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Homomorphisms of Modules over Two Skew Generalized Power Series Rings

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Abstract. Let R and R' be rings with identity and let S be a strictly ordered monoid equipped with twisting homomorphisms into the endomorphism rings of R and R'. Using these data, we consider two skew generalized power series rings associated with R and R'. Starting from an (R, R')-module M, we construct an induced module consisting of generalized power series with coefficients in M and show that it naturally becomes a module over the two skew generalized power series rings through a suitable trilinear action. We then study homomorphisms in this framework. In particular, we prove that every (R, R')-module homomorphism between two modules induces a corresponding homomorphism between their associated generalized power series modules. Furthermore, we provide sufficient conditions describing when a generalized power series belongs to the kernel of the induced homomorphism in terms of the coefficientwise behavior of the original map. These results extend the theory of skew generalized power series modules from the classical single-ring setting to a two-ring context and provide a foundation for further developments, including generalized isomorphism results and related algebraic applications.

Key words and Phrases: kernel characterization, skew generalized power series rings, two-sided SGPSM, (R, R')-modules, module homomorphisms

1. INTRODUCTION

A ring is defined as a nonempty set equipped with two binary operations satisfying specific axioms [1]. Ribenboim [2] introduced the Generalized Power Series Rings (GPSRs) $[[R^{S,\leq}]]$, extending the semigroup ring $R[S]$ [3], the polynomial ring $R[X]$, and the power series ring $R[[X]]$ [4]. The GPSR $[[R^{S,\leq}]]$ imposes additional restrictions, requiring the support $\mathrm{supp}(f)$ to be both Artinian and narrow. A partially ordered set $(S, \leq)$ is called Artinian if every strictly ordered sequence of elements in S is finite, and narrow if every trivially ordered subset of S is finite [5].

Properties of GPSRs have been extensively studied [6, 7, 8, 9, 10, 11]. Varadarajan [12] introduced the Generalized Power Series Module (GPSM) $M[[S, \leq]]$, which

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generalizes the polynomial module $M[X]$ over polynomial rings [13]. Further studies on GPSMs and their Noetherian properties can be found in [14, 15, 16].

Mazurek and Ziembowski [17] extended GPSRs by introducing a monoid homomorphism $\omega : S \to End(R)$, forming the Skew Generalized Power Series Ring (SGPSR) $R[[S, \leq, \omega]]$. Subsequent studies [18, 19, 20, 21, 22] and related works [23, 24, 25, 26, 27, 28] investigated various algebraic properties of SGPSRs. Faisol et al. [29, 30] later constructed the Skew Generalized Power Series Module (SGPSM) $M[[S, \leq, \omega]]$, extending the theory of generalized power series to module structures over SGPSRs.

While much work has been devoted to GPSRs and SGPSRs, existing studies have predominantly focused on module structures over a single generalized or skew generalized power series ring. In many algebraic settings, however, it is natural to consider module structures involving two distinct skew generalized power series rings equipped with possibly different skewing homomorphisms. In such situations, the classical SGPSR-module framework is no longer sufficient, since module actions and homomorphisms must simultaneously respect two independent skew structures.

Khumprapussorn et al. [31] generalized (R, R')-bimodule structures by weakening compatibility conditions and introducing (R, R')-modules, while homomorphisms of such modules were further studied by Yuwaningsih et al. [32]. Nevertheless, these works do not address the case where the underlying rings themselves are skew generalized power series rings. In particular, when considering module homomorphisms between modules over two distinct SGPSRs $R[[S, \leq, \omega]]$ and $R'[[S, \leq, \omega']]$, additional compatibility conditions arise from the simultaneous interaction of the two skewing homomorphisms ω and ω'. These constraints do not appear in the classical single-ring SGPSR setting and cannot, in general, be reduced to previously studied cases.

This paper aims to fill this gap by developing a systematic framework for modules over *two distinct skew generalized power series rings*. In contrast to existing approaches confined to a single skew generalized power series ring, our construction captures bimodule-like structures governed by two independent skew actions. We investigate the resulting module homomorphisms and establish fundamental properties, including sufficient conditions for elements to belong to their kernels, thereby extending skew generalized power series module theory beyond the classical single-ring setting.

To illustrate that the obtained results are not merely formal extensions, we also provide explicit and nontrivial examples demonstrating how module homomorphisms over two skew generalized power series rings induce concrete, coefficientwise kernel characterizations.

2. Skew Generalized Power Series Structures

In this section we briefly recall the construction of skew generalized power series rings and their associated modules. These structures will serve as the algebraic framework used in the next section to construct a $\big(R[[S,\leq,\omega]],R'[[S,\leq,\omega']]\big)$-module.

Throughout this paper, $(S,\leq)$ denotes a strictly ordered monoid. Let R' be a ring with identity and let $\omega' : S \to \mathrm{End}(R')$ be a monoid homomorphism. For a function $f' : S \to R'$, we write

$$\mathrm{supp}(f') = \{s \in S \mid f'(s) \neq 0\}.$$

Generalized power series rings were introduced by Ribenboim [2]. Given the homomorphism ω', the skew version developed by Mazurek and Ziembowski [17] is defined as

$$R'[[S,\leq,\omega']] = \{f' : S \to R' \mid \mathrm{supp}(f') \text{ is artinian and narrow}\},$$

with multiplication

$$(f'g')(s) = \sum_{xy=s} f'(x)\,\omega'_x\big(g'(y)\big),$$

where the sum is finite. Here the twisting acts on the second (right) factor of the convolution, and we refer to this product as the standard right skew convolution. If $\omega'_s = \mathrm{id}_{R'}$ for all $s \in S$, one recovers the ordinary generalized power series ring.

For $r' \in R'$ and $s \in S$, define the elements $c_{r'}$ and f'_s in $R'[[S,\leq,\omega']]$ by

$$c_{r'}(u) = \begin{cases} r', & \text{if } u = 0,\\ 0, & \text{otherwise,}\end{cases} \tag{1}$$

and

$$f'_s(u) = \begin{cases} 1, & \text{if } u = s,\\ 0, & \text{otherwise,}\end{cases} \tag{2}$$

for every $u \in S$. These elements play the role of coefficient and basis elements in the skew generalized power series ring.

The corresponding module construction, considered for instance in [12, 29], is obtained as follows. Let M be a right R'-module and define

$$M[[S,\leq,\omega']] = \{\alpha : S \to M \mid \mathrm{supp}(\alpha) \text{ is artinian and narrow}\}.$$

With pointwise addition, $M[[S,\leq,\omega']]$ becomes a right $R'[[S,\leq,\omega']]$-module under

$$(\alpha g')(s) = \sum_{xy=s} \alpha(x)\,\omega'_x\big(g'(y)\big), \tag{3}$$

where again the twisting occurs on the right factor of the convolution.

For $m \in M$ and $s \in S$, define $d^s_m \in M[[S,\leq,\omega']]$ by

$$d^s_m(u) = \begin{cases} m, & \text{if } u = s,\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

For a fixed $s \in S$, the assignment

$$\iota_M : M \longrightarrow M[[S, \leq, \omega']], \qquad \iota_M(m) = d_m^s,$$

defines an injective R'-module homomorphism.

By analogy with the above construction, one may also consider a left module structure. Let R be another ring with identity and let $\omega : S \to \mathrm{End}(R)$ be a monoid homomorphism. If M is a left R-module, then $M[[S, \leq, \omega]]$ becomes a left $R[[S, \leq, \omega]]$-module under

$$(f\alpha)(s) = \sum_{xy=s} \omega_x\big(f(x)\big)\, \alpha(y),$$

for $f \in R[[S, \leq, \omega]]$ and $\alpha \in M[[S, \leq, \omega]]$. Here the twisting acts on the left coefficient of the convolution, which is compatible with the left R-module structure of M.

These constructions provide the algebraic setting for combining two skew generalized power series rings, $R[[S, \leq, \omega]]$ and $R'[[S, \leq, \omega']]$, in order to construct the $\big(R[[S, \leq, \omega]], R'[[S, \leq, \omega']]\big)$-module structure developed in the next section.

3. The Structure of Modules Over Two Skew Generalized Power Series Rings

To proceed, we recall the definition of an (R, R')-module as introduced in [31].

Definition 3.1. [31] *Let R and R' be rings, and let M be an abelian group under addition. The structure M is called an (R, R')-module if there exists a function*

$$\_ \cdot \_ \bullet \_ : R \times M \times R' \longrightarrow M$$

satisfying the following axioms:

(i) $r \cdot (m + n) \bullet r' = (r \cdot m \bullet r') + (r \cdot n \bullet r')$,
(ii) $(r + t) \cdot m \bullet r' = (r \cdot m \bullet r') + (t \cdot m \bullet r')$,
(iii) $r \cdot m \bullet (r' + t') = (r \cdot m \bullet r') + (r \cdot m \bullet t')$,
(iv) $r \cdot (t \cdot m \bullet r') \bullet t' = (rt) \cdot m \bullet (r't')$,

for all $r, t \in R$, $m, n \in M$, and $r', t' \in R'$.

The following examples illustrate Definition 3.1.

Example 3.2 (Canonical example)**.** *Let R and R' be rings, and let M be an (R, R')-bimodule in the classical sense. Define*

$$r \cdot m \bullet r' := rmr',$$

for all $r \in R$, $m \in M$, and $r' \in R'$. Then M is an (R, R')-module in the sense of Definition 3.1.

Example 3.3 (Non-bimodule example)**.** *Let $R = R' = \mathbb{R}$ and let $M = \mathbb{R}^3$. Define the ternary operation*

$$r \cdot u \bullet s := rs\, u,$$

for all $r, s \in \mathbb{R}$ and $u \in \mathbb{R}^3$. Then M is an (R, R')-module that does not arise from a classical bimodule structure.

The first example shows that the notion of an (R, R')-module extends the classical bimodule framework, while the second example demonstrates that the definition admits genuinely non-bimodule structures.

The following proposition records basic consequences of Definition 3.1, extending the axioms to finite sums in each variable.

Proposition 3.4. *Let M be an (R, R')-module. Then, for any $r, r_1, r_2, \ldots, r_k \in R$, $m, m_1, m_2, \ldots, m_k \in M$, and $r', r'_1, r'_2, \ldots, r'_k \in R'$, the following properties hold:*

(i.) $r \cdot \left(\sum_{i=1}^{k} m_i \right) \bullet r' = \sum_{i=1}^{k} \left(r \cdot m_i \bullet r' \right)$

(ii.) $\left(\sum_{i=1}^{k} r_i \right) \cdot m \bullet r' = \sum_{i=1}^{k} \left(r_i \cdot m \bullet r' \right)$

(iii.) $r \cdot m \bullet \left(\sum_{i=1}^{k} r'_i \right) = \sum_{i=1}^{k} \left(r \cdot m \bullet r'_i \right)$

(iv.) $r \cdot \left(\sum_{i=1}^{k} t_i \cdot m_i \bullet r'_i \right) \bullet t' = \sum_{i=1}^{k} \left((rt_i) \cdot m_i \bullet (r'_i t') \right)$

Proof. We prove each property as follows:

(i) For any $r \in R, r' \in R'$, and $m_1, m_2, \ldots, m_k \in M$,

$$\begin{aligned} r \cdot \left(\sum_{i=1}^{k} m_i \right) \bullet r' &= r \cdot \big(m_1 + m_2 + \cdots + m_k \big) \bullet r' \\ &= (r \cdot m_1 \bullet r') + (r \cdot m_2 \bullet r') + \cdots + (r \cdot m_k \bullet r') \\ &= \sum_{i=1}^{k} \Big(r \cdot m_i \bullet r' \Big). \end{aligned}$$

(ii) For any $r_1, r_2, \ldots, r_k \in R, r' \in R'$, and $m \in M$,

$$\begin{aligned} \left(\sum_{i=1}^{k} r_i \right) \cdot m \bullet r' &= \big(r_1 + r_2 + \cdots + r_k \big) \cdot m \bullet r' \\ &= (r_1 \cdot m \bullet r') + (r_2 \cdot m \bullet r') + \cdots + (r_k \cdot m \bullet r') \\ &= \sum_{i=1}^{k} \Big(r_i \cdot m \bullet r' \Big). \end{aligned}$$

(iii) Similar reasoning applies to $r \cdot m \bullet \left(\sum_{i=1}^{k} r'_i \right)$, establishing the desired equality.

(iv) The distributive properties of the operation ensure the validity of the fourth property.

$\square$

Throughout this paper, let R and R' be rings, $(S, \leq)$ a strictly ordered monoid, and let $\omega : S \to \mathrm{End}(R)$ and $\omega' : S \to \mathrm{End}(R')$ be monoid homomorphisms.

For the left-hand ring $R[[S, \leq, \omega]]$, we adopt a right-twisted skew convolution compatible with the left module construction described in Section 2. Accordingly, multiplication in $R[[S, \leq, \omega]]$ is defined by

$$(fg)(s) = \sum_{(x,y) \in \chi_s(f,g)} \omega_y\big(f(x)\big) g(y), \tag{5}$$

for all $f, g \in R[[S, \leq, \omega]]$.

For the right-hand ring $R'[[S, \leq, \omega']]$, we use the standard skew convolution recalled in Section 2. Thus multiplication is defined by

$$(f'g')(s) = \sum_{(z,w) \in \chi_s(f',g')} f'(z) \omega'_z\big(g'(w)\big), \tag{6}$$

for all $f', g' \in R'[[S, \leq, \omega']]$.

If M is an (R, R')-module as in Definition 3.1, we define

$$M[[S, \leq]] = \{\alpha : S \to M \mid \mathrm{supp}(\alpha) \text{ is artinian and narrow}\},$$

which is an abelian group under pointwise addition

$$(\alpha + \beta)(s) = \alpha(s) + \beta(s), \tag{7}$$

for all $\alpha, \beta \in M[[S, \leq]]$.

We now extend the (R, R')-module structure of M to the level of skew generalized power series. To this end, we construct an operation combining the skew multiplications in $R[[S, \leq, \omega]]$ and $R'[[S, \leq, \omega']]$ with the (R, R')-action on M given in Definition 3.1.

The construction is designed so that the twisting maps ω and ω' act compatibly with the two-sided module structure, thereby producing a $\big(R[[S, \leq, \omega]], R'[[S, \leq, \omega']]\big)$-module.

For any $f \in R[[S, \leq, \omega]]$, $\alpha \in M[[S, \leq]]$, $f' \in R'[[S, \leq, \omega']]$, and $s \in S$, define

$$\chi_s(f, \alpha, f') = \{(x, y, z) \in \mathrm{supp}(f) \times \mathrm{supp}(\alpha) \times \mathrm{supp}(f') \mid xyz = s\}. \tag{8}$$

We emphasize that the ternary operation introduced below is a primary construction. It is not obtained by iterating (5) and (6), but is designed to encode simultaneously the skew multiplications and the (R, R')-module action.

With this notation in place, define the ternary operation

$$\_ * \_ *' \_ : R[[S, \leq, \omega]] \times M[[S, \leq]] \times R'[[S, \leq, \omega']] \longrightarrow M[[S, \leq]]$$

by

$$(f * \alpha *' f')(s) = \sum_{(x,y,z) \in \chi_s(f,\alpha,f')} \omega_y\big(f(x)\big) \cdot \alpha(y) \bullet \omega'_y\big(f'(z)\big). \tag{9}$$

Here the middle index y plays the role of a pivot. Both twisting maps ω and ω' are evaluated at the same element y, ensuring compatibility of the two-sided action.

The following lemma ensures that the above operation is well defined.

Lemma 3.5. *If* $f \in R[[S, \le, \omega]]$*,* $\alpha \in M[[S, \le]]$*, and* $f' \in R'[[S, \le, \omega']]$*, then* $f * \alpha *' f' \in M[[S, \le]]$*.*

Proof. By assumption, the supports of f, α, and f' are artinian and narrow subsets of $(S, \le)$. For any $s \in S$, the sum defining $(f*\alpha*' f')(s)$ involves only triples (x, y, z) such that $x \in \mathrm{supp}(f)$, $y \in \mathrm{supp}(\alpha)$, and $z \in \mathrm{supp}(f')$ with $xyz = s$. Hence,

$$\mathrm{supp}(f * \alpha *' f') \subseteq \mathrm{supp}(f) + \mathrm{supp}(\alpha) + \mathrm{supp}(f').$$

Since finite sums of artinian and narrow subsets of a strictly ordered monoid remain artinian and narrow (see [7]), it follows that $\mathrm{supp}(f * \alpha *' f')$ is artinian and narrow. Therefore, $f * \alpha *' f' \in M[[S, \le]]$. □

The next theorem shows that the above construction yields a genuine module structure over two skew generalized power series rings.

Theorem 3.6. *If* M *is an* (R, R')*-module, then the abelian group* $M[[S, \le]]$*, endowed with the operations defined in* (7) *and* (9)*, is a* $\big(R[[S, \le, \omega]], R'[[S, \le, \omega']]\big)$*-module.*

Proof. To prove that $M[[S, \le]]$ is an $\big(R[[S, \le, \omega]], R'[[S, \le, \omega']]\big)$-module, we verify the axioms in Definition 3.1. Let $f, g \in R[[S, \le, \omega]]$, $\alpha, \beta \in M[[S, \le]]$, $f', g' \in R'[[S, \le, \omega']]$, and $s \in S$;

(i) Additivity of α:

$$\begin{aligned}
(f * (\alpha + \beta) *' f')(s) &= \sum_{xyz=s} \omega_y(f(x)) \cdot (\alpha + \beta)(y) \bullet \omega'_y(f'(z)) \\
&= \sum_{xyz=s} \omega_y(f(x)) \cdot (\alpha(y) + \beta(y)) \bullet \omega'_y(f'(z)) \\
&= \sum_{xyz=s} \omega_y(f(x)) \cdot \alpha(y) \bullet \omega'_y(f'(z)) \\
&+ \sum_{xyz=s} \omega_y(f(x)) \cdot \beta(y) \bullet \omega'_y(f'(z)) \\
&= (f * \alpha *' f')(s) + (f * \beta *' f')(s).
\end{aligned}$$

Therefore, $f * (\alpha + \beta) *' f' = f * \alpha *' f' + f * \beta *' f'$.

(ii) Additivity of f:

$$\begin{aligned}
((f + g) * \alpha *' f')(s) &= \sum_{xyz=s} \omega_y((f + g)(x)) \cdot \alpha(y) \bullet \omega'_y(f'(z)) \\
&= \sum_{xyz=s} \omega_y(f(x) + g(x)) \cdot \alpha(y) \bullet \omega'_y(f'(z)) \\
&= \sum_{xyz=s} \big(\omega_y(f(x)) + \omega_y(g(x))\big) \cdot \alpha(y) \bullet \omega'_y(f'(z)) \\
&= \sum_{xyz=s} \omega_y(f(x)) \cdot \alpha(y) \bullet \omega'_y(f'(z))
\end{aligned}$$

$$+ \sum_{xyz=s} \omega_y(g(x)) \cdot \alpha(y) \bullet \omega'_y(f'(z))$$
$$= (f * \alpha *' f')(s) + (g * \alpha *' f')(s).$$

Hence, $(f+g) * \alpha *' f' = f * \alpha *' f' + g * \alpha *' f'$.

(iii) Additivity of f':

$$\begin{aligned}
(f * \alpha *' (f'+g'))(s) &= \sum_{xyz=s} \omega_y(f(x)) \cdot \alpha(y) \bullet \omega'_y((f'+g')(z)) \\
&= \sum_{xyz=s} \omega_y(f(x)) \cdot \alpha(y) \bullet \omega'_y(f'(z)+g'(z)) \\
&= \sum_{xyz=s} \omega_y(f(x)) \cdot \alpha(y) \bullet \big(\omega'_y(f'(z)) + \omega'_y(g'(z))\big) \\
&= \sum_{xyz=s} \omega_y(f(x)) \cdot \alpha(y) \bullet \omega'_y(f'(z)) \\
&+ \sum_{xyz=s} \omega_y(f(x)) \cdot \alpha(y) \bullet \omega'_y(g'(z)) \\
&= (f * \alpha *' f')(s) + (f * \alpha *' g')(s).
\end{aligned}$$

Thus, $f * \alpha *' (f'+g') = f * \alpha *' f' + f * \alpha *' g'$.

(iv) Associativity: For any $s \in S$, we compute

$$(f * (g * \alpha *' f') *' g')(s) = \sum_{xyz=s} \omega_y(f(x)) \cdot (g * \alpha *' f')(y) \bullet \omega'_y(g'(z)).$$

By the definition of the ternary operation, each term $(g * \alpha *' f')(y)$ is given by a finite sum over triples (a,b,c) with $abc = y$. Substituting this expression and rearranging the finite sums, we obtain a sum over all tuples (x,a,b,c,z) such that $xabcz = s$.

Using the monoid homomorphism properties of ω and ω', namely $\omega_{uv} = \omega_u \circ \omega_v$ and $\omega'_{uv} = \omega'_u \circ \omega'_v$, together with the associativity of the product in S, the summands can be regrouped to yield

$$\omega_b\big((fg)(t)\big) \cdot \alpha(b) \bullet \omega'_b\big((f'g')(w)\big),$$

where $t = xa$ and $w = cz$ with $tbw = s$.

Therefore,

$$(f * (g * \alpha *' f') *' g')(s) = ((fg) * \alpha *' (f'g'))(s),$$

which shows that the associativity axiom holds.

□

Example 3.7. *Let $S = (\mathbb{N}, \leq)$ be the additive monoid of nonnegative integers endowed with the natural order. Let $R = \mathbb{Z}$ and $R' = \mathbb{Z}[x]$. Define homomorphisms*

$$\omega : S \to \mathrm{End}(R), \qquad \omega(n) = \mathrm{id}_{\mathbb{Z}},$$

and

$$\omega' : S \to \mathrm{End}(R'), \qquad \omega'(n)(f'(x)) = f'(x^{2^n}),$$

for all $n \in S$ *and* $f'(x) \in \mathbb{Z}[x]$.

Then $R[[S, \leq, \omega]]$ *and* $R'[[S, \leq, \omega']]$ *are skew generalized power series rings with different twisting behaviours: the left ring has trivial twisting, while the right ring carries a nontrivial endomorphism action determined by* ω'.

Let $M = \mathbb{Z}[x]$, *viewed as an* (R, R')*-module via the usual left and right scalar multiplications. By Theorem 3.6, the abelian group* $M[[S, \leq]]$ *becomes a* $\big(R[[S, \leq, \omega]], R'[[S, \leq, \omega']]\big)$*-module.*

In this case, for $f \in R[[S, \leq, \omega]]$, $\alpha \in M[[S, \leq]]$, $f' \in R'[[S, \leq, \omega']]$, *and* $n \in S$, *the ternary operation becomes*

$$(f * \alpha * f')(n) = \sum_{i+j+k=n} f(i) \cdot \alpha(j) \bullet \omega'_j\big(f'(k)\big).$$

Since $\omega'_j\big(f'(k)\big) = f'(k)(x^{2^j})$, *the right action depends essentially on the skew structure induced by* ω'.

The following example provides an explicit computation illustrating the interaction between the two skew structures in the ternary operation.

Example 3.8. *Let* $S = (\mathbb{N}, +)$, $R = \mathbb{Z}$, *and* $R' = \mathbb{Z}[x]$. *Define* $\omega_n = \mathrm{id}_{\mathbb{Z}}$ *and*

$$\omega'_n(f'(x)) = f'(x^{2^n}).$$

Let $M = \mathbb{Z}[x]$ *be an* (R, R')*-module with the usual scalar multiplications. Recall that the elements* c_r *and* f_s *in* $R[[S, \leq, \omega]]$ *and* $c'_{r'}$ *and* f'_s *in* $R'[[S, \leq, \omega']]$ *are defined in* (1) *and* (2), *while the elements* d^s_m *in* $M[[S, \leq, \omega']]$ *are defined in* (4).

Consider the elements

$$f = f_1, \qquad \alpha = d^2_x, \qquad f' = f'_1.$$

Since $1 + 2 + 1 = 4$, *the only triple contributing to* $(f * \alpha *' f')(4)$ *is* $(1, 2, 1)$. *Hence*

$$(f * \alpha *' f')(4) = \omega_2\big(f(1)\big)\, \alpha(2)\, \omega'_2\big(f'(1)\big).$$

Because $f(1) = 1$, $\alpha(2) = x$, *and* $f'(1) = 1$, *we obtain*

$$(f * \alpha *' f')(4) = \omega_2(1) \cdot x \cdot \omega'_2(1).$$

Since $\omega_2(1) = 1$ *and* $\omega'_2(1) = 1$, *it follows that*

$$(f * \alpha *' f')(4) = x.$$

This computation illustrates how the skew action determined by ω' *appears in the convolution process.*

The twisting map ω' is defined on the coefficient ring $R' = \mathbb{Z}[x]$, hence no support condition is involved at this level. The artinian and narrow support assumption appears only in the construction of the skew generalized power series ring $R'[[S, \leq, \omega']]$ and ensures that the sums arising in the multiplication and in the induced module operations are finite.

Example 3.9. *Let $R'' = \mathbb{Z}$ and let $S = \mathbb{N} \times \mathbb{N}$ be the additive monoid endowed with the product order. Define*

$$\omega''_{(m,n)}(k) = 2^{m+n}k, \qquad k \in \mathbb{Z}.$$

Then $\omega'' : S \to \mathrm{End}(R'')$ is a monoid homomorphism, and the skew generalized power series ring $R''[[S, \leq, \omega'']]$ is well defined.

This example shows that the theory applies to ordered monoids and twisting actions that differ substantially from those appearing in Example 3.7.

Examples 3.7 and 3.9 illustrate the robustness of the $\big(R[[S, \leq, \omega]], R'[[S, \leq, \omega']]\big)$-module framework with respect to variations in both the twisting mechanisms and the underlying ordered monoid structures.

4. On $\big(R[[S, \leq, \omega]], R'[[S, \leq, \omega']]\big)$-module Homomorphisms

We begin by recalling the definition of an (R, R')-module homomorphism, as introduced in [32].

Definition 4.1. *Let M and N be (R, R')-modules. An additive map $\mu : M \to N$ is called an (R, R')-module homomorphism if it satisfies*

$$\mu(r \cdot m \bullet r') = r \cdot \mu(m) \bullet r'$$

for all $r \in R$, $m \in M$, and $r' \in R'$.

Before defining homomorphisms between $\big(R[[S, \leq, \omega]], R'[[S, \leq, \omega']]\big)$-modules, we establish the following lemma.

Lemma 4.2. *Let M and N be (R, R')-modules, and let $M[[S, \leq]]$ and $N[[S, \leq]]$ be $\big(R[[S, \leq, \omega]], R'[[S, \leq, \omega']]\big)$-modules. If $\mu : M \to N$ is an (R, R')-module homomorphism, then:*

(i) $\mu \circ (\alpha_1 + \alpha_2) = (\mu \circ \alpha_1) + (\mu \circ \alpha_2)$,
(ii) $\mu \circ (f * \alpha *' f') = f * (\mu \circ \alpha) *' f'$,

for all $\alpha_1, \alpha_2 \in M[[S, \leq]]$, $f \in R[[S, \leq, \omega]]$, and $f' \in R'[[S, \leq, \omega']]$.

Proof. (i) follows directly from the additivity of μ.

(ii) Let $s \in S$. By definition of the ternary operation, we have

$$(f * \alpha *' f')(s) = \sum_{xyz=s} \omega_y(f(x)) \cdot \alpha(y) \bullet \omega'_y(f'(z)),$$

where the sum is finite. Since μ is additive, we may apply μ termwise to this sum. Moreover, because μ is an (R, R')-module homomorphism, it satisfies

$$\mu(r \cdot m \bullet r') = r \cdot \mu(m) \bullet r' \quad \text{for all } r \in R,\ m \in M,\ r' \in R'.$$

Therefore,

$$(\mu \circ (f * \alpha *' f'))(s) = \sum_{xyz=s} \mu\big(\omega_y(f(x)) \cdot \alpha(y) \bullet \omega'_y(f'(z))\big)$$

$$= \sum_{xyz=s} \omega_y(f(x)) \cdot \mu(\alpha(y)) \bullet \omega'_y(f'(z))$$
$$= (f * (\mu \circ \alpha) *' f')(s).$$

Hence, $\mu \circ (f * \alpha *' f') = f * (\mu \circ \alpha) *' f'$. $\square$

The following diagram describes how an (R, R')-module homomorphism induces a homomorphism between the corresponding skew generalized power series modules:

$$\begin{array}{ccc} M & \xrightarrow{\mu} & N \\ \downarrow{\scriptstyle \iota_M} & & \downarrow{\scriptstyle \iota_N} \\ M[[S,\le,\omega]] & \xrightarrow{\sigma} & N[[S,\le,\omega']] \end{array}$$

Here ι_M and ι_N denote the canonical embeddings defined in (4), which identify each element with the skew generalized power series supported at a fixed $s \in S$. Explicitly,

$$\iota_M(m) = d^s_m, \qquad \iota_N(n) = d^s_n.$$

With these identifications, the diagram is commutative; that is,

$$\sigma\big(\iota_M(m)\big) = \iota_N\big(\mu(m)\big) \quad \text{for all } m \in M,$$

or equivalently,

$$\sigma(d^s_m) = d^s_{\mu(m)}.$$

Thus the assignment $M \mapsto M[[S,\le,\omega]]$ is compatible with module homomorphisms.

Using Lemma 4.2, we can extend the notion of an (R, R')-module homomorphism to the setting of $\big(R[[S,\le,\omega]], R'[[S,\le,\omega']]\big)$-modules.

Theorem 4.3. *Let M and N be (R, R')-modules, and let $M[[S,\le]]$ and $N[[S,\le]]$ be $\big(R[[S,\le,\omega]], R'[[S,\le,\omega']]\big)$-modules. If $\mu : M \to N$ is an (R, R')-module homomorphism, then there exists an $\big(R[[S,\le,\omega]], R'[[S,\le,\omega']]\big)$-module homomorphism*

$$\sigma : M[[S,\le]] \to N[[S,\le]].$$

Proof. Define a map $\sigma : M[[S,\le]] \to N[[S,\le]]$ for every $\alpha \in M[[S,\le]]$ as $\sigma(\alpha) = \mu\circ\alpha$. To prove σ is an $(R[[S,\le,\omega]], R'[[S,\le,\omega']])$-module homomorphism, we verify the following:

(i) Additivity: For any $\alpha_1, \alpha_2 \in M[[S,\le]]$,

$$\begin{aligned} \sigma(\alpha_1+\alpha_2) &= \mu \circ (\alpha_1+\alpha_2) \\ &= (\mu\circ\alpha_1) + (\mu\circ\alpha_2) \quad \text{by Lemma 4.2}(i) \\ &= \sigma(\alpha_1) + \sigma(\alpha_2). \end{aligned}$$

(ii) Scalar Multiplication: For any $\alpha \in M[[S,\le]]$, $f \in R[[S,\le,\omega]]$, and $f' \in R'[[S,\le,\omega']]$,

$$\begin{aligned} \sigma(f * \alpha *' f') &= \mu \circ (f * \alpha *' f') \\ &= f * (\mu\circ\alpha) *' f' \quad \text{by Lemma 4.2}(ii) \end{aligned}$$

$$= f * \sigma(\alpha) *' f'.$$

Thus, σ preserves addition and scalar multiplication. Therefore, σ is an $(R[[S,\leq,\omega]], R'[[S,\leq,\omega']])$-module homomorphism. $\square$

Next, we define the kernel of an $\big(R[[S,\leq,\omega]], R'[[S,\leq,\omega']]\big)$-module homomorphism, extending the notion of the kernel for (R,R')-module homomorphisms introduced in [32].

Definition 4.4. *Let $\sigma : M[[S,\leq]] \to N[[S,\leq]]$ be an $\big(R[[S,\leq,\omega]], R'[[S,\leq,\omega']]\big)$-module homomorphism. The kernel of σ is defined by*

$$Ker(\sigma) = \{\alpha \in M[[S,\leq]] \mid \sigma(\alpha) = 0_{N[[S,\leq]]}\}.$$

The following proposition gives a straightforward coefficientwise criterion for membership in the kernel of the induced homomorphism.

Proposition 4.5. *Let $\mu : M \to N$ be an (R,R')-module homomorphism, and let*

$$\sigma : M[[S,\leq,\omega]] \to N[[S,\leq,\omega]]$$

be the induced $\big(R[[S,\leq,\omega]], R'[[S,\leq,\omega']]\big)$-module homomorphism defined by

$$\sigma(\alpha) = \mu \circ \alpha.$$

If $\alpha \in M[[S,\leq]]$ satisfies

$$\alpha(s) \in \ker(\mu) \quad \textit{for all } s \in S,$$

then $\alpha \in \ker(\sigma)$.

Proof. If $\alpha(s) \in \ker(\mu)$ for all $s \in S$, then $\mu(\alpha(s)) = 0$ for every $s \in S$. Hence

$$\sigma(\alpha)(s) = (\mu \circ \alpha)(s) = \mu(\alpha(s)) = 0 \quad \text{for all } s \in S.$$

Therefore $\sigma(\alpha) = 0$ in $N[[S,\leq]]$, so $\alpha \in \ker(\sigma)$. $\square$

We conclude this section with examples illustrating the coefficientwise kernel criterion in different algebraic settings.

Example 4.6. *Let $R = \mathbb{Z}$ and $R' = \mathbb{Z}/2\mathbb{Z}$. Consider the $\mathbb{Z}$-modules $M = N = \mathbb{Z}$ and define the (R,R')-module homomorphism*

$$\mu : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}, \qquad \mu(n) = \overline{n}.$$

Then $Ker(\mu) = 2\mathbb{Z}$.

Let $(S,\leq)$ be a totally ordered monoid, for instance $(\mathbb{N},\leq)$, and let

$$\sigma : M[[S,\leq]] \to N[[S,\leq]]$$

be the homomorphism induced by μ, defined by

$$(\sigma(\alpha))(s) = \mu(\alpha(s)) \quad \textit{for all } s \in S.$$

Define $\alpha \in M[[S,\leq]]$ by

$$\alpha(s) = 2s \quad \textit{for all } s \in \mathbb{N}.$$

Then $\alpha(s) \in Ker(\mu)$ for every $s \in S$, since $2s \in 2\mathbb{Z}$. Hence, by Proposition 4.5, we obtain $\alpha \in Ker(\sigma)$. Indeed, $\sigma(\alpha)$ is the zero element of $N[[S,\leq]]$.

This example shows that the kernel of the induced homomorphism is determined entirely coefficientwise. The presence of twisting maps does not affect this criterion, since the induced map σ acts pointwise on coefficients.

We now turn to situations where the twisting maps are nontrivial and the skew structure plays an essential role.

Example 4.7 (A genuinely skew example)**.** *Let $S = (\mathbb{N}, +)$ and let k be a field. Set $R = k[x]$ and $R' = k[y]$. Define monoid homomorphisms*

$$\omega_n(f(x)) = f(x+n), \qquad \omega'_n(g(y)) = g(y+n),$$

for all $n \in \mathbb{N}$. Then ω and ω' are nontrivial homomorphisms from S into $\mathrm{End}(R)$ and $\mathrm{End}(R')$, respectively.

For $\alpha, \beta \in R[[S, \leq, \omega]]$, the convolution product is given by

$$(\alpha * \beta)(n) = \sum_{i+j=n} \alpha(i)\, \omega_i(\beta(j)) = \sum_{i+j=n} \alpha(i)\, \beta(j)(x+i).$$

Thus each coefficient $\beta(j)$ is shifted by i before multiplication, and the product depends essentially on the monoid index.

Therefore, the multiplication in $R[[S, \leq, \omega]]$ cannot be reduced to the classical generalized power series multiplication. The same phenomenon occurs in $R'[[S, \leq, \omega']]$, where coefficients are shifted in the y-variable. This provides a concrete commutative example in which the skew structure is genuinely present.

The previous example remains commutative at the level of coefficients. We next exhibit a genuinely noncommutative situation.

Example 4.8 (A noncommutative genuinely skew example)**.** *Let $S = (\mathbb{N}, +)$ and let k be a field. Set $R = R' = M_2(k)$, the ring of 2×2 matrices over k.*

Fix an invertible matrix $P \in M_2(k)$ and define

$$\omega_n(A) = P^n A P^{-n}, \qquad \omega'_n(B) = P^{-n} B P^n,$$

for all $n \in \mathbb{N}$ and $A, B \in M_2(k)$. Then ω and ω' are monoid homomorphisms from S into $\mathrm{Aut}(M_2(k))$ given by inner automorphisms.

For $\alpha, \beta \in R[[S, \leq, \omega]]$, the convolution product is

$$(\alpha * \beta)(n) = \sum_{i+j=n} \alpha(i)\, \omega_i(\beta(j)) = \sum_{i+j=n} \alpha(i)\, P^i \beta(j) P^{-i}.$$

Thus each coefficient $\beta(j)$ is conjugated by P^i before multiplication with $\alpha(i)$. Since matrix multiplication in $M_2(k)$ is noncommutative, the order of twisting and multiplication is essential, and the product cannot be simplified to a classical form.

Similarly, in $R'[[S, \leq, \omega']]$ the coefficients are conjugated by P^{-i}, so that the left and right twisting mechanisms act in opposite directions. Consequently, the two skew generalized power series rings interact in a genuinely noncommutative manner, reflecting the essential role of inner automorphisms in the skew convolution structure.

Taken together, the preceding examples illustrate three distinct aspects of the theory. The first shows that the kernel of the induced homomorphism is determined coefficientwise. The second demonstrates that nontrivial twisting already produces skew behavior even when the coefficient rings are commutative. The third exhibits a genuinely noncommutative skew interaction arising from inner automorphisms. These examples clarify that the framework of two skew generalized power series rings extends the classical theory both at the level of module homomorphisms and at the level of convolution multiplication.

5. CONCLUDING REMARKS

In this paper, we developed a systematic framework for modules over *two distinct skew generalized power series rings.* Starting from the notion of (R, R')-modules, we constructed the induced module $M[[S, \leq]]$ endowed with a trilinear action

$$\_ * \_ *' \_ : R[[S, \leq, \omega]] \times M[[S, \leq]] \times R'[[S, \leq, \omega']] \longrightarrow M[[S, \leq]],$$

and proved that it satisfies the axioms of an $\big(R[[S, \leq, \omega]], R'[[S, \leq, \omega']]\big)$-module. This construction extends the classical skew generalized power series module theory to a two-ring setting, providing a natural algebraic framework that captures bimodule-type behavior governed by two independent skew actions.

We further investigated the behavior of homomorphisms in this context. Specifically, we showed that every (R, R')-module homomorphism $\mu : M \to N$ induces a homomorphism $\sigma : M[[S, \leq]] \to N[[S, \leq]]$ between the corresponding modules over skew generalized power series rings. In addition, we established a coefficientwise sufficient condition for elements of $M[[S, \leq]]$ to belong to $\ker(\sigma)$ in terms of membership in $\ker(\mu)$. Although this characterization follows directly from the induced coefficientwise action, it confirms the structural compatibility of the construction within the two-ring skew framework. The examples provided illustrate how the presence of two independent skew actions influences the surrounding module structure.

More precisely, the main contributions of this paper consist of: (i) the construction of $\big(R[[S, \leq, \omega]], R'[[S, \leq, \omega']]\big)$-module structures from (R, R')-modules; (ii) the extension of (R, R')-module homomorphisms to homomorphisms between the associated skew generalized power series modules; and (iii) a coefficientwise sufficient condition describing elements in the kernel of such induced homomorphisms.

The framework developed here provides a concrete algebraic foundation for further investigations of modules over two skew generalized power series rings. In particular, it offers a natural setting for studying isomorphism theorems, quotient constructions, and exact sequences in this context, as well as for analyzing the transfer of structural properties (for example, finiteness conditions and Noetherian-type behavior) from (R, R')-modules to their associated skew generalized power series modules. These directions constitute natural avenues for future research.

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Author Contributions. Ahmad Faisol: Conceptualization, formal analysis, investigation, methodology, project administration, resources, validation, writing-original draft. Fitriani: Formal analysis, investigation, validation, writing-review & editing. Muslim Ansori: Formal analysis and validation. Budi Surodjo: Formal analysis, methodology, validation. All authors discussed the results and contributed to the final manuscript.

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