'}X^{\circ}), \bar{J}^{'}Z^{\circ}) \\
+ 3g^{'}(\Omega_{X^{\circ}}\bar{J}^{'}Y^{\circ}, Z^{\circ}),
\end{aligned}$$

for any $X^{\circ}, Y^{\circ} \in \Gamma(B)$, $Z^{\circ} \in \Gamma(B^{'})$ and $N^{\circ} \in \Gamma(ltr(TM^{\circ}))$.

Proof. Since $\bar{M}^{\circ}$ is a locally bronze semi-Riemannian manifold equipped with an (l, m)-type connection $\bar{\Omega}$, therefore

$$\begin{aligned}
\bar{g}^{'}([X^{\circ}, Y^{\circ}], Z^{\circ}) = {} & \bar{g}^{'}(\bar{\Omega}_{X^{\circ}}\bar{J}^{'}(\bar{J}^{'}Y^{\circ}), Z^{\circ}) - 3\bar{g}^{'}(\bar{\Omega}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, Z^{\circ}) \\
& - \bar{g}^{'}(\bar{\Omega}_{Y^{\circ}}\bar{J}^{'}(\bar{J}^{'}X^{\circ}), Z^{\circ}) + 3\bar{g}^{'}(\bar{\Omega}_{Y^{\circ}}\bar{J}^{'}X^{\circ}, Z^{\circ}) \\
& - l\theta(Y^{\circ})\bar{g}^{'}(X^{\circ}, Z^{\circ}) - m\theta(Y^{\circ})\bar{g}^{'}(\bar{J}^{'}X^{\circ}, Z^{\circ}) \\
& + l\theta(X^{\circ})\bar{g}^{'}(Y^{\circ}, Z^{\circ}) + m\theta(X^{\circ})\bar{g}^{'}(\bar{J}^{'}Y^{\circ}, Z^{\circ}),
\end{aligned}$$

Now, using (22) and (26), we obtain

$$\begin{aligned}
\bar{g}^{'}([X^{\circ}, Y^{\circ}], Z^{\circ}) = {} & g^{'}(\Omega_{X^{\circ}}\bar{J}^{'}Y^{\circ}, f^{'}Z^{\circ}) + g^{'}(\bar{h^{\circ}}^{s}(X^{\circ}, \bar{J}^{'}Y^{\circ}), w^{'}Z^{\circ}) \\
& + 3l\theta(\bar{J}^{'}Y^{\circ})g^{'}(X^{\circ}, Z^{\circ}) - l\theta(\bar{J}^{'}Y^{\circ})g^{'}(\bar{J}^{'}X^{\circ}, Z^{\circ}) \\
& - m\theta(\bar{J}^{'}Y^{\circ})g^{'}(X^{\circ}, Z^{\circ}) - 3g^{'}(\Omega_{X^{\circ}}\bar{J}^{'}Y^{\circ}, Z^{\circ}) \\
& - g^{'}(\Omega_{Y^{\circ}}\bar{J}^{'}X^{\circ}, f^{'}Z^{\circ}) - g^{'}(\bar{h^{\circ}}^{s}(Y^{\circ}, \bar{J}^{'}X^{\circ}), w^{'}Z^{\circ}) \\
& - 3l\theta(\bar{J}^{'}X^{\circ})g^{'}(Y^{\circ}, Z^{\circ}) + l\theta(\bar{J}^{'}X^{\circ})g^{'}(\bar{J}^{'}Y^{\circ}, Z^{\circ}) \\
& + m\theta(\bar{J}^{'}X^{\circ})g^{'}(Y^{\circ}, Z^{\circ}) + 3g^{'}(\Omega_{Y^{\circ}}\bar{J}^{'}X^{\circ}, Z^{\circ}),
\end{aligned}$$

Since B is integrable, therefore $\bar{g}^{'}([X^{\circ}, Y^{\circ}], Z^{\circ}) = 0$. Also, M° is SG-lightlike submanifold.
Thus,

$$g^{'}(\Omega_{X^{\circ}}\bar{J}^{'}Y^{\circ}, f^{'}Z^{\circ}) + g^{'}(\bar{h^{\circ}}^{s}(X^{\circ}, \bar{J}^{'}Y^{\circ}), \bar{J}^{'}Z^{\circ}) + 3g^{'}(\Omega_{Y^{\circ}}\bar{J}^{'}X^{\circ}, Z^{\circ})$$
$$- g^{'}(\Omega_{Y^{\circ}}\bar{J}^{'}X^{\circ}, f^{'}Z^{\circ}) - g^{'}(\bar{h^{\circ}}^{s}(Y^{\circ}, \bar{J}^{'}X^{\circ}), \bar{J}^{'}Z^{\circ}) - 3g^{'}(\Omega_{X^{\circ}}\bar{J}^{'}Y^{\circ}, Z^{\circ}) = 0.$$

□

Theorem 5.10. *Let M° be a SG-lightlike submanifold of a locally bronze semi-Riemannian manifold $\bar{M}^{\circ}$ with an (l, m)- type connection $\bar{\Omega}$. Then, the distribution B_o is parallel if and only if the following conditions hold:*

(i) $g^{'}(\bar{h^{\circ}}^{*}(X^{\circ}, \bar{J}^{'}Y^{\circ}), \bar{J}^{'}N^{\circ}) = 3g^{'}(\bar{h^{\circ}}^{*}(X^{\circ}, \bar{J}^{'}Y^{\circ}), N^{\circ})$,

(ii) $g^{'}(\Omega^{*}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, f^{'}Z^{\circ}) + g^{'}(\bar{h^{\circ}}^{s}(X^{\circ}, \bar{J}^{'}Y^{\circ}), \bar{J}^{'}Z^{\circ}) = 3g^{'}(\Omega^{*}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, Z^{\circ})$,

for all $X^{\circ}, Y^{\circ} \in \Gamma(B_o)$, $Z^{\circ} \in \Gamma(B^{'})$ and $N^{\circ} \in \Gamma(ltr(TM^{\circ}))$.

Proof. Since the distribution B_o is parallel, therefore for all $X^{\circ}, Y^{\circ} \in \Gamma(B_o)$, $Z^{\circ} \in \Gamma(B^{'})$ and $N^{\circ} \in \Gamma(ltr(TM^{\circ}))$,

$$g^{'}(\Omega_{X^{\circ}}Y^{\circ}, Z^{\circ}) = 0,\ g^{'}(\Omega_{X^{\circ}}Y^{\circ}, N^{\circ}) = 0, \tag{45}$$

The concept of locally bronze semi-Riemannian manifold leads to
$g^{'}(\Omega_{X^{\circ}}Y^{\circ}, N^{\circ}) = \bar{g}^{'}(\bar{\Omega}_{X^{\circ}}\bar{J}^{'}(\bar{J}^{'}Y^{\circ}), N^{\circ}) - 3\bar{g}^{'}(\bar{\Omega}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, N^{\circ})$,
Employing equations (14) and (25), we obtain

$$\begin{aligned} g^{'}(\Omega_{X^{\circ}}Y^{\circ}, N^{\circ}) &= g^{'}(\Omega_{X^{\circ}}\bar{J}^{'}Y^{\circ}, \bar{J}^{'}N^{\circ}) + 3l\theta(\bar{J}^{'}Y^{\circ})g^{'}(X^{\circ}, N^{\circ}) \\ &\quad + l\theta(Y^{\circ})g^{'}(X^{\circ}, N^{\circ}) - l\theta(\bar{J}^{'}Y^{\circ})g^{'}(\bar{J}^{'}X^{\circ}, N^{\circ}) \\ &\quad + m\theta(Y^{\circ})g^{'}(\bar{J}^{'}X^{\circ}, N^{\circ}) - m\theta(\bar{J}^{'}Y^{\circ})g^{'}(X^{\circ}, N^{\circ}) \\ &\quad - 3g^{'}(\Omega_{X^{\circ}}\bar{J}^{'}Y^{\circ}, N^{\circ}), \end{aligned}$$

Equations (39) and (45) leads to

$$\begin{aligned} g^{'}(\Omega_{X^{\circ}}Y^{\circ}, N^{\circ}) &= g^{'}(\Omega^{*}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, \bar{J}^{'}N^{\circ}) + g^{'}(\bar{h^{\circ}}^{*}(X^{\circ}, \bar{J}^{'}Y^{\circ}), \bar{J}^{'}N^{\circ}) \\ &\quad - 3g^{'}(\Omega^{*}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, \bar{J}^{'}N^{\circ}) - 3g^{'}(\bar{h^{\circ}}^{*}(X^{\circ}, \bar{J}^{'}Y^{\circ}), \bar{J}^{'}N^{\circ}) \\ &= 0. \end{aligned}$$

Similarly, from equations (13), (14), (22) and (25), we get

$$\begin{aligned} g^{'}(\Omega_{X^{\circ}}Y^{\circ}, Z^{\circ}) &= \bar{g}^{'}(\bar{\Omega}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, f^{'}Z^{\circ}) + \bar{g}^{'}(\bar{\Omega}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, w^{'}Z^{\circ}) \\ &\quad - 3\bar{g}^{'}(\bar{\Omega}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, Z^{\circ}) + 3l\theta(\bar{J}^{'}Y^{\circ})\bar{g}^{'}(X^{\circ}, Z^{\circ}) \\ &\quad + l\theta(Y^{\circ})\bar{g}^{'}(X^{\circ}, Z^{\circ}) - l\theta(\bar{J}^{'}Y^{\circ})\bar{g}^{'}(\bar{J}^{'}X^{\circ}, Z^{\circ}) \\ &\quad + m\theta(Y^{\circ})\bar{g}^{'}(\bar{J}^{'}X^{\circ}, Z^{\circ}) - m\theta(\bar{J}^{'}Y^{\circ})\bar{g}^{'}(X^{\circ}, Z^{\circ}), \end{aligned}$$

From (26), (39) and (45), we attain
$g^{'}(\Omega_{X^{\circ}}Y^{\circ}, Z^{\circ}) = g^{'}(\Omega^{*}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, f^{'}Z^{\circ}) + g^{'}(\bar{h^{\circ}}^{s}(X^{\circ}, \bar{J}^{'}Y^{\circ}), \bar{J}^{'}Z^{\circ}) - 3g^{'}(\Omega^{*}_{X^{\circ}}\bar{J}^{'}Y^{\circ}, Z^{\circ})$
$= 0$. □

Theorem 5.11. *For a locally bronze semi-Riemannian manifold $\bar{M}^\circ$ with an (l,m)-type connection $\bar{\Omega}$, the distribution $B^{'}$ of a SG-lightlike submanifold M° is parallel if and only if*

(i) $g^{'}(\bar{h^\circ}^*(Y^\circ, f^{'}Z^\circ), \bar{J}^{'}N^\circ) + 3g^{'}(\bar{A}_{w^{'}Z^\circ}Y^\circ, N^\circ) = 3g^{'}(\bar{h^\circ}^*(Y^\circ, f^{'}Z^\circ), N^\circ) + g^{'}(\bar{A}_{w^{'}Z^\circ}Y^\circ, \bar{J}^{'}N^\circ)$,

(ii) $g^{'}(\Omega^*_{Y^\circ}f^{'}Z^\circ, \bar{J}^{'}X^\circ) + 3g^{'}(\bar{A}_{w^{'}Z^\circ}Y^\circ, X^\circ) = g^{'}(\bar{A}_{w^{'}Z^\circ}Y^\circ, \bar{J}^{'}X^\circ) + 3g^{'}(\Omega^*_{Y^\circ}f^{'}Z^\circ, X^\circ)$,

for any $Y^\circ, Z^\circ \in \Gamma(B^{'})$, $X^\circ \in \Gamma(B_o)$ and $N^\circ \in \Gamma(ltr(TM^\circ))$.

Proof. Let the distribution $B^{'}$ is parallel. Therefore, for all $Y^\circ, Z^\circ \in \Gamma(B^{'})$, $X^\circ \in \Gamma(B_o)$ and $N^\circ \in \Gamma(ltr(TM^\circ))$,
$g^{'}(\Omega_{Y^\circ}Z^\circ, X^\circ) = 0$, $g^{'}(\Omega_{Y^\circ}Z^\circ, N^\circ) = 0$,
Now, using equations (13) and (25), we obtain

$$\begin{aligned} g^{'}(\Omega_{Y^\circ}Z^\circ, N^\circ) &= \bar{g}^{'}(\bar{J}^{'}(\bar{\Omega}_{Y^\circ}\bar{J}^{'}Z^\circ), N^\circ) + \bar{g}^{'}(l\{3\theta(\bar{J}^{'}Z^\circ)Y^\circ + \theta(Z^\circ)Y^\circ \\ &\quad - \theta(\bar{J}^{'}Z^\circ)\bar{J}^{'}Y^\circ\}, N^\circ) + \bar{g}^{'}(m\{\theta(Z^\circ)\bar{J}^{'}Y^\circ \\ &\quad - \theta(\bar{J}^{'}Z^\circ)Y^\circ\}, N^\circ) - 3\bar{g}^{'}(\bar{\Omega}_{Y^\circ}\bar{J}^{'}Z^\circ, N^\circ), \end{aligned}$$

Employing (14), (22), (26) and (33), we attain

$$\begin{aligned} g^{'}(\Omega_{Y^\circ}Z^\circ, N^\circ) &= g^{'}(\Omega_{Y^\circ}f^{'}Z^\circ, \bar{J}^{'}N^\circ) + g^{'}(\bar{h^\circ}^l(Y^\circ, f^{'}Z^\circ), \bar{J}^{'}N^\circ) \\ &\quad + g^{'}(\bar{h^\circ}^s(Y^\circ, f^{'}Z^\circ), \bar{J}^{'}N^\circ) + g^{'}(-\bar{A}_{w^{'}Z^\circ}Y^\circ, \bar{J}^{'}N^\circ) \\ &\quad + g^{'}(\bar{\Omega}^s_{Y^\circ}w^{'}Z^\circ, \bar{J}^{'}N^\circ) + g^{'}(\bar{D^\circ}^l(Y^\circ, w^{'}Z^\circ), \bar{J}^{'}N^\circ) \\ &\quad - 3g^{'}(\Omega_{Y^\circ}f^{'}Z^\circ, N^\circ) - 3g^{'}(\bar{h^\circ}^l(Y^\circ, f^{'}Z^\circ), N^\circ) \\ &\quad - 3g^{'}(\bar{h^\circ}^s(Y^\circ, f^{'}Z^\circ), N^\circ) - 3g^{'}(-\bar{A}_{w^{'}Z^\circ}Y^\circ, N^\circ) \\ &\quad - 3g^{'}(\bar{\Omega}^s_{Y^\circ}w^{'}Z^\circ, N^\circ) - 3g^{'}(\bar{D^\circ}^l(Y^\circ, w^{'}Z^\circ), N^\circ), \end{aligned}$$

Further, from equation(39), we get

$$\begin{aligned} g^{'}(\Omega_{Y^\circ}Z^\circ, N^\circ) &= g^{'}(\Omega^*_{Y^\circ}f^{'}Z^\circ, \bar{J}^{'}N^\circ) + g^{'}(\bar{h^\circ}^*(Y^\circ, f^{'}Z^\circ), \bar{J}^{'}N^\circ) \\ &\quad + g^{'}(-\bar{A}_{w^{'}Z^\circ}Y^\circ, \bar{J}^{'}N^\circ) - 3g^{'}(\Omega^*_{Y^\circ}f^{'}Z^\circ, N^\circ) \\ &\quad - 3g^{'}(\bar{h^\circ}^*(Y^\circ, f^{'}Z^\circ), N^\circ) - 3g^{'}(-\bar{A}_{w^{'}Z^\circ}Y^\circ, N^\circ) \\ &= 0. \end{aligned}$$

Similarly,

$$\begin{aligned} g^{'}(\Omega_{Y^\circ}Z^\circ, X^\circ) &= g^{'}(\Omega_{Y^\circ}\bar{J}^{'}Z^\circ, \bar{J}^{'}X^\circ) + 3l\theta(\bar{J}^{'}Z^\circ)g^{'}(Y^\circ, X^\circ) \\ &\quad + l\theta(Z^\circ)g^{'}(Y^\circ, X^\circ) - l\theta(\bar{J}^{'}Z^\circ)g^{'}(\bar{J}^{'}Y^\circ, X^\circ) \\ &\quad + m\theta(Z^\circ)g^{'}(\bar{J}^{'}Y^\circ, X^\circ) - m\theta(\bar{J}^{'}Z^\circ)g^{'}(Y^\circ, X^\circ) \\ &\quad - 3g^{'}(\Omega_{Y^\circ}\bar{J}^{'}Z^\circ, X^\circ), \end{aligned}$$

Equations (26), (33) and (39) leads to

$$\begin{aligned} g^{'}(\Omega_{Y^\circ}Z^\circ, X^\circ) &= g^{'}(\Omega^*_{Y^\circ}f^{'}Z^\circ + \bar{h^\circ}^*(Y^\circ, f^{'}Z^\circ), \bar{J}'X^\circ) + g^{'}(-\bar{A}_{w'Z^\circ}Y^\circ, \bar{J}'X^\circ) \\ &\quad - 3g^{'}(\Omega^*_{Y^\circ}f^{'}Z^\circ + \bar{h^\circ}^*(Y^\circ, f^{'}Z^\circ), X^\circ) - 3g^{'}(-\bar{A}_{w'Z^\circ}Y^\circ, X^\circ) \\ &= 0. \end{aligned}$$

□

Definition 5.12. [11] *M° is a screen B-geodesic SG-lightlike submanifold if its second fundamental form satisfies*

$$h^\circ(X^\circ, Y^\circ) = 0, \quad \forall\, X^\circ, Y^\circ \in \Gamma(B), \tag{46}$$

Remark 5.13. *M° is B-geodesic SG-lightlike submanifold of $\bar{M}^\circ$ endowed with an (l, m)-type connection if*

$$\bar{h^\circ}^l(X^\circ, Y^\circ) = \bar{h^\circ}^s(X^\circ, Y^\circ) = 0,\ \forall\, X^\circ, Y^\circ \in \Gamma(B). \tag{47}$$

Theorem 5.14. *The distribution B of a SG-lightlike submanifold of a locally bronze semi-Riemannian manifold with an (l, m)-type connection $\bar{\Omega}$ defines a totally geodesic foliation in $\bar{M}^\circ$ if and only if M° is B-geodesic and B is parallel with respect to $\bar{\Omega}$ on M°.*

Proof. If B defines totally geodesic foliation in $\bar{M}^\circ$, then for all $X^\circ, Y^\circ \in \Gamma(B)$, $\bar{\Omega}_{X^\circ}Y^\circ \in \Gamma(B)$ and $\bar{g}^{'}(\bar{\Omega}_{X^\circ}Y^\circ, \xi^\circ) = 0$, $\bar{g}^{'}(\bar{\Omega}_{X^\circ}Y^\circ, W^\circ) = 0$, $\bar{g}^{'}(\bar{\Omega}_{X^\circ}Y^\circ, Z^\circ) = 0$, for all $\xi^\circ \in \Gamma(Rad(TM^\circ))$, $Z^\circ \in \Gamma(B^{'})$ and $W^\circ \in \Gamma(S(TM^{\circ\perp}))$.
Using equations (26), (28) and (29), we get

$$\begin{aligned} \bar{g}^{'}(\bar{\Omega}_{X^\circ}Y^\circ, \xi^\circ) &= g^{'}(\bar{h^\circ}^l(X^\circ, Y^\circ), \xi^\circ) \\ &= g^{'}(h^{\circ l}(X^\circ, Y^\circ), \xi^\circ) + m\theta(Y^\circ)g^{'}(w_l^{'}X^\circ, \xi^\circ), \end{aligned}$$

$$\begin{aligned} \bar{g}^{'}(\bar{\Omega}_{X^\circ}Y^\circ, W^\circ) &= g^{'}(\bar{h^\circ}^s(X^\circ, Y^\circ), W^\circ) \\ &= g^{'}(h^{\circ s}(X^\circ, Y^\circ), W^\circ) + m\theta(Y^\circ)g^{'}(w_s^{'}X^\circ, W^\circ), \end{aligned}$$

Since M° is SG-lightlike submanifold of $\bar{M}^\circ$. Therefore, the distribution B is invariant with respect to bronze structure $\bar{J}'$. Hence, $w_l^{'}X^\circ$ and $w_s^{'}X^\circ$ vanishes. Thus, $\bar{h^\circ}^s(X^\circ, Y^\circ) = \bar{h^\circ}^l(X^\circ, Y^\circ) = 0$, which implies that M° is B-geodesic and B is parallel with respect to Ω on M°.

Conversely, let M° be B-geodesic and parallel with respect to Ω on M°. Therefore, $\bar{h^\circ}^s(X^\circ, Y^\circ) = \bar{h^\circ}^l(X^\circ, Y^\circ) = 0\ \ \forall\, X^\circ, Y^\circ \in \Gamma(B)$, which implies that $\bar{\Omega}_{X^\circ}Y^\circ \in \Gamma(B)$.

□

Definition 5.15. [11] *"M° is a mixed geodesic SG-lightlike submanifold of a locally bronze semi-Riemannian manifold if its second fundamental form satisfies*

$$h^\circ(X^\circ, Y^\circ) = 0,\ \forall\, X^\circ \in \Gamma(B),\ Y^\circ \in \Gamma(B^{'})\text{"} \tag{48}$$

Remark 5.16. *If its second fundamental forms satisfies the conditions*

$$\bar{h^\circ}^l(X^\circ, Y^\circ) = \bar{h^\circ}^s(X^\circ, Y^\circ) = 0,\ \forall\ X^\circ \in \Gamma(B),\ Y^\circ \in \Gamma(B^{'}), \tag{49}$$

then M° is said to be mixed geodesic SG-lightlike submanifold with an (l, m)-type connection.

Theorem 5.17. *Let M° be a SG-lightlike submanifold of a locally bronze semi-Riemannian manifold $(\bar{M}^\circ, \bar{g}^{'}, \bar{J}')$ equipped with an (l, m)-type connection $\bar{\Omega}$. Then M° is mixed geodesic if and only if*

(i) $g^{'}(\bar{h^\circ}^l(X^\circ, f^{'}Z^\circ), \xi^\circ) + l\theta(Z^\circ)g^{'}(\bar{J}'X^\circ, \xi^\circ) + 3m\theta(Z^\circ)g^{'}(\bar{J}'X^\circ, \xi^\circ) + m\theta(Z^\circ)g^{'}(X^\circ, \xi^\circ) = -g^{'}(\bar{D^\circ}^l(X^\circ, w^{'}Z^\circ), \xi^\circ) + l\theta(\bar{J}'Z^\circ)g^{'}(X^\circ, \xi^\circ) + m\theta(\bar{J}'Z^\circ)g^{'}(\bar{J}'X^\circ, \xi^\circ),$

(ii) $3g^{'}(\bar{h^\circ}^s(X^\circ, f^{'}Z^\circ) + \bar{\Omega}^s_{X^\circ}w^{'}Z^\circ, W^\circ) + g^{'}(\bar{A}_{w'Z^\circ}X^\circ, f^{'}W^\circ) = g^{'}(\Omega_{X^\circ}f^{'}Z^\circ + \bar{h^\circ}^s(X^\circ, f^{'}Z^\circ), \bar{J}'W^\circ) + g^{'}(\bar{\Omega}^s_{X^\circ}w^{'}Z^\circ, w^{'}W^\circ),$

for all $X^\circ \in \Gamma(B)$, $Z^\circ \in \Gamma(B^{'})$ and $W^\circ \in \Gamma(S(TM^{\circ\perp}))$.

Proof. Since M° is mixed geodesic, therefore
$g^{'}(\bar{h^\circ}^l(X^\circ, Z^\circ), \xi^\circ) = 0, g^{'}(\bar{h^\circ}^s(X^\circ, Z^\circ), W^\circ) = 0$, for all $X^\circ \in \Gamma(B)$, $Z^\circ \in \Gamma(B^{'})$, $\xi^\circ \in \Gamma(Rad(TM^\circ))$ and $W^\circ \in \Gamma(S(TM^{\circ\perp}))$.
Using Gauss formula, we derive
$\bar{g}^{'}(\bar{\Omega}_{X^\circ}Z^\circ, \xi^\circ) = 0,\ \bar{g}^{'}(\bar{\Omega}_{X^\circ}Z^\circ, W^\circ) = 0$
From equations (19), (22) and (25), we infer

$$\begin{aligned}
\bar{g}^{'}(\bar{\Omega}_{X^\circ}Z^\circ, \xi^\circ) &= \bar{g}^{'}(\bar{\Omega}_{X^\circ}Z^\circ, \bar{J}'\xi^\circ) \\
&= \bar{g}^{'}(\bar{\Omega}_{X^\circ}f^{'}Z^\circ, \xi^\circ) + \bar{g}^{'}(\bar{\Omega}_{X^\circ}w^{'}Z^\circ, \xi^\circ) + l\theta(Z^\circ)\bar{g}^{'}(\bar{J}'X^\circ, \xi^\circ) \\
&\quad + 3m\theta(Z^\circ)\bar{g}^{'}(\bar{J}'X^\circ, \xi^\circ) + m\theta(Z^\circ)\bar{g}^{'}(X^\circ, \xi^\circ) - l\theta(\bar{J}'Z^\circ)\bar{g}^{'}(X^\circ, \xi^\circ) \\
&\quad - m\theta(\bar{J}'Z^\circ)\bar{g}^{'}(\bar{J}'X^\circ, \xi^\circ) \\
&= 0,
\end{aligned}$$

Similarly, using equations (13), (14), (22) and (25), we get

$$\begin{aligned}
\bar{g}^{'}(\bar{\Omega}_{X^\circ}Z^\circ, W^\circ) &= \bar{g}^{'}(\bar{\Omega}_{X^\circ}\bar{J}'(\bar{J}'Z^\circ), W^\circ) - 3\bar{g}^{'}(\bar{\Omega}_{X^\circ}\bar{J}'Z^\circ, W^\circ) \\
&= \bar{g}^{'}(\bar{\Omega}_{X^\circ}f^{'}Z^\circ, f^{'}W^\circ) + \bar{g}^{'}(\bar{\Omega}_{X^\circ}f^{'}Z^\circ, w^{'}W^\circ) + \bar{g}^{'}(\bar{\Omega}_{X^\circ}w^{'}Z^\circ, f^{'}W^\circ) \\
&\quad + \bar{g}^{'}(\bar{\Omega}_{X^\circ}w^{'}Z^\circ, w^{'}W^\circ) - 3\bar{g}^{'}(\bar{\Omega}_{X^\circ}f^{'}Z^\circ, W^\circ) - 3\bar{g}^{'}(\bar{\Omega}_{X^\circ}w^{'}Z^\circ, W^\circ),
\end{aligned}$$

Employing equations (26) and (33), we attain

$$\begin{aligned}
\bar{g}^{'}(\bar{\Omega}_{X^\circ}Z^\circ, W^\circ) &= g^{'}(\Omega_{X^\circ}f^{'}Z^\circ, f^{'}W^\circ) + g^{'}(\bar{h^\circ}^s(X^\circ, f^{'}Z^\circ), w^{'}W^\circ) \\
&\quad + g^{'}(-\bar{A}_{w'Z^\circ}X^\circ, f^{'}W^\circ) + g^{'}(\bar{\Omega}^s_{X^\circ}w^{'}Z^\circ, w^{'}W^\circ) \\
&\quad - 3g^{'}(\bar{h^\circ}^s(X^\circ, f^{'}Z^\circ), W^\circ) - 3g^{'}(\bar{\Omega}^s_{X^\circ}w^{'}Z^\circ, W^\circ) \\
&= 0.
\end{aligned}$$

□

Theorem 5.18. *Let M° be a SG-lightlike submanifold of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$ equipped with an (l,m)-type connection $\bar{\Omega}$. Then, for any $X^\circ \in \Gamma(B_o)$, $Z^\circ \in \Gamma(B^{'})$,*
$f^{'}(\Omega_{X^\circ} f^{'} Z^\circ - \bar{A}_{w^{'} Z^\circ} X^\circ) + B^{'}(\bar{h^\circ}^s(X^\circ, f^{'} Z^\circ) + \bar{\Omega}^s_{X^\circ} w^{'} Z^\circ) + 3\bar{A}_{w^{'} Z^\circ} X^\circ + 3l\theta(\bar{J}^{'} Z^\circ) X^\circ + l\theta(Z^\circ) X^\circ + m\theta(Z^\circ)\bar{J}^{'} X^\circ = 3\Omega_{X^\circ} f^{'} Z^\circ + l\theta(\bar{J}^{'} Z^\circ)\bar{J}^{'} X^\circ + m\theta(\bar{J}^{'} Z^\circ) X^\circ + \Omega_{X^\circ} Z^\circ.$

Proof. Using (13), (22) and (25), we get

$$\begin{aligned}
\bar{\Omega}_{X^\circ} Z^\circ = {} & \bar{J}^{'}(\bar{\Omega}_{X^\circ} f^{'} Z^\circ) + \bar{J}^{'}(\bar{\Omega}_{X^\circ} w^{'} Z^\circ) + 3l\theta(\bar{J}^{'} Z^\circ) X^\circ + l\theta(Z^\circ) X^\circ - l\theta(\bar{J}^{'} Z^\circ)\bar{J}^{'} X^\circ \\
& + m\theta(Z^\circ)\bar{J}^{'} X^\circ - m\theta(\bar{J}^{'} Z^\circ) X^\circ - 3\bar{\Omega}_{X^\circ} f^{'} Z^\circ - 3\bar{\Omega}_{X^\circ} w^{'} Z^\circ,
\end{aligned}$$

From (17), (18), (26) and (33), we obtain

$$\begin{aligned}
\bar{\Omega}_{X^\circ} Z^\circ = {} & f^{'}(\Omega_{X^\circ} f^{'} Z^\circ) + w^{'}(\Omega_{X^\circ} f^{'} Z^\circ) + B^{'}(\bar{h^\circ}^l(X^\circ, f^{'} Z^\circ)) + C^{'}(\bar{h^\circ}^l(X^\circ, f^{'} Z^\circ)) \\
& + B^{'}(\bar{h^\circ}^s(X^\circ, f^{'} Z^\circ)) + C^{'}(\bar{h^\circ}^s(X^\circ, f^{'} Z^\circ)) + f^{'}(-\bar{A}_{w^{'} Z^\circ} X^\circ) \\
& + w^{'}(-\bar{A}_{w^{'} Z^\circ} X^\circ) + B^{'}(\bar{\Omega}^s_{X^\circ} w^{'} Z^\circ) + C^{'}(\bar{\Omega}^s_{X^\circ} w^{'} Z^\circ) + B^{'}(\bar{D^\circ}^l(X^\circ, w^{'} Z^\circ)) \\
& + C^{'}(\bar{D^\circ}^l(X^\circ, w^{'} Z^\circ)) + 3l\theta(\bar{J}^{'} Z^\circ) X^\circ + l\theta(Z^\circ) X^\circ - l\theta(\bar{J}^{'} Z^\circ)\bar{J}^{'} X^\circ \\
& + m\theta(Z^\circ)\bar{J}^{'} X^\circ - m\theta(\bar{J}^{'} Z^\circ) X^\circ - 3\Omega_{X^\circ} f^{'} Z^\circ - 3\bar{h^\circ}^l(X^\circ, f^{'} Z^\circ) \\
& - 3\bar{h^\circ}^s(X^\circ, f^{'} Z^\circ) + 3\bar{A}_{w^{'} Z^\circ} X^\circ - 3\bar{\Omega}^s_{X^\circ} w^{'} Z^\circ - 3\bar{D^\circ}^l(X^\circ, w^{'} Z^\circ),
\end{aligned}$$

Comparing tangential parts in the above equation,

$$\begin{aligned}
\Omega_{X^\circ} Z^\circ = {} & f^{'}(\Omega_{X^\circ} f^{'} Z^\circ - \bar{A}_{w^{'} Z^\circ} X^\circ) + B^{'}(\bar{h^\circ}^s(X^\circ, f^{'} Z^\circ) + \bar{\Omega}^s_{X^\circ} w^{'} Z^\circ) + 3\bar{A}_{w^{'} Z^\circ} X^\circ \\
& + 3l\theta(\bar{J}^{'} Z^\circ) X^\circ + l\theta(Z^\circ) X^\circ + m\theta(Z^\circ)\bar{J}^{'} X^\circ - 3\Omega_{X^\circ} f^{'} Z^\circ - l\theta(\bar{J}^{'} Z^\circ)\bar{J}^{'} X^\circ \\
& - m\theta(\bar{J}^{'} Z^\circ) X^\circ.
\end{aligned}$$

□

Theorem 5.19. *For a locally bronze semi-Riemannian manifold $(\bar{M}^\circ, \bar{g}^{'}, \bar{J}^{'})$ with an (l,m)-type connection $\bar{\Omega}$, a SG-lightlike submanifold M° is mixed geodesic if and only if the following conditions hold*

(i) $C^{'}(\bar{h^\circ}^l(X^\circ, f^{'} Z^\circ) + \bar{D^\circ}^l(X^\circ, w^{'} Z^\circ)) = 0,$
(ii) $w^{'}(\Omega_{X^\circ} f^{'} Z^\circ - \bar{A}_{w^{'} Z^\circ} X^\circ) = -C^{'}(\bar{h^\circ}^s(X^\circ, f^{'} Z^\circ) + \bar{\Omega}^s_{X^\circ} w^{'} Z^\circ),$
(iii) $\bar{h^\circ}^s(X^\circ, f^{'} Z^\circ) + \bar{D^\circ}^l(X^\circ, w^{'} Z^\circ) = -\bar{h^\circ}^l(X^\circ, f^{'} Z^\circ) - \bar{\Omega}^s_{X^\circ} w^{'} Z^\circ,$

for any $X^\circ \in \Gamma(B)$, $Z^\circ \in \Gamma(B^{'})$.

Proof. Let M° be a mixed geodesic SG-lightlike submanifold.
From (1), (13), (22) and (23), we get

$$\begin{aligned}
h^\circ(X^\circ, Z^\circ) = {} & \bar{J}^{'}(\bar{\Omega}_{X^\circ} f^{'} Z^\circ) + \bar{J}^{'}(\bar{\Omega}_{X^\circ} w^{'} Z^\circ) + 3l\theta(\bar{J}^{'} Z^\circ) X^\circ + l\theta(Z^\circ) X^\circ \\
& - l\theta(\bar{J}^{'} Z^\circ)\bar{J}^{'} X^\circ + m\theta(Z^\circ)\bar{J}^{'} X^\circ - m\theta(\bar{J}^{'} Z^\circ) X^\circ \\
& - 3\bar{\Omega}_{X^\circ} f^{'} Z^\circ - 3\bar{\Omega}_{X^\circ} w^{'} Z^\circ - \Omega_{X^\circ} Z^\circ,
\end{aligned}$$

Employing (17), (18), (26) and (33) on above equation and comparing the transversal parts, we obtain

$$\begin{aligned}h^\circ(X^\circ,Z^\circ) = &\, w^{'}(\Omega_{X^\circ}f^{'}Z^\circ) + C^{'}(\bar{h^\circ}^{l}(X^\circ,f^{'}Z^\circ)) + C^{'}(\bar{h^\circ}^{s}(X^\circ,f^{'}Z^\circ))\\ &+ w^{'}(-\bar{A}_{w^{'}Z^\circ}X^\circ) + C^{'}(\bar{\Omega}^{s}_{X^\circ}w^{'}Z^\circ) + C^{'}(\bar{D^\circ}^{l}(X^\circ,f^{'}Z^\circ))\\ &- 3\bar{h^\circ}^{l}(X^\circ,f^{'}Z^\circ) - 3\bar{h^\circ}^{s}(X^\circ,f^{'}Z^\circ)\\ &- 3\bar{\Omega}^{s}_{X^\circ}w^{'}Z^\circ - 3\bar{D^\circ}^{l}(X^\circ,w^{'}Z^\circ)\\ = &\, 0.\end{aligned}$$

□

6. TOTALLY UMBILICAL SG-LIGHTLIKE SUBMANIFOLDS

Definition 6.1. [16] *"Consider a lightlike submanifold $(M^\circ, g^{'})$ of a semi-Riemannian manifold $(\bar{M}^\circ, \bar{g}^{'})$ is said to be totally umbilical in $\bar{M}^\circ$ if there exist a smooth transversal vector field $H^\circ \in \Gamma(ltr(TM^\circ))$ on M°, called the transversal curvature vector field of M°, such that, for all $X^\circ, Y^\circ \in \Gamma(TM^\circ)$,*

$$h^\circ(X^\circ,Y^\circ) = H^\circ g^{'}(X^\circ,Y^\circ),\text{''} \tag{50}$$

The Gauss and Weingarten equations for $\bar{M}^\circ$ implies that M° is totally umbilical if and only if on each coordinate neighborhood U°, there exist smooth vector fields $H^{\circ l} \in \Gamma(ltr(TM^\circ))$ and $H^{\circ s} \in \Gamma(S(TM^{\circ\perp}))$, such that

$$h^{\circ l}(X^\circ,Y^\circ) = H^{\circ l}g^{'}(X^\circ,Y^\circ),\ h^{\circ s}(X^\circ,Y^\circ) = H^{\circ s}g^{'}(X^\circ,Y^\circ),\ D^{\circ l}(X^\circ,W^\circ) = 0, \tag{51}$$

for all $X^\circ, Y^\circ \in \Gamma(TM^\circ), W^\circ \in \Gamma(S(TM^{\circ\perp}))$.

Lemma 6.2. *Let M° be a totally umbilical proper SG-lightlike submanifold of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$ with an (l,m)-type connection $\bar{\Omega}$. Then,*

(i) $\Omega_{X^\circ}f^{'}Y^\circ - \bar{A}_{w_l^{'}Y^\circ}X^\circ - \bar{A}_{w_s^{'}Y^\circ}X^\circ - l\theta(\bar{J}^{'}Y^\circ)X^\circ - m\theta(\bar{J}^{'}Y^\circ)f^{'}X^\circ$
$= f^{'}\Omega_{X^\circ}Y^\circ + B^{'}\bar{h^\circ}^{s}(X^\circ,Y^\circ) - l\theta(Y^\circ)f^{'}X^\circ - 3m\theta(Y^\circ)f^{'}X^\circ - m\theta(Y^\circ)X^\circ,$

(ii) $\bar{h^\circ}^{l}(X^\circ,f^{'}Y^\circ) + \bar{\Omega}^{l}_{X^\circ}w_l^{'}Y^\circ - m\theta(\bar{J}^{'}Y^\circ)w_l^{'}X^\circ$
$= w_l^{'}\Omega_{X^\circ}Y^\circ + C^{'}\bar{h^\circ}^{l}(X^\circ,Y^\circ) - l\theta(Y^\circ)w_l^{'}X^\circ - 3m\theta(Y^\circ)w_l^{'}X^\circ,$

(iii) $\bar{h^\circ}^{s}(X^\circ,f^{'}Y^\circ) + \bar{D^\circ}^{s}(X^\circ,w_l^{'}Y^\circ) + \bar{\Omega}^{s}_{X^\circ}w_s^{'}Y^\circ - m\theta(\bar{J}^{'}Y^\circ)w_s^{'}X^\circ$
$= w_s^{'}\Omega_{X^\circ}Y^\circ + C^{'}\bar{h^\circ}^{s}(X^\circ,Y^\circ) - l\theta(Y^\circ)w_s^{'}X^\circ - 3m\theta(Y^\circ)w_s^{'}X^\circ,$

for any $X^\circ, Y^\circ \in \Gamma(TM^\circ)$.

Proof. Since $\bar{M}^\circ$ is a locally bronze semi-Riemannian manifold, it implies that $\bar{\nabla}^\circ_{X^\circ}\bar{J}^{'}Y^\circ = \bar{J}^{'}(\bar{\nabla}^\circ_{X^\circ}Y^\circ)$,
Following equation (23), we obtain
$\bar{\Omega}_{X^\circ}\bar{J}^{'}Y^\circ - l\theta(\bar{J}^{'}Y^\circ)X^\circ - m\theta(\bar{J}^{'}Y^\circ)\bar{J}^{'}X^\circ = \bar{J}^{'}(\bar{\Omega}_{X^\circ}Y^\circ) - l\theta(Y^\circ)\bar{J}^{'}X^\circ - m\theta(Y^\circ)\bar{J}^{'2}X^\circ,$

Using equations (13), (17) and (18), we get

$$
\begin{aligned}
&\bar{\Omega}_{X^\circ} f^{'} Y^\circ + \bar{\Omega}_{X^\circ} w_l^{'} Y^\circ + \bar{\Omega}_{X^\circ} w_s^{'} Y^\circ - l\theta(\bar{J}^{'} Y^\circ) X^\circ \\
&\qquad - m\theta(\bar{J}^{'} Y^\circ) f^{'} X^\circ - m\theta(\bar{J}^{'} Y^\circ) w_l^{'} X^\circ - m\theta(\bar{J}^{'} Y^\circ) w_s^{'} X^\circ \\
&\qquad = \bar{J}^{'}(\bar{\Omega}_{X^\circ} Y^\circ) - l\theta(Y^\circ) f^{'} X^\circ - l\theta(Y^\circ) w_l^{'} X^\circ - l\theta(Y^\circ) w_s^{'} X^\circ \\
&\qquad - 3m\theta(Y^\circ) f^{'} X^\circ - 3m\theta(Y^\circ) w_l^{'} X^\circ - 3m\theta(Y^\circ) w_s^{'} X^\circ - m\theta(Y^\circ) X^\circ,
\end{aligned}
$$

From equations (26), (32), (33) and (51), we attain

$$
\begin{aligned}
&\Omega_{X^\circ} f^{'} Y^\circ + \bar{h^\circ}^l(X^\circ, f^{'} Y^\circ) + \bar{h^\circ}^s(X^\circ, f^{'} Y^\circ) - \bar{A}_{w_l^{'} Y^\circ} X^\circ + \bar{\Omega}^l_{X^\circ} w_l^{'} Y^\circ \\
&\qquad + \bar{D^\circ}^s(X^\circ, w_l^{'} Y^\circ) - \bar{A}_{w_s^{'} Y^\circ} X^\circ + \bar{\Omega}^l_{X^\circ} w_s^{'} Y^\circ - l\theta(\bar{J}^{'} Y^\circ) X^\circ \\
&\qquad - m\theta(\bar{J}^{'} Y^\circ) f^{'} X^\circ - m\theta(\bar{J}^{'} Y^\circ) w_l^{'} X^\circ - m\theta(\bar{J}^{'} Y^\circ) w_s^{'} X^\circ \\
&\qquad = f^{'} \Omega_{X^\circ} Y^\circ + w_l^{'} \Omega_{X^\circ} Y^\circ + w_s^{'} \Omega_{X^\circ} Y^\circ + C^{'} \bar{h^\circ}^l(X^\circ, Y^\circ) \\
&\qquad + B^{'} \bar{h^\circ}^s(X^\circ, Y^\circ) + C^{'} \bar{h^\circ}^s(X^\circ, Y^\circ) - l\theta(Y^\circ) f^{'} X^\circ - l\theta(Y^\circ) w_l^{'} X^\circ \\
&\qquad - l\theta(Y^\circ) w_s^{'} X^\circ - 3m\theta(Y^\circ) f^{'} X^\circ - 3m\theta(Y^\circ) w_l^{'} X^\circ - 3m\theta(Y^\circ) w_s^{'} X^\circ \\
&\qquad - m\theta(Y^\circ) X^\circ,
\end{aligned}
$$

Thus the required result is obtained by comparing the tangential and normal parts of above equation. □

Theorem 6.3. *If M° is a totally umbilical proper SG-lightlike submanifold of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$ with an (l,m)-type connection $\bar{\Omega}$, then $H^{\circ s} \notin \Gamma(\mu^\circ)$.*

Proof. Let M° be a totally umbilical proper SG-lightlike submanifold of a locally bronze semi-Riemannian manifold, then $\bar{\nabla}^\circ_{X^\circ} \bar{J}^{'} Y^\circ = \bar{J}^{'}(\bar{\nabla}^\circ_{X^\circ} Y^\circ)$.
Now, using equations (13), (17), (18), (23), (26) and comparing transversal parts, we obtain
$h^{\circ s}(X^\circ, \bar{J}^{'} X^\circ) + m\theta(\bar{J}^{'} X^\circ) w_s^{'} X^\circ = C^{'} h^{\circ s}(X^\circ, X^\circ) + C^{'} m\theta(X^\circ) w_s^{'} X^\circ, \forall\ X^\circ, Y^\circ \in \Gamma(B)$,
From the concept of totally umbilical submanifold M° of $\bar{M}^\circ$, we get
$g^{'}(X^\circ, \bar{J}^{'} X^\circ) H^{\circ s} + m\theta(\bar{J}^{'} X^\circ) w_s^{'} X^\circ = g^{'}(X^\circ, X^\circ) C^{'} H^{\circ s} + C^{'} m\theta(X^\circ) w_s^{'} X^\circ$,
Since M° is a SG-lightlike submanifold of $\bar{M}^\circ$, it is inferred that the distribution B is invariant with respect to $\bar{J}^{'}$. Also, if we take $X^\circ = Y^\circ$, then $g^{'}(X^\circ, X^\circ) C^{'} H^{\circ s} = 0$, $w_s^{'} X^\circ = 0$.
Hence, $H^{\circ s} \notin \Gamma(\mu^\circ)$. □

Theorem 6.4. *Let M° be a totally umbilical proper SG-lightlike submanifold of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$ with an (l,m)-type connection $\bar{\Omega}$. Then the necessary and sufficient condition for induced connection to be metric is $l\theta(Y^\circ) g^{'}(X^\circ, Z^\circ) + m\theta(Y^\circ) g^{'}(f^{'} X^\circ, Z^\circ) = -l\theta(Z^\circ) g^{'}(Y^\circ, X^\circ) - m\theta(Z^\circ) g^{'}(Y^\circ, f^{'} Z^\circ)$, $\forall\ X^\circ, Y^\circ, Z^\circ \in \Gamma(B_o)$.*

Proof. Lemma (6.2) implies
$\bar{h^{\circ}}^{l}(X^{\circ}, f^{'}Y^{\circ}) + \bar{\Omega}^{l}_{X^{\circ}} w^{'}_{l} Y^{\circ} - m\theta(\bar{J}^{'}Y^{\circ}) w^{'}_{l} X^{\circ} = w^{'}_{l} \Omega_{X^{\circ}} Y^{\circ} + C^{'} \bar{h^{\circ}}^{l}(X^{\circ}, Y^{\circ}) - l\theta(Y^{\circ}) w^{'}_{l} X^{\circ} - 3m\theta(Y^{\circ}) w^{'}_{l} X^{\circ}$,
Since M° is a totally umbillical proper SG-lightlike submanifold, therefore
$h^{\circ l}(X^{\circ}, \bar{J}^{'}Y^{\circ}) + m\theta(\bar{J}^{'}Y^{\circ}) w^{'}_{l} X^{\circ} = C^{'} h^{\circ l}(X^{\circ}, Y^{\circ}) + C^{'} m\theta(Y^{\circ}) w^{'}_{l} X^{\circ}$,
Since the distribution B_o is invariant with respect to bronze structure $\bar{J}^{'}$, we have $w^{'}_{l} X^{\circ} = 0$. Also, from equations (50) and (51), we attain
$g^{'}(X^{\circ}, \bar{J}^{'}Y^{\circ}) H^{\circ l} = g^{'}(X^{\circ}, Y^{\circ}) C^{'} H^{\circ l}$, which leads to $2g^{'}(X^{\circ}, \bar{J}^{'}Y^{\circ}) H^{\circ l} = 0$.
If we take $X^{\circ} = \bar{J}^{'}Y^{\circ}$, then $H^{\circ l} = 0$, so equation (51) implies that $h^{\circ l} = 0$,
Employing equation (30), we have
$(\Omega_{X^{\circ}} g^{'})(Y^{\circ}, Z^{\circ}) = -l(\theta(Y^{\circ}) g^{'}(X^{\circ}, Z^{\circ}) + \theta(Z^{\circ}) g^{'}(Y^{\circ}, X^{\circ})) - m(\theta(Y^{\circ}) g^{'}(f^{'}X^{\circ}, Z^{\circ}) + \theta(Z^{\circ}) g^{'}(Y^{\circ}, f^{'}Z^{\circ})) = 0.$
□

7. MINIMAL SG-LIGHTLIKE SUBMANIFOLDS

Definition 7.1. [17] *A lightlike submanifold* $(M^{\circ}, g^{'}, S(TM^{\circ}), S(TM^{\circ\perp}))$, *of a semi-Riemannian manifold* $(\bar{M}^{\circ}, \bar{g}^{'})$, *is said to be minimal if :*
(i) $h^{\circ s} = 0$ *on* $Rad(TM^{\circ})$ *and*
(ii) trace $h^{\circ} = 0$, *where trace is written with respect to* $g^{'}$ *restricted to* $S(TM^{\circ})$.

The above definition is independent of $S(TM^{\circ})$ and $S(TM^{\circ\perp})$, but is dependent on $tr(TM^{\circ})$.

Example 7.2. *let* $\bar{M}^{\circ} = \mathbb{R}^{11}_{1}$ *be a semi-Riemannian space of signature* $(-,+,+,+,+,+,+,+,+,+,+)$, *with canonical basis*

$$\{\partial y^{\circ}_1, \partial y^{\circ}_2, \partial y^{\circ}_3, \partial y^{\circ}_4, \partial y^{\circ}_5, \partial y^{\circ}_6, \partial y^{\circ}_7, \partial y^{\circ}_8, \partial y^{\circ}_9, \partial y^{\circ}_{10}, \partial y^{\circ}_{11}\}.$$

Let M° *be a submanifold of* $(\mathbb{R}^{11}_{1}, \bar{J}^{'})$ *given by*

$$y^{\circ}_1 = 0,\ y^{\circ}_2 = \frac{1}{2}(\sqrt{3}t^{'}_4 + t^{'}_2),$$

$$y^{\circ}_3 = \frac{1}{2}(-\sqrt{3}t^{'}_2 + t^{'}_4),\ y^{\circ}_4 = t^{'}_1,\ y^{\circ}_5 = t^{'}_4,$$

$$y^{\circ}_6 = sint^{'}_5\ sinht^{'}_6,\ y^{\circ}_7 = 0,\ y^{\circ}_8 = sint^{'}_5\ cosht^{'}_6,$$

$$y^{\circ}_9 = 0,\ y^{\circ}_{10} = \sqrt{2}\ cost^{'}_5\ cosht^{'}_6,\ y^{\circ}_{11} = 0.$$

Consider the bronze structure $\bar{J}^{'}$ *defined by* $\bar{J}^{'}(y^{\circ}_1, y^{\circ}_2, y^{\circ}_3, y^{\circ}_4, y^{\circ}_5, y^{\circ}_6, y^{\circ}_7, y^{\circ}_8, y^{\circ}_9, y^{\circ}_{10}, y^{\circ}_{11}) = (\omega y^{\circ}_1, (3-\omega) y^{\circ}_2, (3-\omega) y^{\circ}_3, \omega y^{\circ}_4, (3-\omega) y^{\circ}_5, 3y^{\circ}_6 + y^{\circ}_7, y^{\circ}_6, 3y^{\circ}_8 + y^{\circ}_9, y^{\circ}_8, 3y^{\circ}_{10} + y^{\circ}_{11}, y^{\circ}_{10})$.

Then, TM° is spanned by $\{B_1, B_2, B_3, B_4, B_5\}$, where

$$
\begin{aligned}
B_1 &= \frac{1}{2}(-\sqrt{3}\partial y_3^\circ + \partial y_2^\circ),\\
B_2 &= \partial y_4^\circ,\\
B_3 &= \partial y_5^\circ + \frac{1}{2}(\sqrt{3}\partial y_2^\circ + \partial y_3^\circ),\\
B_4 &= cost_5^{'}\ sinht_6^{'}\ \partial y_6^\circ + cost_5^{'}\ cosht_6^{'}\ \partial y_8^\circ - \sqrt{2}\ sint_5^{'}\ cosht_6^{'}\ \partial y_{10}^\circ,\\
B_5 &= sint_5^{'}\ cosht_6^{'}\ \partial y_6^\circ + sint_5^{'}\ sinht_6^{'}\ \partial y_8^\circ + \sqrt{2}\ cost_5^{'}\ sinht_6^{'}\ \partial y_{10}^\circ.
\end{aligned}
$$

where $Rad(TM^\circ) = Span\{B_3\}$ and $B_o = Span\{B_1, B_2\}$. It is proved that $\bar{J}'B_3 = (3-\omega)B_3$, which shows that $Rad(TM^\circ)$ is invariant under $\bar{J}'$. Since, $\bar{J}'B_1 = (3-\omega)B_1$, $\bar{J}'B_2 = \omega B_1$. Therefore, $B_o = Span\{B_1, B_2\}$, is invariant under $\bar{J}'$.

Upon calculation, we derive that $ltr(TM^\circ)$ is spanned by

$$N = \frac{1}{2}\{-\partial y_5^\circ + \frac{1}{2}\partial y_3^\circ + \frac{\sqrt{3}}{2}\partial y_2^\circ\},$$

such that $\bar{J}'N = (3-\omega)N$, which implies that $ltr(TM^\circ)$ is invariant under $\bar{J}'$. Also, $S(TM^{\circ\perp})$ is spanned by

$$
\begin{aligned}
W_1 &= \partial y_1^\circ,\\
W_2 &= cost_5^{'}\ sinht_6^{'}\ \partial y_7^\circ + cost_5^{'}\ cosht_6^{'}\ \partial y_9^\circ - \sqrt{2}\ sint_5^{'}\ cosht_6^{'}\ \partial y_{11}^\circ,\\
W_3 &= sint_5^{'}\ cosht_6^{'}\ \partial y_7^\circ + sint_5^{'}\ sinht_6^{'}\ \partial y_9^\circ + \sqrt{2}\ cost_5^{'}\ sinht_6^{'}\ \partial y_{11}^\circ,\\
W_4 &= -\sqrt{2}\ sinht_6^{'}\ cosht_6^{'}\ \partial y_7^\circ + \sqrt{2}\ (sin^2t_5^{'} + sinh^2t_6^{'})\ \partial y_9^\circ + sint_5^{'}\ cost_5^{'}\ \partial y_{11}^\circ,\\
W_5 &= -\sqrt{2}\ sinht_6^{'}\ cosht_6^{'}\ \partial y_6^\circ + \sqrt{2}\ (sin^2t_5^{'} + sinh^2t_6^{'})\ \partial y_8^\circ + sint_5^{'}\ cost_5^{'}\ \partial y_{10}^\circ,
\end{aligned}
$$

Since, $\bar{J}'W_4 = W_5$, $\bar{J}'W_5 = 3W_5 + W_4$, we see that $\mu^\circ = Span\{W_4, W_5\}$ is invariant under $\bar{J}'$. Now, $B^{'} = Span\{B_4, B_5\}$ such that $\bar{J}'B_4 = 3B_4 + W_2$ and $\bar{J}'B_5 = 3B_5 + W_3$, which shows that $\bar{J}'(B^{'}) \not\subset S(TM^\circ)$ and $\bar{J}'(B^{'}) \not\subset S(TM^{\circ\perp})$.
Therefore, M° is a proper SG-1-lightlike submanifold of $\mathbb{R}_1^{11}$.
Also, we derive upon calculation

$$\bar{\nabla}^\circ_{B_i}B_j = 0,\ i = 1,2,3,\ 1 \le j \le 5,\ and\ h^{\circ l}(B_4, B_4) = 0,\ h^{\circ l}(B_5, B_5) = 0,$$

$$h^{\circ s}(B_4, B_4) = -\frac{\sqrt{2}\ sint_5^{'}\ cosht_6^{'}}{(sin^2t_5^{'} + 2sinh^2t_6^{'})(1 + sin^2t_5^{'} + 2sinh^2t_6^{'})}W_5,$$

$$h^{\circ s}(B_5, B_5) = \frac{\sqrt{2}\ sint_5^{'}\ cosht_6^{'}}{(sin^2t_5^{'} + 2sinh^2t_6^{'})(1 + sin^2t_5^{'} + 2sinh^2t_6^{'})}W_5,$$

Hence, for all $X^\circ \in \Gamma(TM^\circ)$, $h^{\circ s}(X^\circ, B_3) = 0$,
That is $h^{\circ s} = 0$ on $Rad(TM^\circ)$ and $traceh^\circ|_{S(TM^\circ)} = h^{\circ s}(B_4, B_4) + h^{\circ s}(B_5, B_5) = 0$.
Therefore, M° is minimal proper SG-lightlike submanifold of $\mathbb{R}_1^{11}$.

Theorem 7.3. *The distribution B_o in a SG-lightlike submanifold M° of a locally bronze semi-Riemannian manifold $(\bar{M}^\circ, \bar{g}^{'}, \bar{J}')$ with an (l,m)-type connection $\bar{\Omega}$ is minimal if and only if*

$$3\{g^{'}(f^{'}X^\circ, \bar{A}_{W^\circ}f^{'}X^\circ) + l\theta(W^\circ)g^{'}(X^\circ, f^{'^2}X^\circ) + m\theta(W^\circ)g^{'}(f^{'}X^\circ, f^{'^2}X^\circ)\}$$
$$= -2\{g^{'}(\bar{A}_{W^\circ}f^{'}X^\circ, X^\circ) + l\theta(W^\circ)g^{'}(X^\circ, f^{'}X^\circ) + m\theta(W^\circ)g^{'}(X^\circ, f^{'^2}X^\circ)\},$$

for any $X^\circ \in \Gamma(B_o)$ and $W^\circ \in \Gamma(S(TM^{\circ\perp}))$.

Proof. B_o is minimal if and only if

$$\begin{aligned}\Omega_{X^\circ}X^\circ + \Omega_{\bar{J}'X^\circ}\bar{J}'X^\circ - l\{\theta(X^\circ)X^\circ + \theta(\bar{J}'X^\circ)\bar{J}'X^\circ\}&\\ - m\{\theta(X^\circ)f^{'}X^\circ + \theta(\bar{J}'X^\circ)f^{'^2}X^\circ\} \in \Gamma(B_o),&\end{aligned} \tag{52}$$

for all $X^\circ \in \Gamma(B_o)$.
Since M° is a SG-lightlike submanifold of $\bar{M}^\circ$. Therefore, the distribution B_o is invariant w.r.t $\bar{J}'$. Also using equations (27), (3), (16), (14), (5) and (36), we get

$$g^{'}(\Omega_{X^\circ}X^\circ, \bar{J}'W^\circ) - l\theta(X^\circ)g^{'}(X^\circ, \bar{J}'W^\circ) - m\theta(X^\circ)g^{'}(f^{'}X^\circ, \bar{J}'W^\circ)$$
$$= g^{'}(\bar{J}'X^\circ, \bar{A}_{W^\circ}X^\circ) + l\theta(W^\circ)g^{'}(X^\circ, \bar{J}'X^\circ) + m\theta(W^\circ)g^{'}(\bar{J}'X^\circ, f^{'}X^\circ),$$

and

$$\begin{aligned}g^{'}(\Omega_{\bar{J}'X^\circ}\bar{J}'X^\circ, \bar{J}'W^\circ) - l\theta(\bar{J}'X^\circ)g^{'}(\bar{J}'X^\circ, \bar{J}'W^\circ) - m\theta(\bar{J}'X^\circ)g^{'}(f^{'^2}X^\circ, \bar{J}'W^\circ)&\\ = g^{'}(3\bar{J}'X^\circ + X^\circ, \bar{A}_{W^\circ}\bar{J}'X^\circ) + l\theta(W^\circ)g^{'}(3\bar{J}'X^\circ + X^\circ, \bar{J}'X^\circ)&\\ + m\theta(W^\circ)g^{'}(3\bar{J}'X^\circ + X^\circ, f^{'^2}X^\circ),&\end{aligned}$$

for any $X^\circ \in \Gamma(B_o)$ and $W^\circ \in \Gamma(S(TM^{\circ\perp}))$.
Also, since the shape operator is symmetric on $S(TM^\circ)$, we obtain

$$\begin{aligned}g^{'}\big(\Omega_{X^\circ}X^\circ + \Omega_{\bar{J}'X^\circ}\bar{J}'X^\circ - l\{\theta(X^\circ)X^\circ + \theta(\bar{J}'X^\circ)\bar{J}'X^\circ\}&\\ -m\{\theta(X^\circ)f^{'}X^\circ + \theta(\bar{J}'X^\circ)f^{'^2}X^\circ\}, \bar{J}'W^\circ\big)&\\ = g^{'}(\bar{J}'X^\circ, \bar{A}_{W^\circ}X^\circ + l\theta(W^\circ)X^\circ + m\theta(W^\circ)f^{'}X^\circ)&\\ + g^{'}(3\bar{J}'X^\circ + X^\circ, \bar{A}_{W^\circ}\bar{J}'X^\circ + l\theta(W^\circ)\bar{J}'X^\circ + m\theta(W^\circ)f^{'^2}X^\circ),&\end{aligned}$$

for any $X^\circ \in \Gamma(B_o)$ and $W^\circ \in \Gamma(S(TM^{\circ\perp}))$.
Employing (52) on above equation, we attain

$$3\{g^{'}(f^{'}X^\circ, \bar{A}_{W^\circ}f^{'}X^\circ) + l\theta(W^\circ)g^{'}(X^\circ, f^{'^2}X^\circ) + m\theta(W^\circ)g^{'}(f^{'}X^\circ, f^{'^2}X^\circ)\}$$
$$= -2\{g^{'}(\bar{A}_{W^\circ}f^{'}X^\circ, X^\circ) + l\theta(W^\circ)g^{'}(X^\circ, f^{'}X^\circ) + m\theta(W^\circ)g^{'}(X^\circ, f^{'^2}X^\circ)\}.$$

□

Theorem 7.4. *A SG-lightlike submanifold M° of a locally bronze semi-Riemannian manifold $\bar{M}^\circ$ with an (l,m)-type connection $\bar{\Omega}$ is minimal if and only if* *(i)* $trace\bar{A}^*_{\xi^\circ_k}|_{S(TM^\circ)} = trace\bar{A}_{W^\circ_j}|_{S(TM^\circ)} = 0,$

(ii) $\bar{g}^{'}(\bar{D^\circ}^l(X^\circ, W^\circ), Y^\circ) = l\theta(W^\circ)\bar{g}^{'}(X^\circ, Y^\circ) + m\theta(W^\circ)\bar{g}^{'}(f^{'}X^\circ, Y^\circ),$

$\forall\ X^\circ, Y^\circ \in \Gamma(Rad(TM^\circ)),\ W^\circ \in \Gamma(S(TM^{\circ\perp})),$ where

$\dim(TM^\circ) = m,\ \dim(tr(TM^\circ)) = n,\ \dim(Rad(TM^\circ)) = r,\ \{\xi^\circ_k\}$ is a basis of $Rad(TM^\circ)$ and $W^\circ_j \in \Gamma(S(TM^{\circ\perp}))$ for $k \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n-r\}$.

Proof. M° being minimal submanifold implies that
$trace\bar{h}^\circ|_{S(TM^\circ)} = trace\bar{h}^\circ|_{B_o} + trace\bar{h}^\circ|_{B'} = \Sigma^a_{i=1}\bar{h}^\circ(B_i, B_i) + \Sigma^b_{j=1}\bar{h}^\circ(E_j, E_j) = 0,$
and $\bar{h^\circ}^s|_{Rad(TM^\circ)} = 0.$

If we choose an orthonormal basis of $S(TM^\circ)$ as $\{e_i\}^{m-r}_{i=1}$, then we get

$$\begin{aligned}
trace\bar{h}^\circ|_{S(TM^\circ)} &= \sum_{i=1}^{a}\epsilon_i(\bar{h^\circ}^l(e_i, e_i) + \bar{h^\circ}^s(e_i, e_i)) + \sum_{j=1}^{b}\epsilon_j(\bar{h^\circ}^l(e_j, e_j) + \bar{h^\circ}^s(e_j, e_j)) \\
&= \sum_{i=1}^{a}\epsilon_i\left[\frac{1}{r}\Sigma^r_{k=1}\bar{g}^{'}(\bar{h^\circ}^l(e_i, e_i), \xi^\circ_k)N^\circ_k\right. \\
&\qquad \left. + \frac{1}{n-r}\Sigma^{n-r}_{j=1}\bar{g}^{'}(\bar{h^\circ}^s(e_i, e_i), W^\circ_j)W^\circ_j\right] \\
&\quad + \sum_{j=1}^{b}\epsilon_j\left[\frac{1}{r}\Sigma^r_{k=1}\bar{g}^{'}(\bar{h^\circ}^l(e_j, e_j), \xi^\circ_k)N^\circ_k\right. \\
&\qquad \left. + \frac{1}{n-r}\Sigma^{n-r}_{j=1}\bar{g}^{'}(\bar{h^\circ}^s(e_j, e_j), W^\circ_j)W^\circ_j\right],
\end{aligned}$$

where $\{W^\circ_1, W^\circ_2, ..., W^\circ_{n-r}\}$is an orthonormal basis of $S(TM^{\circ\perp})$.

Since, $\bar{g}^{'}(\bar{h^\circ}^l(e_i, e_i), \xi^\circ_k)N^\circ_k = g^{'}(\bar{A}^*_{\xi^\circ_k}e_i, e_i)N^\circ_k$ and $\bar{g}^{'}(\bar{h^\circ}^s(e_i, e_i), W^\circ_j)W^\circ_j = g^{'}(\bar{A}_{W^\circ_j}e_i, e_i)W^\circ_j$, we derive
$trace\bar{h}^\circ|_{S(TM^\circ)} = trace\bar{A}^*_{\xi^\circ_k}|_{B_o \oplus B'} + trace\bar{A}_{W^\circ_j}|_{B_o \oplus B'},$
Therefore, $trace\bar{A}^*_{\xi^\circ_k}|_{B_o \oplus B'}$ and $trace\bar{A}_{W^\circ_j}|_{B_o \oplus B'} = 0.$
On the other hand, using concept of minimal SG-lightlike submanifold with equation (38), we obtain

$$\begin{aligned}
\bar{g}^{'}(\bar{h^\circ}^s(X^\circ, Y^\circ), W^\circ) = -\bar{g}^{'}(\bar{D^\circ}^l(X^\circ, W^\circ), Y^\circ) + l\theta(W^\circ)\bar{g}^{'}(X^\circ, Y^\circ) \\
+ m\theta(W^\circ)\bar{g}^{'}(f^{'}X^\circ, Y^\circ)
\end{aligned}$$

$\forall\ X^\circ, Y^\circ \in \Gamma(Rad(TM^\circ)), W^\circ \in \Gamma(S(TM^{\circ\perp}))$, which proof is completed. □

8. CONCLUDING REMARKS

This research work has initiated the lightlike geometry of screen generic lightlike submanifolds in the bronze semi-Riemannian manifold equipped with an (l, m)-type connection.

The bronze Riemannian manifolds and bronze means are the impoprtant classes of metallic means family, are instrumental as the basis of proportion to design sculptures. These manifolds are of great use to physicists to analyze the behaviour of nonlinear dynamic systems in the transition from periodicity to semi-periodicity. The study undertaken suggests that there is a potential for further investigation of various metric or non-metric connections derived from (l, m)-type connection equipped with bronze semi-Riemannian manifold. Also, the geometry of screen generic lightlike submanifolds can be examined for various classes of metallic semi-Riemannian manifold equipped with distinct connections inherent to their structure. Also, the bronze semi-Riemannian manifolds, being intrinsically related to the theoretical explanation of behavior in quantum physics, can motivate the geometers to delve into the applications of this noteworthy area of study.

Data Availability Statement. There were no data analyzed in this project.

Declarations. The authors declare no conflicts of interest.

Author Contributions. Conceptualization, R.K. and J.K.; methodology, J.K.; validation, R.K.; formal analysis, R.K. and J.K.; investigation, J.K.; resources, R.K. and J.K.; writing—original draft preparation, R.K.and J.K.; writing—review and editing, R.K., and J.K.; visualization, R.K.; supervision, J.K.; project administration, R.K. and J.K. All authors have read and agreed to the published version of the manuscript.

Funding Statement. This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Acknowledgment. The authors are deeply grateful to the anonymous referees for their thorough review and constructive feedback, which led to a substantially improved version.

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