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Energy and Degree Sum Energy of Non-coprime Graphs on Dihedral Groups

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Abstract. Research on graphs has increasingly garnered attention in recent years.This research focuses on graph representations, with particular emphasis on non-coprime graphs within the dihedral group D_{2n} with $n = p^k$, p prime numbers, $k \in \mathbb{Z}^+$. The non-coprime graph of a group G is defined as a graph in which the vertex set is $G \setminus \{e\}$, and two distinct vertices r and s are connected by an edge if $\gcd(|r|, |s|) \neq 1$. Specifically, this research examines the adjacency matrix energy and the degree sum energy of non-coprime graphs on dihedral groups. With the extensive application of chemical topological graphs in the field of chemistry, it is hoped that they can assist in the numerical analysis of chemical compounds used in healthcare, such as the analysis of vaccines for the COVID-19 epidemic.

Key words and Phrases: non-coprime graph, degree sum energy, adjacency matrix energy, epidemic, vaccine.

1. INTRODUCTION

Research on graphs has gained significant attention from researchers in recent years. A graph is a mathematical structure consisting of a non-empty set of vertices and a set of edges that connect those vertices. Graphs can also be utilized to predict the properties of chemical compounds using the concept of graph homomorphisms [1].

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Many researchers explore graph representations of algebraic structures known as groups. The study by [2] focuses on graph representations on the non-coprime graph of a group G, which is defined as a graph in which the vertex $G \setminus \{e\}$, where two distinct vertices r and s are connected by an edge if $\gcd(|r|, |s|) \neq 1$. Non-coprime graphs have significant potential applications in various fields, such as chemical graph theory, which is used for modeling atomic interactions in complex molecules, as well as network science, which analyzes the structure and dynamics of large systems with unique properties. These graphs can also be applied in network optimization, including infrastructure design and security algorithms, leveraging their unique spectral properties. In the past, many studies on topological indices have focused on exploring their role in characterizing graph structures and their wide-ranging applications in mathematics and science [3].

The research by [4] explored the characteristics of the coprime graph formed by groups modulo integers. On the same groups, [5] research is further expanded by specifically focusing on the structure and properties of non-coprime graphs. The works of [6] did on the generalized quaternion groups and [7], did on the dihedral group is denoted by D_{2n}. In addition, [8] focused on identifying and analyzing the specific circumstances that enable the coprime graph of dihedral groups to exhibit distinct structural forms. It explored the conditions under which these graphs can be classified as complete bipartite graphs, complete tripartite graphs, or, more generally, complete k-partite graphs. The adjacency matrix energy and degree sum energy are important concepts in graph theory, used to analyze graph structures through the spectra of related matrices. The adjacency matrix energy measures the sum of the absolute values of the eigenvalues of a graph's adjacency matrix, providing insights into its connectivity and relationship distribution. Meanwhile, the degree sum energy is based on a graph's degree matrix, reflecting the contribution of vertex degrees to the total energy. These two concepts bridge spectral theory with practical applications, such as network analysis, theoretical chemistry, transportation networks, cybersecurity, and structural optimization. Some direct applications of chemical topological graphs include measuring entropy [9], and analyzing vaccine for COVID-19 treatment[10]. Building on these foundational ideas, this study aims to deepen the understanding of the interplay between group properties and their graphical representations. Specifically, it focuses on calculating the energy and degree sum energy of the non-coprime graph of the dihedral group D_{2n} where $n = p^k, p$ is a prime number, and $k \in \mathbb{Z}$. The dihedral group was chosen for this study because of its simple yet rich algebraic structure, which includes both rotational and reflectional symmetry. This makes it an ideal framework for exploring the properties of non-coprime graphs in a context that allows for in-depth analysis.

2. MAIN RESULTS

In exploring the properties of the dihedral group, D_{2n} as discussed by[11], the group is composed of both rotational and reflectional elements, which are associated with symmetries of a regular polygon with n sides, where $\{x\}$ denotes a rotation by an angle of $\frac{360^\circ}{n}$ and $\{y\}$ represents a reflection. The group representation of the dihedral group is expressed as:

$$D_{2n} = \{x, y | x^n = y^2 = e, x^{-1} = bab^{-1}\}, n \in \mathbb{N}, n \neq 1, 2. \quad (1)$$

Fundamental in graph theory and connectivity is described by the concept of vertex degree, as defined in the following.

Definition 2.1. [12]*(Vertex Degree). Let Γ be a graph, where $V(\Gamma)$ represents the set of vertices. The degree of $v_i \in V(\Gamma)$ is defined as the number of edges connected to v_i, denoted by d_i.*

The concept of energy and degree sum energy of graphs is often associated with matrix representations, such as degree sum matrices or adjacency matrices. The determinant of a unique matrix can be calculated using methods derived from the following two lemmas.

Lemma 2.2. [13]*If we have a matrix of order p, and μ,ζ are scalars then*

$$\begin{vmatrix} \mu & \zeta & \cdots & \zeta \\ \zeta & \mu & \cdots & \zeta \\ \vdots & \vdots & \ddots & \vdots \\ \zeta & \zeta & \cdots & \mu \end{vmatrix} = (\mu - \zeta)^{p-1}[\mu - (p-1)\zeta]. \quad (2)$$

Lemma 2.3. [13]*Let P be of matrix a size $m \times m$, composed of submatrix $J, K, L, 0$ arranged in a specific matrix form $P = \begin{pmatrix} J & K \\ 0 & L \end{pmatrix}$, then $det(P) = det(J).det(L)$ with matrices J and L are square matrices.*

Definition 2.4. [14]*If Γ is a graph and ϕ is an eigenvalue of the graph matrix of Γ, then the energy of Γ is defined as*

$$E(\Gamma) = \sum_{i=1}^{n} |\phi_i|. \quad (3)$$

2.1. **Adjacency Matrix Energy.**

Definition 2.5. [14]*Let Γ be a graph with vertices v_i for $i = 1, 2, ..., n$, where n is total number of vertices. The adjacency matrix of Γ is $AM(\Gamma) = [a_{ij}]$, where*

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \text{adjacency in } \Gamma, \\ 0, & \text{else.} \end{cases} \quad (4)$$

Theorem 2.6. *Let D_{2n} be a dihedral group with $n = 2^k$, where $k \in \mathbb{Z}^+$, then the adjacency matrix energy of $\Gamma_{D_{2n}}$ is*

$$E_{AM}(\Gamma_{D_{2(2^k)}}) = 4(2^k - 1). \quad (5)$$

Proof. Let $D_{2(2^k)} = \{a, b \mid a^{2^k} = b^2 = e, a^{-1} = bab^{-1}\}, k \in \mathbb{Z}^+$, then the order of any element rotations $(a) \in D_{2(2^k)}$ is of the form 2^p, where $p \in \mathbb{Z}^+$. However, for all reflections $(b) \in D_{2(2^k)}$, the order is 2. The $\gcd(|x|, |y|) = 2$, $\forall x, y \in D_{2(2^k)}$ then $\{a, b\}$ adjacency in Γ. Since the number of elements in $D_{2(2^k)}$ is $2(2^k)$ and the non-coprime graph has $D_{2(2^k)} \setminus \{e\}$ vertices, the number of vertices of the graph Γ is $2(2^k) - 1$. Thus, the adjacency matrix of Γ will be of order $(2(2^k) - 1) \times (2(2^k) - 1)$ and based on **Definition 2.5** the adjacency matrix of $D_{2(2^k)}$ is

$$AM(\Gamma_{D_{2(2^k)}}) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}. \tag{6}$$

The eigenvalues of the adjacency matrix are determined by solving the corresponding characteristic equation:

$$|AM(\Gamma_{D_{2(2^k)}}) - \phi I| = |\phi I - AM(\Gamma_{D_{2(2^k)}})| = 0. \tag{7}$$

$$\begin{vmatrix} \phi & -1 & \cdots & -1 \\ -1 & \phi & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \phi \end{vmatrix} = 0. \tag{8}$$

Based on **Lemma 2.2**,

$$\begin{vmatrix} \phi & -1 & \cdots & -1 \\ -1 & \phi & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \phi \end{vmatrix} = (\phi - (-1))^{(2(2^k)-1)-1}[\phi - ((2(2^k) - 1) - 1)(-1))] = 0.$$

Thus we have

$$(\phi + 1)^{2(2^k)-2}[\phi + (2(2^k) - 2)] = 0.$$

$$(\phi + 1)^{2(2^k)-2} = 0 \text{ or } [\phi + (2(2^k) - 2)] = 0.$$

As a result, we obtain $\phi = -1$ with multiplicity $2(2k) - 2$ and $\phi = -(2(2^k) - 2)$ with a multiplicity 1. Using **Definition 2.4**, the adjacency matrix energy of the graph can be calculated as follows:

$$\begin{aligned} E_{AM}(\Gamma_{D_{2(2^k)}}) &= \sum_{i=1}^{n} |\phi_i| \\ &= (2(2^k(-2)| - 1| + | - (2(2^k) - 2)| \\ &= (2(2^k) - 2) + (2(2^k) - 2) \\ &= 4(2^k) - 4 \\ &= 4((2^k) - 1). \end{aligned}$$

□

Theorem 2.7. *Let D_{2n} be a dihedral group with $n = p^k$, where p is a prime number (except 2), $k \in \mathbb{Z}^+$, then the adjacency matrix energy of $\Gamma_{D_{2n}}$ is*

$$E_{AM}(\Gamma_{D_{2(p^k)}}) = 2(2p^k - 3). \tag{9}$$

Proof. Let $D_{2(p^k)} = \{a, b \mid a^{p^k} = b^2 = e, a^{-1} = bab^{-1}\}$, where p is a prime number (except 2), $k \in \mathbb{Z}^+$, then the order of any element rotations $(a) \in D_{2(p^k)}$ is of the form p^m, where $m \in \mathbb{Z}^+$. However, for all reflections $(b) \in D_{2(p^k)}$, the order is 2. The $\gcd(|x|, |y|) = 1$, $\forall x \in \{a\}$ and $\forall y \in \{b\}$ then $\{a, b\}$ not adjencent in Γ. Since the number of rotations elements in $D_{2(p^k)}$ is $p^k - 1$, the number of reflections elements in $D_{2(p^k)}$ is p^k and the non-coprime graph has $D_{2(p^k)} \setminus \{e\}$ vertices, the number of vertices of the graph Γ is $2n - 1$. Since the elements type a and b are always non-adjacent, the adjacency matrix can be decomposed into two submatrix, say matrix A and matrix B. Thus, the adjacency submatrix A of will be of order $(p^k - 1) \times (p^k - 1)$, the adjacency submatrix B of will be of order $(p^k) \times (p^k)$ and based on **Definition 2.5** the adjacency matrix of $D_{2(p^k)}$ is

$$AM(\Gamma_{D_{2(p^k)}}) = \begin{pmatrix} 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 \end{pmatrix}. \tag{10}$$

The eigenvalues of the adjacency matrix are determined by solving the corresponding characteristic equation:

$$|AM(\Gamma_{D_{2(p^k)}}) - \phi I| = |\phi I - AM(\Gamma_{D_{2(p^k)}})| = 0. \tag{11}$$

$$\begin{vmatrix} \phi & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ -1 & \phi & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \phi & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \phi & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 & -1 & \phi & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & \phi \end{vmatrix} = 0. \tag{12}$$

Based on **Lemma 2.3**,

$$\underbrace{\begin{vmatrix} \phi & -1 & \cdots & -1 \\ -1 & \phi & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \phi \end{vmatrix}}_{detA}\underbrace{\begin{vmatrix} \phi & -1 & \cdots & -1 \\ -1 & \phi & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \phi \end{vmatrix}}_{detB} = 0. \tag{13}$$

Based on **Lemma 2.2**,

$$[(\phi-(-1))^{(p^k-1)-1}[\phi-((p^k-1)-1)(-1))]][(\phi-(-1))^{p^k-1}[\phi-(p^k-1)(-1)]] = 0.$$

$$[(\phi+1)^{p^k-2}[\phi+(p^k-2)]][(\phi+1)^{p^k-1}[\phi+(p^k-1)]] = 0.$$

$$(\phi+1)^{p^k-2} = 0 \text{ or } [\phi+(p^k-2)] = 0 \text{ or } (\phi+1)^{p^k-1} = 0 \text{ or } [\phi+(p^k-1)] = 0.$$

As a result, we obtain $\phi = -1$ with multiplicity $(p^k-2)+(p^k-1)$, $\phi = -(p^k-2)$ with multiplicity 1, and $\phi = -(p^k-1)$ with multiplicity 1. Using **Definition 2.4**, we find that the adjacency matrix energy of the graph can be calculated as follows:

$$\begin{aligned} E_{AM}(\Gamma_{D_{2(p^k)}}) &= \sum_{i=1}^{n} |\phi_i| \\ &= [(p^k-2)+(p^k-1)]|-1| + |-(p^k-2)| + |-(p^k-1)| \\ &= (2p^k-3)+(p^k-2)+(p^k-1) \\ &= 4p^k-6 \\ &= 2(2p^k-3). \end{aligned}$$

□

2.2. **Degree Sum Energy.**

Definition 2.8. [15][16] *Let Γ be a graph with vertices v_i for $i = 1, 2, ..., n$, where n is total number of vertices. The degree sum matrix of Γ is $DS(\Gamma) = [d_{ij}]$*

$$d_{ij} = \begin{cases} d_i + d_j, & \textit{if } \{v_i, v_j\} \textit{adjacency in } \Gamma, \\ 0, & \textit{else.} \end{cases} \tag{14}$$

Theorem 2.9. *Let D_{2n} be a dihedral group with $n = 2^k$, where $k \in \mathbb{Z}^+$, then the degree sum energy of $\Gamma_{D_{2n}}$ is*

$$E_{DS}(\Gamma_{D_{2(2^k)}}) = 16(2^{2k} - 2^{k+1} + 1). \tag{15}$$

Proof. Let $D_{2(2^k)} = \{a, b \mid a^{2^k} = b^2 = e, a^{-1} = bab^{-1}\}, k \in \mathbb{Z}^+$, then the order of any element rotations $(a) \in D_{2(2^k)}$ is of the form 2^p, where $p \in \mathbb{Z}^+$. However, for all reflections $(b) \in D_{2(2^k)}$, the order is 2. The $\gcd(|x|, |y|) = 2$, $\forall x, y \in D_{2(2^k)}$ then $\{a, b\}$ adjacency in Γ. Since the number of elements in $D_{2(2^k)}$ is $2(2^k)$ and the non-coprime graph has $D_{2(2^k)} \setminus \{e\}$ vertices, the number of vertices of the graph Γ is $2(2^k)-1$. Thus, the adjacency matrix of Γ will be of order $(2(2^k)-1)\times(2(2^k)-1)$ and based on **Definition 2.8** then the degree sum matrix of $D_{2(2^k)}$ is

$$DS(\Gamma_{D_{2(2^k)}}) = \begin{pmatrix} 0 & 2^{k+2}-4 & \cdots & 2^{k+2}-4 \\ 2^{k+2}-4 & 0 & \cdots & 2^{k+2}-4 \\ \vdots & \vdots & \ddots & \vdots \\ 2^{k+2}-4 & 2^{k+2}-4 & \cdots & 0 \end{pmatrix}. \tag{16}$$

The eigenvalues of the adjacency matrix are determined by solving the corresponding characteristic equation:

$$|DS(\Gamma_{D_{2(2^k)}}) - \phi I| = |\phi I - DS(\Gamma_{D_{2(2^k)}})| = 0. \tag{17}$$

$$\begin{vmatrix} \phi & -(2^{k+2}-4) & \cdots & -(2^{k+2}-4) \\ -(2^{k+2}-4) & \phi & \cdots & -(2^{k+2}-4) \\ \vdots & \vdots & \ddots & \vdots \\ -(2^{k+2}-4) & -(2^{k+2}-4) & \cdots & \phi \end{vmatrix} = 0. \tag{18}$$

Based on **Lemma 2.2**,

$$(\phi-(-(2^{k+2}-4)))^{(2(2^k)-1)-1}[\phi-(((2(2^k)-1)-1)(-(2^{k+2}-4)))] = 0.$$

$$(\phi+(2^{k+2}-4)^(2^{k+1}-2)[\phi+(2^{k+1}-2)(2^{k+2}-4)] = 0.$$

$$(\phi+(2^{k+2}-4))^(2^{k+1}-2)[\phi+(2^{2k+3}-2^{k+4}+8)] = 0.$$

$$(\phi+(2^{k+2}-4))^{2^{k+1}-2} = 0 \text{ or } [\phi+(2^{2k+3}-2^{k+4}+8)] = 0.$$

As a result, we obtain $\phi = -(2^{k+2}-4)$ with multiplicity $2^{k+1}-2$ and $\phi = -(2^{2k+3}-2^{k+4}+8)$ with multiplicity 1. Using **Definition 2.4**, we find that the degree sum energy of the graph can be calculated as follows:

$$\begin{aligned} E_{DS}(\Gamma_{D_{2n}}) &= \sum_{i=1}^{n} |\phi_i| \\ &= (2^{k+1}-2)|-(2^{k+2}-4)| + |-(2^{2k+3}-2^{k+4}+8)| \\ &= (2^{k+1}-2)(2^{k+2}-4) + (2^{2k+3}-2^{k+4}+8) \\ &= (2^{2k+3}-2^{k+4}+8) + (2^{2k+3}-2^{k+4}+8) \\ &= (2^{2k+4}-2^{k+5}+16) \\ &= 16(2^{2k}-2^{k+1}+1). \end{aligned}$$

□

Theorem 2.10. *Let D_{2n} be a dihedral group with $n = p^k$, where p is a prime number (except 2), $k \in \mathbb{Z}^+$, then the degree sum energy of $\Gamma_{D_{2n}}$ is*

$$E_{DS}(\Gamma_{D_{2(p^k)}}) = 4(2p^{2k}-6p^k+5). \tag{19}$$

Proof. Let $D_{2(p^k)} = \{a, b \mid a^{p^k} = b^2 = e, a^{-1} = bab^{-1}\}$, where p is a prime number (except 2), $k \in \mathbb{Z}^+$, then The order of any element rotations $(a) \in D_{2(p^k)}$ is of the form p^m, where $m \in \mathbb{Z}^+$. However, for all reflections $(b) \in D_{2(p^k)}$, the order is 2. The $\gcd(|x|, |y|) = 1, \forall x \in \langle a \rangle$ and $\forall y \in \langle b \rangle$ then $\{a, b\}$ not adjencent in Γ. Since the number of rotations elements in $D_{2(p^k)}$ is $p^k - 1$, the number of reflections elements in $D_{2(p^k)}$ is p^k and the non-coprime graph has $D_{2(p^k)} \setminus \{e\}$ vertices, the number of vertices of the graph Γ is $2n - 1$. Since the elements type a and b are always non-adjacent, the degree sum matrix can be decomposed into two submatrix, say matrix A and matrix B. Thus, the degree sum submatrix A of will be of order $(p^k - 1) \times (p^k - 1)$, the degree sum submatrix B of will be of order $(p^k) \times (p^k)$ and based on **Definition 2.8** the degree sum matrix of $D_{2(p^k)}$ is

$$DS(\Gamma_{D_{2(p^k)}}) =$$

$$\begin{pmatrix} 0 & 2p^k-4 & \cdots & 2p^k-4 & 0 & 0 & \cdots & 0 \\ 2p^k-4 & 0 & \cdots & 2p^k-4 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2p^k-4 & 2p^k-4 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 2p^k-2 & \cdots & 2p^k-2 \\ 0 & 0 & \cdots & 0 & 2p^k-2 & 0 & \cdots & 2p^k-2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 2p^k-2 & 2p^k-2 & \cdots & 0 \end{pmatrix}. \quad (20)$$

The eigenvalues of the adjacency matrix are determined by solving the corresponding characteristic equation:

$$|DS(\Gamma_{D_{2(p^k)}}) - \phi I| = |\phi I - DS(\Gamma_{D_{2(p^k)}})| = 0. \quad (21)$$

$$\begin{vmatrix} \phi & -(2p^k-4) & \cdots & -(2p^k-4) & 0 & 0 & \cdots & 0 \\ -(2p^k-4) & \phi & \cdots & -(2p^k-4) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(2p^k-4) & -(2p^k-4) & \cdots & \phi & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \phi & -(2p^k-2) & \cdots & -(2p^k-2) \\ 0 & 0 & \cdots & 0 & -(2p^k-2) & \phi & \cdots & -(2p^k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -(2p^k-2) & -(2p^k-2) & \cdots & \phi \end{vmatrix}$$

$$= 0. \quad (22)$$

Based on **Lemma 2.3**,

$$\underbrace{\begin{vmatrix} \phi & -(2p^k-4) & \cdots & -(2p^k-4) \\ -(2p^k-4) & \phi & \cdots & -(2p^k-4) \\ \vdots & \vdots & \ddots & \vdots \\ -(2p^k-4) & -(2p^k-4) & \cdots & \phi \end{vmatrix}}_{detA} \underbrace{\begin{vmatrix} \phi & -(2p^k-2) & \cdots & -(2p^k-2) \\ -(2p^k-2) & \phi & \cdots & -(2p^k-2) \\ \vdots & \vdots & \ddots & \vdots \\ -(2p^k-2) & -(2p^k-2) & \cdots & \phi \end{vmatrix}}_{detB}$$

$$= 0. \tag{23}$$

Based on **Lemma 2.2**,

$[(\phi - (-(2p^k - 4)))^{(p^k-1)-1}[\phi - ((p^k - 1) - 1)(-(2p^k - 4)))]][(\phi - (-(2p^k - 2)))^{p^k-1}$
$[\phi - (p^k - 1)(-(2n - p^k))]] = 0.$

$[(\phi + (2p^k - 4))^{p^k-2}[\phi + (p^k - 2)(2p^k - 4)]][(\phi + (2p^k - 2))^{p^k-1}[\phi + (p^k - 1)(2p^k - 2)]]$
$= 0.$

$[(\phi + (2p^k - 4))^{p^k-2}[\phi + (2p^{2k} - 8p^k + 8)]][(\phi + (2p^k - 2))^{p^k-1}[\phi + (2p^{2k} - 4p^k + 2)]]$
$= 0.$

$$(\phi + (2p^k - 4))^{p^k-2} = 0 \text{ or } [\phi + (2p^{2k} - 8p^k + 8)] = 0.$$

or

$$(\phi + (2p^k - 2))^{p^k-1} = 0 \text{ or } [\phi + (2p^{2k} - 4p^k + 2)] = 0.$$

As a result, we obtain $\phi = -(2p^k - 4)$ with multiplicity $p^k - 2$,$\phi = -(2p^k - 2)$ with multiplicity $p^k - 1$, $\phi = -(2p^{2k} - 8p^k + 8)$ with multiplicity 1, and $\phi = -(2p^{2k} - 4p^k + 2)$ with multiplicity 1. Using **Definition 2.4**, we find that the degree sum energy of the graph can be calculated as follows:

$$\begin{aligned}
E_{DS}(\Gamma_{D_{2n}}) &= \sum_{i=1}^{n} |\phi_i| \\
&= [(p^k - 2)]| - (2p^k - 4)| + [(p^k - 1)]| - (2p^k - 2)| + | - (2p^{2k} - 8p^k \\
&\quad + 8)| + | - (2p^{2k} - 4p^k + 2)| \\
&= (p^k - 2)(2p^k - 4) + (p^k - 1)(2p^k - 2) + (2p^{2k} - 8p^k + 8) + (2p^{2k} \\
&\quad - 4p^k + 2) \\
&= (2p^{2k} - 8p^k + 8) + (2p^{2k} - 4p^k + 2) + (2p^{2k} - 8p^k + 8) + (2p^{2k} \\
&\quad - 4p^k + 2) \\
&= (8p^{2k} - 24p^k + 20) \\
&= 4(2p^{2k} - 6p^k + 5).
\end{aligned}$$

□

3. CONCLUDING REMARKS

Based on the results of the discussion above, the energy of the adjacency matrix and the degree sum energy of the non-coprime graph on the dihedral group D_{2n}, where $n = p^k, p$ is a prime number, and $k \in \mathbb{Z}^+$, are given respectively as follows:

$$E_{AM}(\Gamma_{D_{2n}}) = \begin{cases} 4(2^k - 1), & \text{if } n = 2^k, \\ 2(2p^k - 3), & \text{if } n = p^k, p \neq 2. \end{cases} \tag{24}$$

and

$$E_{DS}(\Gamma_{D_{2n}}) = \begin{cases} 16(2^{2k} - 2^{k+1} + 1), & \text{if } n = 2^k, \\ 4(2p^{2k} - 6p^k + 5), & \text{if } n = p^k, p \neq 2. \end{cases} \tag{25}$$

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