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ON THE CUBIC EDGE-TRANSITIVE GRAPHS OF ORDER $58p^2$

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Abstract. A graph is called edge-transitive, if its full automorphism group acts transitively on its edge set. In this paper, we inquire the existence of connected edge-transitive cubic graphs of order $58p^2$ for each prime p. It is shown that only for $p = 29$, there exists a unique edge-transitive cubic graph of order $58p^2$.

Key words: Edge-transitive graphs, Symmetric graphs, Semisymmetric graphs, s-Regular graphs, Regular coverings.

Abstrak. Sebuah graf disebut transitif-sisi jika grup automorfisma penuhnya berlaku secara transitif pada himpunan sisinya. Pada paper ini, kami meneliti keberadaan graf kubik transitif-sisi terhubung berorde $58p^2$ untuk setiap bilangan prima p. Kami tunjukkan bahwa hanya untuk $p = 29$ terdapat graf kubik transitif-sisi unik berorde $58p^2$.

Kata kunci: Graf transitif-sisi, graf simetris, graf reguler-s, selimut reguler.

1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For the group-theoretic concepts and notations not defined here we refer to Rose [20].

For a graph X, we denote its vertex set, edge set, arc set and full automorphism group of X by $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$, respectively. For $u, v \in V(X)$, denote by $\{u, v\}$ the edge incident to u and v in X.

Let G be a finite group and S a subset of G such that $1 \notin S$ and $S = S^{-1}$.

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The Cayley graph $X = Cay(G, S)$ on G with respect to S is defined to have vertex set $V(X) = G$ and edge set $E(X) = \{(g, sg) | g \in G, s \in S\}$. Clearly, $Cay(G, S)$ is connected if and only if S generates the group G. The automorphism group $\mathrm{Aut}(X)$ of X contains the right regular representation G_R of G, the acting group of G by right multiplication, as a subgroup, and G_R is regular on $V(X)$, that is, G_R is transitive on $V(X)$ with trivial vertex stabilizers. A graph X is isomorphic to a Cayley graph on a group G if and only if its automorphism group $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on the vertex set.

An *s-arc* in a graph X is an ordered $(s + 1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ acts transitively on the set of its s-arcs. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph X is said to be *s-regular*, if $\mathrm{Aut}(X)$ acts regularly on the set of its s-arcs. Tutte [22] showed that every finite connected cubic symmetric graph is s-regular for $1 \leq s \leq 5$. A subgroup of $\mathrm{Aut}(X)$ is said to be s-regular, if it acts regularly on the set of s-arcs of X. If a subgroup G of $\mathrm{Aut}(X)$ acts transitively on $V(X)$ and $E(X)$, we say that X is *G-vertex-transitive* and *G-edge-transitive*, respectively. In the special case, when $G = \mathrm{Aut}(X)$, we say that X is vertex-transitive and edge-transitive, respectively. It can be shown that a G-edge-transitive but not G-vertex-transitive graph X is necessarily bipartite, where the two parts of the bipartition are orbits of $G \leq \mathrm{Aut}(X)$. Moreover, if X is regular then these two parts have the same cardinality. A regular G-edge-transitive but not G-vertex-transitive graph will be referred to as a *G-semisymmetric* graph. In particular, if $G = \mathrm{Aut}(X)$ the graph is said to be semisymmetric.

The classification of cubic symmetric graphs of different orders is given in many papers. Ronald M. Foster started collecting specimens of small cubic symmetric graphs prior to 1934, maintaining a census of all such graphs. In 1988 the then current version of the census was published in a book entitled *The Foster Census* Foster [15], and contained data for the graphs on up to 512 vertices. By Conder [3, 4], the cubic s-regular graphs up to order 10000 are classified. Throughout this paper, p and q are prime numbers. The s-regular cubic graphs of some orders such as $2p^2$, $4p^2$, $6p^2$, $10p^2$ were classified in Feng [9, 10, 11, 12]. Also, cubic s-regular graphs of order $2pq$ were classified in Zhou [27]. Also, we classified the cubic edge-transitive graphs of order $18p$ in Alaeiyan [1]. Furthermore, the study of semisymmetric graphs was initiated by Folkman [14]. For example, cubic semisymmetric graphs of orders $6p^2$, $28p^2$ and $2pq$ are classified in Lu, Alaeiyan and Du [18, 2, 8].

Now suppose that p is an odd prime. Let $N(p, p, p) = \langle x^p = y^p = z^p = 1, [x, y] = z, [z, x] = [z, y] = 1\rangle$ be a finite group of order p^3 and $G = \langle a, b, c, d \mid a^2 = b^p = c^p = d^p = [a, d] = [b, d] = [c, d] = 1, d = [b, c], aba = b^{-1}, aca = c^{-1}\rangle$ be a group of order $2p^3$ and $S = \{a, ab, ac\}$. We write $C(N(p, p, p)) = Cay(G, S)$. By Feng [1, Theorem 3.2], $C(N(p, p, p))$ is a 2-regular graph of order $2p^3$.
In this paper, we classify all the connected cubic edge-transitive (symmetric and also semisymmetric) graphs of order $58p^2$ as follows.

Theorem 1.1. Let p be a prime. Then the only connected cubic edge-transitive graph of order $58p^2$ is the 2-regular graph $C(N(29, 29, 29))$.

2. PRELIMINARIES

Let X be a graph and let N be a subgroup of $\mathrm{Aut}(X)$. For u, $v \in V(X)$, denote by $\{u, v\}$ the edge incident to u and v in X, and by $N_X(u)$ the set of vertices adjacent to u in X. The *quotient graph* X/N or X_N induced by N is defined as the graph such that the set Σ of N-orbits in $V(X)$ is the vertex set of X/N and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $\{u, v\} \in E(X)$.

A graph $\widetilde{X}$ is called a *covering* of a graph X with a projection $\wp : \widetilde{X} \to X$ if there is a surjection $\wp : V(\widetilde{X}) \to V(X)$ such that $\wp|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in \wp^{-1}(v)$. The graph X is often called the *base graph.* A covering graph $\widetilde{X}$ of X with a projection $\wp$ is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition $\wp h$ of $\wp$ and h.

Proposition 2.1. Lorimer [17, Theorem 9] Let X be a connected symmetric graph of prime valency and let G be an s-regular subgroup of $\mathrm{Aut}(X)$ for some $s \geq 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-regular subgroup of $\mathrm{Aut}(X_N)$, where X_N is the quotient graph of X corresponding to the orbits of N. Furthermore, X is an N-regular covering of X_N.

The next proposition is a special case of Wang [24, Proposition 2.5].

Proposition 2.2. Let X be a G-semisymmetric cubic graph with bipartition sets $U(X)$ and $W(X)$, where $G \leq A := Aut(X)$. Moreover, suppose that N is a normal subgroup of G. Then,
(1) If N is intransitive on bipartition sets, then N acts semiregularly on both $U(X)$ and $W(X)$, and X is an N-regular covering of a G/N-semisymmetric graph X_N.
(2) If 3 does not divide $|A/N|$, then N is semisymmetric on X.

Proposition 2.3. Djoković [7, Propositions 2-5] Let X be a connected cubic symmetric graph and G be an s-regular subgroup of $\mathrm{Aut}(X)$. Then, the stabilizer G_v of $v \in V(X)$ is isomorphic to $\mathbb{Z}_3$, S_3, $S_3 \times \mathbb{Z}_2, S_4$, or $S_4 \times \mathbb{Z}_2$ for $s = 1, 2, 3, 4$ or 5, respectively.

Proposition 2.4. Malnič [19, Proposition 2.4] The vertex stabilizers of a connected G-semisymmetric cubic graph X have order $2^r \cdot 3$, where $0 \leq r \leq 7$. Moreover, if u and v are two adjacent vertices, then the edge stabilizer $G_u \cap G_v$ is a common Sylow 2-subgroup of G_u and G_v.

Let G be a group. If $a, b \in G$, then the commutator of a and b is the element $aba^{-1}b^{-1}$. The commutator subgroup or derived subgroup of G is the subgroup generated by all the commutators of G and it is denoted by G' or $[G, G]$. Now, we have the following obvious facts in group theory.

Proposition 2.5. Let G be a finite group and let p be a prime. If G has an Abelian Sylow p-subgroup, then p does not divide $|G' \cap Z(G)|$.

Proposition 2.6. Wielandt [26, Proposition 4.4] Every transitive Abelian group G on a set Ω is regular and the centralizer of G in the symmetric group on Ω is G.

For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G.

Proposition 2.7. Rose [20, Lemme 4.36] Let G be a finite group, and $H \leqslant G$. Then $C_G(H)$ is normal in $N_G(H)$, and $N_G(H)/C_G(H)$ is isomorphic to a subgroup of Aut$H$.

3. MAIN RESULTS

Let X be a cubic edge-transitive graph of order $58p^2$. By Tutte [22], X is either symmetric or semisymmetric. We now consider the symmetric case, and then we have the following lemma.

Lemma 3.1. Let p be a prime and let X be a cubic symmetric graph of order $58p^2$. Then X is isomorphic to the 2-regular graph $C(N(29, 29, 29))$.

Proof. By Conder [3, 4] there is no symmetric graph of order $58p^2$, where $p < 7$. If $p = 29$, then by Feng [13, Theorem 3.2], X is isomorphic to the 2-regular graph $C(N(29, 29, 29))$.

To prove the lemma, we only need to show that no cubic symmetric graph of order $58p^2$ exists, for $p \geq 7$ and $p \neq 29$. We suppose to the contrary, that X is such a graph. Set $A := Aut(X)$. Since X is symmetric, by Tutte [23], X is at most 5-regular and by Proposition 2.3, $|A_v| = 2^{s-1} \cdot 3$ for some integer, $1 \leq s \leq 5$ and hence $|A| = 2^s \cdot 3 \cdot 29 \cdot p^2$. Let $Q := O_p(A)$ be the maximal normal p-subgroup of A. If $|Q| = p^2$, then by Proposition 2.1, the quotient graph X_Q of X corresponding to the orbits of Q is a cubic symmetric graph of order 58, which is impossible by Conder [3]. Thus $|Q| = 1$ or p.

First, suppose that $|Q| = 1$ and let N be a minimal normal subgroup of A. If N is unsolvable, then $N \cong T \times T \times \cdots \times T$, where T is a non-Abelian simple group. Since $|A| = 2^s \cdot 3 \cdot 29 \cdot p^2$, thus $N \cong T$. Suppose that N has more than two orbits in $V(X)$. By Proposition 2.1, N is semiregular on $V(X)$. Thus $|N| \mid 58p^2$. This forces that N is solvable, a contradiction. It follows N has at most two orbits in

$V(X)$, implying $29p^2 \mid |N|$. Since N is unsolvable, it is not a $\{p,q\}$-group. Thus, $|N| = 2^t \cdot 29 \cdot p^2$ or $2^t \cdot 3 \cdot 29 \cdot p^2$, where $1 \leq t \leq s$. Let q be a prime .Then by Gorenstein [16, pp. 12-14] and Conway [6], a non-Abelian simple $\{2,p,q\}$-group is one of the following groups.

$$A_5, A_6, PSL(2,7), PSL(2,8), PSL(2,17), PSL(3,3), PSU(3,3), PSU(4,2), \quad (\mathbf{1})$$

with orders $2^2 \cdot 3 \cdot 5, 2^3 \cdot 3^2 \cdot 5, 2^3 \cdot 3 \cdot 7, 2^3 \cdot 3^2 \cdot 7, 2^4 \cdot 3^2 \cdot 17, 2^4 \cdot 3^3 \cdot 13, 2^5 \cdot 3^3 \cdot 7, 2^6 \cdot 3^4 \cdot 5$, respectively. This implies that for $p \geq 7$, there is no simple group of order $2^t \cdot 29 \cdot p^2$. Hence $|N| = 2^t \cdot 3 \cdot 29 \cdot p^2$.

Assume that T is a proper subgroup of N. If T is unsolvable, then T has a non-Abelian simple composite factor T_1/T_2. Since $|T_1/T_2| \mid 2^t.3.29.p^2$, by simple group listed in $(\mathbf{1})$, T_1/T_2 cannot be a $\{2,3,29\}$-, $\{2,3,p\}$- or $\{2,29,p\}$-group. Thus, T_1/T_2 is a $\{2,3,29,p\}$-group. One may assume that $|T| = 2^r \cdot 3 \cdot 29 \cdot p^2$ or $2^r \cdot 3 \cdot 29 \cdot p$, where $r \geq 2$. Let $|T| = 2^r \cdot 3 \cdot 29 \cdot p^2$. Then $|N : T| \leq 8$ because $|N| = 2^t \cdot 3 \cdot 29 \cdot p^2$. Consider the action of N on the right cosets of by right multiplication, and the simplicity of N implies that this action is faithful. It follows $N \leq S_8$ and hence $p \leq 7$. Since $p \geq 7$, one has $p = 7$ and hence $N = 2^t \cdot 3 \cdot 29 \cdot 7^2$. But by Conway [6], there is no non-Abelian simple group of order $2^t \cdot 3 \cdot 29 \cdot 7^2$, a contradiction. Thus, T is solvable and hence N is a minimal non-Abelian simple group, that is, N is a non-Abelian simple group and every proper subgroup of N is solvable. By Thompson [21, Corollary 1], N is one of the groups in Table 1. It can be easily verified that the order of the groups in Table 1 is not of the form $2^r \cdot 3 \cdot 29 \cdot p^2$. Thus $|T| = 2^r \cdot 3 \cdot 29 \cdot p$.

By the same argument as in the preceding paragraph (replacing N by T), T is one of the groups in Table 1. Since $|T| = 2^r \cdot 3 \cdot 29 \cdot p$, the possible candidates for T is $PSL(2,m)$. Clearly, $m = p$. We show that $|T| < 10^{25}$. If $29 \nmid (p-1)/2$, then $(p-1)/2|96$, which implies that $p \leq 193$. If $p = 193$, then $2^6||T|$, a contradiction. Thus $p < 193$ and hence $p \leq 97$ because $(p-1)/2|96$. It follows that $|T| \leq 96 \cdot 29 \cdot 97 = 270048$. If $29|(p-1)/2$, then $p+1|96$. Consequently $p \leq 47$, implying $|T| \leq 96 \cdot 29 \cdot 47 < 270048$. Thus, $|T| \leq 214176$. Hence, by Conway [6, pp. 239], is isomorphic to $PSL(2,23)$ or $PSL(2,47)$.

Table I. The possible for non-Abelian simple group N

N	$\lvert N\rvert$
$PSL(2,m), m > 3$ a prime and $m^2 \neq 3 \pmod{p^2}$	$\frac{1}{2}m(m-1)(m+1)$
$PSL(2,2^n), n$ a prime	$2^n(2^{2n}-1)$
$PSL(2,3^n), n$ an odd prime	$\frac{1}{2}3^n(3^{2n}-1)$
$PSL(3,3), n$ a prime	$\frac{1}{3} \cdot 3^3 \cdot 2^4$
Suzuki group $Sz(2^n), n$ an odd prime	$2^{2n}(2^{2n}+1)(2n-1)$

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It follows that $p = 11$ or 47 and hence $|N| = 2^t \cdot 3 \cdot 29 \cdot 11^2$ or $2^t \cdot 3 \cdot 29 \cdot 47^2$, which is impossible by Conway [6, pp. 239].

Hence, N is solvable and so elementary Abelian. Since X has order $58p^2$, by Proposition 2.1, N is semiregular on $V(X)$, implying $|N| \mid 58p^2$. Consequently, $N \cong \mathbb{Z}_2$, or $\mathbb{Z}_{29}$, because $|Q| = 1$. If $N \cong \mathbb{Z}_2$, then by Proposition 2.1, X_N is

a cubic symmetric graph of odd order $29p^2$, a contradiction. If $N \cong \mathbb{Z}_{29}$, then by Proposition 2.1 the quotient graph X_N is a cubic symmetric graph of order $2p^2$. Let M/N be a minimal normal subgroup of A/N. Since $p \geq 7$ and $|A/N| = 2^t \cdot 3 \cdot p^2$, by the simple group listed in (**1**), M/N is solvable and so elementary Abelian. Again by Proposition 2.1, M/N is semiregular on $V(X_N)$, which implies that $M/N \cong \mathbb{Z}_2, \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p$. For the former by Proposition 2.1, the quotient graph X_M of X corresponding to the orbits of M is a cubic graph with an odd order p^2, a contradiction. Thus $M/N \cong \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p$. If $p \neq 7$, then, since $p > 7$, M has a normal subgroup of order p or p^2, which is characteristic in M and hence is normal in A, because M is normal in A. This contradicts our assumption that $|Q| = 1$.
Now, suppose that $p = 7$. Consider the quotient graph X_N. Let T/N be a minimal normal subgroup of A/N. Clearly, T/N is solvable and so elementary Abelian. By Proposition 2.1, T/N is semiregular on $V(X_N)$. It implies $|T/N| \mid 2 \cdot 7^2$. Consequently, $|T/N| = 2, 7$ or 7^2. If $|T/N| = 2$, then $|T| = 58$. So, the quotient graph X_T is a cubic symmetric graph with an odd order, a contradiction. Now suppose that $|T/N| = 7$. Thus $|T| = 7 \cdot 29$. If T be Abelian, then $T \cong \mathbb{Z}_7 \times \mathbb{Z}_{29} \cong \mathbb{Z}_{203}$ and by Proposition 2.1, X is a $\mathbb{Z}_{203}$- covering of the Heawood graph. But by Wang [25, Theorem 1.1], there is no symmetric $\mathbb{Z}_{203}$-covering of the Heawood graph, a contradiction. Thus, T is a non-Abelian group. Let $C = C_A(N)$ be the centralizer N in A. Clearly $C = N$ or $C = T$ because T/N is a simple group. If $C = N$, then by Proposition 2.7, $A/N \leqslant Aut(N) \cong \mathbb{Z}_{28}$, a contradiction. So $C = T$. By Proposition 2.5, $7 \nmid |T' \cap Z(T)|$ and hence $T' \cap N = 1$, where T' is the derived subgroup of T. Also, $T' \neq 1$ and $T' \nleqslant N$. Therefore, $T' \cong T'/(T' \cap N) \cong T'N/N \trianglelefteq T/N$. The simplicity of T/N implies $T' \cong T/N$. As T' is characteristic in T and $T \lhd A$, we have $T' \lhd A$. By Proposition 2.1, the quotient graph $X_{T'}$ is a cubic symmetric graph of order 406. But, by Conder [3] there is no symmetric cubic graph of order 406, a contradiction.

Now, we show that $|T/N| \neq 7^2$. Let $|T/N| = 7^2$, that is $T/N \cong \mathbb{Z}_7 \times \mathbb{Z}_7$. If $T \cong \mathbb{Z}_{29} \times \mathbb{Z}_7 \times \mathbb{Z}_7$, then A has a normal subgroup of order 7 or 7^2. It contradicts with $|Q| = 1$. So T is non-Abelian group. Let $C = C_T(N)$ be the centralizer N in T. Then clearly $N \leqslant C$. Suppose that $N = C$. Then by Proposition 2.7, T/N is isomorphic to a subgroup of $Aut(N) \cong \mathbb{Z}_{28}$, a contradiction. Hence $N \lneqq C$. Since $C/N \lhd T/N \cong \mathbb{Z}_7 \times \mathbb{Z}_7$. So $C/N \cong T/N$ or $\mathbb{Z}_7$. If $C/N = T/N$, then by Proposition 2.5, $7 \nmid |T' \cap Z(T)|$ and hence $T' \cap N = 1$, where T' is the derived subgroup of T. Thus $T' \cong T'N/N \lhd T/N$ and so $T' \cong \mathbb{Z}_7$ or T/N. It implies $|T'| = 7$ or 7^2. If $|T'| = 7$, then by a similar argument on the previous paragraph, we can get a contradiction.
Suppose now $|T'| = 7^2$. As T' is characteristic in T and $T \lhd A$. Thus $T' \lhd A$. By Proposition 2.1, the quotient graph $X_{T'}$ is a cubic symmetric graph of order 58. But, by Conder [3] there is no symmetric cubic graph of order 58, a contradiction. Also, If $C/N \cong \mathbb{Z}_7$, then by a similar argument on the case $|T/N| = 7$, we can get a contradiction.

Suppose now that $Q \cong \mathbb{Z}_p$ and let $C = C_A(Q)$ be the centralizer of Q in A. By Proposition 2.5, $p \nmid |C' \cap Z(C)$ and hence $C' \cap Q = 1$, where C' is the derived subgroup of C. This force $p^2 \nmid |C'|$, because C' is normal in A. It follows that C'

has more than two orbits on $V(X)$. As C' is normal in A, by Proposition 2.1, it is semiregular on $V(X)$. Moreover, the quotient graph $X_{C'}$ is a cubic graph and consequently, has even order. Hence $2 \nmid |C'|$ and since $p^2 \nmid |C'|$, the semiregularity C' implies $|C'| \mid 29p$. Since the Sylow p-subgroups of A are Abelian, one has $p^2 \mid |C|$ and so $|C/C'|$. Now let K/C' be a Sylow p-subgroup of the Abelian group C/C'. As K/C' is characteristic in C/C' and $C/C' \lhd A/C'$, we have that $K/C' \lhd A/C'$. Hence K is normal in A. Clearly $|K| = 29p^2$ because $|Q| = p$. If $p > 7$, then K has a normal subgroup of order p^2, which is characteristic in K, hence is normal in A, contradicting the fact that $Q \cong \mathbb{Z}_p$.

Hence $p = 7$. Consider the quotient graph X_Q. By Proposition 2.1, X_Q is cubic symmetric graph and A/Q is an arc-transitive subgroup of $Aut(X_Q)$. Let T/Q be a minimal normal subgroup of A/Q. If T/Q is unsolvable, then by Conway [6], $T/Q \cong PSL(2,7)$. Let $C = C_T(Q)$ be the centralizer of Q in T. Then $C = Q$ or $Q \leqslant Z(T)$. If $C = Q$, then by Proposition 2.7, T/Q is isomorphic to a subgroup of $Aut(Q) \cong \mathbb{Z}_6$, a contradiction. Thus $Q \leqslant Z(C)$. By Conway [6] the Schur multiplier of $PSL(2,7)$ is isomorphic to $\mathbb{Z}_2$. Thus, we have $T \cong T_1 \times Q$ where T_1 is isomorphic to $PSL(2,7)$. Since T_1 is characteristic in T and $T \lhd A$. one has T_1 is normal in A, implying A has an unsolvable minimal normal subgroup, a contradiction. Again T/Q is solvable and so Abelian elementary. By Proposition 2.1, T/Q is semiregular on $V(X_Q)$ an so $|T/Q| \mid 2 \cdot 29 \cdot 7$. It implies $|T/Q| = 2, 7$ or 29. If $|T/Q| = 2$, then $|T| = 14$. So the quotient graph X_T is a cubic graph of odd order $29 \cdot 7$, a contradiction. Also, if $|T/Q| = 7$, then the quotient graph X_T is a cubic symmetric graph of order 58. But, by Conway [6] there is no symmetric cubic graph of order 58, a contradiction. Therefore, $|T/Q| = 29$. Let $C = C_T(Q)$ be the centralizer of T in Q. Clearly $Q \leqslant C$ because Q is Abelian. If $Q = C$, then by Proposition 2.7, $T/Q \leqslant Aut(Q) \cong \mathbb{Z}_6$, a contradiction. Thus, $Q < C$. Since $C/Q \trianglelefteq T/Q$ and $T/Q \cong \mathbb{Z}_{29}$, one has $C/Q = T/Q$, imply that $Q \leqslant Z(T)$. Let T' be the derived group of T. Since the Schur multiplier of $\mathbb{Z}_{29}$ is trivial (see Atlas by Conway [6]). One has $T' < T$. It follows that $T = T' \times Q$, where $T' \cong \mathbb{Z}_{29}$. Thus $T \cong \mathbb{Z}_{29} \times \mathbb{Z}_7 \cong \mathbb{Z}_{203}$ and by Proposition 2.1, X is $\mathbb{Z}_{203}$-covering of the Heawood graph. But by Wang [25, Theorem 1.1], we get a contradiction. Hence, the result now follows.

Now, we consider the semisymmetric case, and we have the following result.

Lemma 3.2. Let p be a prime. Then, there is no cubic semisymmetric graph of order $58p^2$.

Proof. Let X be a cubic semisymmetric graph of order $58p^2$. Denote by $U(X)$ and $W(X)$ the bipartition sets of X, where $|U(X)| = |W(X)| = 29p^2$. For $p = 2, 3$ by Conder [5] there is no cubic semisymmetric graph of order $58p^2$. Thus we can assume that $p \geq 5$. Set $A := Aut(X)$ and also let $Q := O_p(A)$ be the maximal normal p-subgroup of A. The automorphism group A acts transitively on the set $U(X)$ (and also $W(X)$). So by Proposition 2.4, $|A| = 2^r \cdot 3 \cdot 29 \cdot p^2$, where $0 \leqslant r \leqslant 7$.

Let N be a minimal normal subgroup of A. One can deduce that N is solvable. Because if N is unsolvable, then $N \cong T \times T \times \cdots \times T = T^k$, where T is a non-Abelian $\{2, 3, 29\}$, $\{2, 3, p\}$ or $\{2, 3, 29, p\}$-simple group (see Gorenstein [16]). For the two formers, since $3^2 \nmid |N|$, then $k = 1$. So $N \cong T$. Since $3 \nmid |A/N|$, by Proposition 2.2, N must be semisymmetric on X and then $29p^2 \mid |N|$, a contradiction. For the latter, by a similar argument as in Lemma 3.1, we get a contradiction. Thus, we can assume that N is solvable, so elementary Abelian. Clearly, N acts intransitively on $U(X)$ and $W(X)$ and by Proposition 2.2, it is semiregular on each partition. Hence $|N| \mid 29p^2$. So $|N| = 29, p$ or p^2. We show that $|Q| = p^2$ as follows.

First, Suppose that $|Q| = 1$. It implies that $N \cong \mathbb{Z}_{29}$. Now we consider X_N be the quotient graph of X relative to N, where X_N is a cubic A/N-semisymmetric graph of order $2p^2$. By Folkman [14], X_N is a vertex-transitive graph. So X_N is a cubic symmetric graph of order $2p^2$. Suppose that T/N is a minimal normal subgroup of A/N. If T/N is not solvable, then by Conway [6], $T/N \cong A_5$ or $PSL(2, 7)$. Thus, $|T| = 2^2 \cdot 3 \cdot 5 \cdot 29$ or $2^3 \cdot 3 \cdot 7 \cdot 29$. Since 3 does not divide A/T, then by Proposition 2.2, T is semisymmetric on X. Consequently, 5^2 or $7^2 \mid |T|$, a contradiction. Therefore, T/N is solvable and so elementary Abelian. First, suppose that $p = 7$, by Feng [11, Lemma 3.1], T/N is 7-subgroup Abelian elementary. So $|T/N| = 7$ or 7^2 and hence $|T| = 29 \cdot 7$ or $29 \cdot 7^2$. Now, let H be the Sylow 29-subgroup of T in A. Clearly H is normal in T, since T is characteristic in A. So H is normal in A. By Proposition 2.2, the quotient graph X_H is a cubic A/H- semisymmetric graph of order 98. But, by Conder [5], there is no semisymmetric graph of order 98.
Therefore, we assume that $p \geq 11$. If $|T/N| = p^2$, then $|T| = 29p^2$. It is easily seen the Sylow p-subgroup of T is characteristic and consequently normal in A. It contradicts our assumption that $|Q| = 1$. Therefore, T/N acts intransitively on the bipartition sets of X_N and by Proposition 2.2, it is semiregular on each partition, which forces $|T/N| \mid p^2$. Hence, $|T/N| = p$ and so $|T| = 29p$. We can deduce that A has a normal subgroup of order p, which is a contradiction. Thus $|Q| \neq 1$.

We now suppose that $|Q| = p$. Since $|N| \mid 29p^2$, then we have two cases: $N \cong \mathbb{Z}_{29}$ and $N \cong \mathbb{Z}_p$.

Case I. $N \cong \mathbb{Z}_{29}$.
By Proposition 2.2, X_N is a cubic A/N-semisymmetric graph of order $2p^2$. Let T/N be a minimal normal subgroup of A/N.
Suppose first that $p = 7$. By a similar argument as in the case $|Q| = 1$, we get a contradiction.
Now let $p > 7$. Then by Feng [11, Theorem 3.2], the Sylow p-subgroup of $Aut(X_N)$ is normal, and also we know that $A/N \leqslant Aut(X_N)$. Consequently, the Sylow p-subgroup A/N is normal, say M/N. It is easy to see that $|M| = 29p^2$. Since $p > 7$, the Sylow p-subgroup of M is normal and hence characteristic in M. Thus, A has a normal subgroup of order p^2. It contradicts our assumption that $|Q| = p$.

Case II. $N \cong \mathbb{Z}_p$.
By Proposition 2.2, X_N is a cubic A/N-semisymmetric graph of order $58p$. Let T/N be a minimal normal subgroup of A/N. By a similar way as above, T/N is

solvable and so elementary Abelian. By Proposition 2.2, T/N is semiregular. It implies that $|T/N| \mid 29p$. If $|T/N| = p$, then $|T| = p^2$, a contrary to $|Q| = p$. Hence $|T/N| = 29$ and so $|T| = 29p$. By Proposition 2.2, X_T is a cubic A/T-semisymmetric graph of order $2p$. Thus by a similar way as in Case I, we get a contradiction.

Therefore $|Q| = p^2$ and so by Proposition 2.2, X is a regular Q-covering of an A/Q-semisymmetric graph of order 58. But, it is impossible because by Conder [4, 5] there is no edge-transitive graph of order 58 The result now follows.

Proof of Theorem 1. Now we complete the proof of the main theorem. Let X be a connected cubic edge-transitive graph of order $58p^2$, where p is a prime. We know that every cubic edge-transitive graph is either symmetric or semisymmetric. Therefore, by Lemmas 3.1 and 3.2, the proof is completed.

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