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ON THE SUPER EDGE-MAGIC DEFICIENCY AND α-VALUATIONS OF GRAPHS

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Abstract. A graph G is called super edge-magic if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(u) + f(v) + f(uv)$ is a constant for each $uv \in E(G)$ and $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. The super edge-magic deficiency, $\mu_s(G)$, of a graph G is defined as the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n. In this paper, we prove that if G is a graph without isolated vertices that has an α-valuation, then $\mu_s(G) \leq |E(G)| - |V(G)| + 1$. This leads to $\mu_s(G) = |E(G)| - |V(G)| + 1$ if G has the additional property that G is not sequential. Also, we provide necessary and sufficient conditions for the disjoint union of isomorphic complete bipartite graphs to have an α-valuation. Moreover, we present several results on the super edge-magic deficiency of the same class of graphs. Based on these, we propose some open problems and a new conjecture.

Key words: Super edge-magic labeling, super edge-magic deficiency, sequential labeling, sequential number, α-valuation.

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Abstrak. Suatu graf G disebut sisi-ajaib super jika terdapat sebuah fungsi bijektif $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ sedemikian sehingga $f(u)+f(v)+f(uv)$ adalah sebuah konstanta untuk tiap $uv \in E(G)$ dan $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. Defisiensi sisi-ajaib super, $\mu_s(G)$, dari sebuah graf G didefinisikan sebagai bilangan bulat non negatif terkecil n dengan sifat yaitu graf $G \cup nK_1$ adalah sisi-ajaib super atau $+\infty$ jika tidak terdapat bilangan bulat n yang demikian. Pada paper ini, kami membuktikan bahwa jika G adalah sebuah graf tanpa titik terisolasi yang mempunyai sebuah nilai-α, maka $\mu_s(G) \leq |E(G)| - |V(G)| + 1$. Hal ini menghasilkan $\mu_s(G) = |E(G)| - |V(G)| + 1$ jika G mempunyai sifat tambahan yaitu G adalah tidak berurutan. Kami juga memberikan syarat perlu dan cukup untuk gabungan disjoin dari graf bipartit lengkap isomorfik untuk mempunyai sebuah nilai-α. Lebih jauh, kami menyajikan beberapa hasil pada defisiensi sisi-ajaib dari kelas graf yang sama. Berdasarkan hal-hal tersebut, kami mengusulkan beberapa masalah terbuka dan sebuah konjektur baru.

Kata kunci: Pelabelan sisi-ajaib super, defisiensi sisi-ajaib super, pelabelan secara berurutan, bilangan secara berurutan, nilai-α.

1. Introduction

The notation and terminology of this paper will generally follow closely that of [4]. All graphs considered here are finite, simple and undirected. The *vertex set* of a graph G is denoted by $V(G)$, while the *edge set* is denoted by $E(G)$. A *complete bipartite graph* with partite sets X and Y, where $|X| = s$ and $|Y| = t$, is denoted by $K_{s,t}$. For any graph G, the graph mG denotes the disjoint union of m copies of G. For two integers a and b with $b \geq a$, the set $\{x \in \mathbb{Z} |\, a \leq x \leq b\}$ will be denoted by simply writing $[a, b]$, where $\mathbb{Z}$ denotes the set of all integers.

The first paper in edge-magic labelings was published in 1970 by Kotzig and Rosa [20], who called these labelings: magic valuations; these were later rediscovered by Ringel and Lladó [22], who coined one of the now popular terms for them: edge-magic labelings. More recently, they have also been referred to as edge-magic total labelings by Wallis [24]. For a graph G of order p and size q, a bijective function $f : V(G) \cup E(G) \rightarrow [1, p + q]$ is called an *edge-magic labeling* of G if $f(u) + f(v) + f(uv)$ is a constant k (called the *valence* of f) for each $uv \in E(G)$. If such a labeling exists, then G is called an *edge-magic graph.* In 1998, Enomoto et al. [5] defined an edge-magic labeling f of a graph G to be a *super edge-magic labeling* if f has the additional property that $f(V(G)) = [1, p]$. Thus, a graph possessing a super edge-magic labeling is a *super edge-magic graph.* Lately, super edge-magic labelings and super edge-magic graphs are called by Wallis [24] strong edge-magic total labelings and strongly edge-magic graphs, respectively.

The following lemma taken from [6] provides necessary and sufficient conditions for a graph to be super edge-magic.

Lemma 1.1. *A graph G of order p and size q is super edge-magic if and only if there exists a bijective function $f : V(G) \to [1, p]$ such that the set*

$$S = \{ f(u) + f(v) | \, uv \in E(G)\}$$

consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = p + q + s$, where $s = \min(S)$ and

$$S = [k - (p + q) \, , k - (p + 1)] \, .$$

For every graph G, Kotzig and Rosa [20] proved that there exists an edge-magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This motivated them to define the edge-magic deficiency of a graph. The *edge-magic deficiency*, $\mu(G)$, of a graph G is the smallest nonnegative integer n for which the graph $G \cup nK_1$ is edge-magic. Motivated by the concept of edge-magic deficiency, Figueroa-Centeno et al. [10] analogously defined the super edge-magic deficiency of a graph. The *super edge-magic deficiency*, $\mu_s(G)$, of a graph G is either the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n. Thus, the super edge-magic deficiency of a graph G is a measure of how close G is to being super edge-magic.

An alternative term exists for the super edge-magic deficiency, namely, the vertex dependent characteristic. This term was coined by Hedge and Shetty [16]. In [16], they gave a construction of polygons having same angles and distinct sides using the result on the super edge-magic deficiency of cycles provided in [10].

In 1967, Rosa [23] initiated the study of β-valuations. They were later studied by Golomb [14], who called them graceful labelings, which is the term used in the current literature of graph labelings. A graph G of size q is called *graceful* if there exists an injective function $f : V(G) \to [0, q]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Such a function is called a *graceful labeling*. In [23], Rosa also introduced the notion of α-valuations stemming from his interest in graph decompositions. A graceful labeling f is called an *α-valuation* if there exists an integer λ (called the *critical value* of f) so that $\min \{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. Moreover, he pointed out that a graph that admits an α-valuation f is necessarily bipartite and has the partite sets $\{ v \in V(G) | \, f(v) \leq \lambda\}$ and $\{ v \in V(G) | \, f(v) > \lambda\}$.

The notion of sequential graphs was introduced by Grace [15]. He defined a graph G of size q to be *sequential* if there exists an injective function $f : V(G) \to [0, q-1]$ (with the label q allowed if G is a tree) such that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting set of edge labels is $[m, m + q - 1]$ for some positive integer m. Such a function is called a *sequential labeling*.

We now consider a concept that is somehow related to the super edge-magic deficiency of graphs without isolated vertices as well as α-valuations and sequential labelings. The notion of the sequential number was recently introduced by Figueroa-Centeno and Ichishima [11]. The *sequential number*, $\sigma(G)$, of a graph G of size q without isolated vertices is defined to be either the smallest positive integer n for which it is possible to label the vertices of G with distinct elements from the set $[0, n]$ in such a way that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting edge labels are q consecutive integers or $+\infty$ if there exists no such integer n. Thus, the sequential number of a graph G is a measure of how close G is to being sequential.

Figueroa-Centeno and Ichishima [11] found the following formula for the sequential number of a graph without isolated vertices in terms of its super edge-magic deficiency and order. As a consequence of this theorem, they also determined the exact value of the super edge-magic deficiency of the complete bipartite graph, which is stated in the succeeding corollary. These will later serve as the bases for some remarks and a new conjecture.

Theorem 1.2. *If G is a graph of order p without isolated vertices, then*

$$\sigma(G) = \mu_s(G) + p - 1.$$

Due to Theorem 1.2, the sequential number plays an important role in the study of super edge-magic deficiency of a graph without isolated vertices.

Corollary 1.3. *For all integers s and t with $s \geq 2$ and $t \geq 2$,*

$$\mu_s(K_{s,t}) = (s-1)(t-1).$$

In this paper, we prove that if G is a graph of order p and size q without isolated vertices that has an α-valuation, then $\mu_s(G) \leq q - p + 1$. Additionally, if G is not sequential, then $\mu_s(G) = q - p + 1$. Also, we provide necessary and sufficient conditions for the disjoint union of isomorphic complete bipartite graphs to have an α-valuation. Moreover, we present several results on the super edge-magic deficiency of the same class of graphs. These lead to some open problems and a new conjecture.

The survey by Gallian [12] on graph labeling problems is an excellent source of additional information. More information on super edge-magic graphs and related subjects can be found in the books by Bača and Miller [2], and Wallis [24].

2. Main Results

Our goal of this section is to establish a general formula for the super edge-magic deficiency of graphs without isolated vertices that have α-valuations, but not sequential. To achieve this, we start with the following result.

Theorem 2.1. *If G is a graph of order p and size q without isolated vertices that has an α-valuation, then*

$$\mu_s(G) \leq q - p + 1.$$

Proof. First, assume that G is a graph of size q without isolated vertices that has an α-valuation f with critical value λ. Then G is bipartite and has the partite sets

$$X = \{x \in V(G) | f(x) \leq \lambda\} \text{ and } Y = \{y \in V(G) | f(y) > \lambda\}.$$

Next, define the vertex labeling $g : V(G) \to [0, q]$ such that

$$g(v) = \begin{cases} f(v), & \text{if } v \in X; \\ \lambda + q + 1 - f(v), & \text{if } v \in Y. \end{cases}$$

Now, notice that

$$g(X) \subseteq [0, \lambda] \text{ and } g(Y) \subseteq [\lambda + 1, q].$$

This implies that g is an injective function and

$$g(x) + g(y) = \lambda + q + 1 - (f(y) - f(x))$$

for each $xy \in E(G)$, where $x \in X$ and $y \in Y$. Thus,

$$\lambda + 1 \leq g(x) + g(y) \leq \lambda + q$$

since $1 \leq f(y) - f(x) \leq q$. Finally, notice that since f is an α-valuation of G, it follows that

$$\{ f(y) - f(x) | \, x \in X \text{ and } y \in Y\} = [1, q],$$

implying that $\{ g(x) + g(y) | \, xy \in E(G)\}$ is a set of q consecutive integers. This implies that $\sigma(G) \leq q$; hence, it follows from Theorem 1.2 that $\mu_s(G) \leq q - p + 1$. □

If G is a graph of order p and size q without isolated vertices that is not sequential, then it is clearly true that $\sigma(G) \geq q$. Thus, it follows from Theorem 1.2 that $\mu_s(G) \geq q - p + 1$. Combining this with Theorem 2.1, we have the following result.

Corollary 2.2. *If G is a graph of order p and size q without isolated vertices that has an α-valuation and is not sequential, then*

$$\mu_s(G) = q - p + 1.$$

3. On The Disjoint Union of Complete Bipartite Graphs

In this section, we study the super edge-magic deficiency of the disjoint union of isomorphic complete bipartite graphs. To do this, we first present necessary and sufficient conditions for such graphs to have an α-valuation.

Rosa [23] observed that all complete bipartite graphs have α-valuations. This result is now extended in the following theorem.

Theorem 3.1. *Let m, s and t be integers with $m \geq 1$, $s \geq 2$ and $t \geq 2$. Then the graph $mK_{s,t}$ has an α-valuation if and only if $(m, s, t) \neq (3, 2, 2)$.*

Proof. For every two positive integers s and t, the complete bipartite graph $K_{s,t}$ has shown to admit an α-valuation by Rosa [23]. Also, Abrham and Kotzig [1] have proved that $m = 3$ is the only integer such that the 2-regular graph $mC_4 \cong mK_{2,2}$ does not have an α-valuation. Thus, it suffices to show that for all integers m, s and t such that $m \geq 2$ and $t > s \geq 2$ except $(m, s, t) = (3, 2, 2)$, there exists an α-valuation of $mK_{s,t}$. Let $mK_{s,t}$ have partite sets $X = \bigcup_{i=1}^{m} X_i$ and $Y = \bigcup_{i=1}^{m} Y_i$, where $X_i = \{ x_{i,j} | \, i \in [1, m] \text{ and } j \in [1, s]\}$ and $Y_i = \{ y_{i,j} | \, i \in [1, m] \text{ and } j \in [1, t]\}$ are the partite sets of the i-th component of $mK_{s,t}$. Then define the vertex labeling $f : V(mK_{s,t}) \to [0, mst]$ such that

$$f(x_{i,j}) = (s + 1)(i - 1) - 2 + j,$$

if $i \in [1, m]$ and $j \in [1, s]$; and

$$f(y_{i,j}) = mst - 1 - (st - s - 1)\, i + s\,(j - 1),$$

if $i \in [1, m]$ and $j \in [1, t]$.

To show that f is indeed an α-valuation of $mK_{s,t}$, notice first that for each $i \in [1, m]$,

$$f(X_i) = \{a_i, a_i + 1, \ldots, a_i + s - 1\}$$

is a sequence of s consecutive integers, and

$$f(Y_i) = \{b_i, b_i + s, \ldots, b_i + s\,(t - 1)\}$$

is an arithmetic progression with t terms and common difference s, where $a_i = (s + 1)(i - 1)$ and $b_i = mst - 1 - (st - s - 1)\, i$. Now, it follows that not only $f(X_i) \neq f(X_j)$ for $i \neq j$ and $f(Y_k) \neq f(Y_l)$ for $k \neq l$, but also $f(X_i) \neq f(Y_j)$ for $i \neq j$. Moreover, it follows that

$$f(X) \subseteq [a_1, a_m + s - 1] \text{ and } f(Y) \subseteq [b_m, b_1 + s\,(t - 1)]$$

or, equivalently,

$$f(X) \subseteq [0, m\,(s + 1) - 2] \text{ and } f(Y) \subseteq [m\,(s + 1) - 1, mst]\,.$$

This implies that f is an injective function. Finally, notice that for each $i \in [1, m]$, the induced edge labels in the i-th component of $mK_{s,t}$ are st consecutive integers of the set

$$[b_i - a_i, b_i - a_i + st - 1] = [(m - i)\, st + 1, (m - i + 1)\, st]\,.$$

Thus, the induced edge labels are precisely $[1, mst]$. Therefore, f is an α-valuation of $mK_{s,t}$ with critical value $m\,(s + 1) - 2$. □

An illustration of Theorem 3.1 is given in Figure 1 for $m = 2$, $s = 3$ and $t = 4$.

The remaining part of this section contains results on the super edge-magic deficiency of the graph $mK_{s,t}$.

We first consider the super edge-magic deficiency of the forest $mK_{1,n}$. For all positive integers m and n such that m is odd, Figueroa-Centeno et al. [8] have shown that $\mu_s(mK_{1,n}) = 0$. When m is even, we only know that $\mu_s(mK_{1,1}) = 1$ for $m \geq 2$ (see [10]), and $\mu_s(mK_{1,2}) = 0$ for $m \geq 4$ (see [3]). Thus, the only instance that needs to be settled is when m is even and $n \geq 2$. For this, we have found the following result.

Theorem 3.2. *For all positive integers m and n such that m is even,*

$$\mu_s(mK_{1,n}) \leq 1.$$

Proof. Let $F \cong mK_{1,n} \cup K_1$ be the forest with

$$V(F) = \{\, x_i \mid i \in [1, m]\} \cup \{\, y_{i,j} \mid i \in [1, m] \text{ and } j \in [1, n]\} \cup \{z\}$$

and

$$E(F) = \{\, x_i y_{i,j} \mid i \in [1, m] \text{ and } j \in [1, n]\}\,,$$

and consider two cases.

Case 1: For $m = 2$, define the vertex labeling $f : V(F) \to [1, 2n+3]$ such that $f(x_i) = 2n + 5 - 2i$, if $i \in [1, 2]$; $f(y_{i,j}) = i + 2j - 2$, if $i \in [1, 2]$ and $j \in [1, n]$; and $f(z) = 2n + 2$. Notice then that

$$\{f(y_{i,j}) | \, i \in [1, 2] \text{ and } j \in [1, n]\} = [1, 2n]$$

and

$$\{f(x_1), f(x_2), f(z)\} = [2n+1, 2n+3],$$

which implies that f is a bijective function. Notice also that

$$\{f(x_1) + f(y_{1,j}) | \, j \in [1, n]\} = \{2n + 2 + 2j | \, j \in [1, n]\}$$

and

$$\{f(x_2) + f(y_{2,j}) | \, j \in [1, n]\} = \{2n + 1 + 2j | \, j \in [1, n]\},$$

implying that

$$\{f(u) + f(v) | \, uv \in E(F)\} = [2n+3, 4n+2]$$

is a set of $2n$ consecutive integers. Thus, by Lemma 1.1, f extends to a super edge-magic labeling of F with valence $6n + 6$.

Case 2: For $m = 2k$, where k is an integer with $k \geq 2$, define the vertex labeling $f : V(F) \to [1, 2kn + 2k + 1]$ such that

$$f(x_i) = \begin{cases} 2k(n+1) + 3 - 2i, & \text{if } i \in [1, k]; \\ 2k(n+2) + 2 - 2i, & \text{if } i \in [k+1, 2k]; \end{cases}$$

$$f(y_{i,j}) = \begin{cases} i + k(j-1), & \text{if } i \in [1, k] \text{ and } j \in [1, n]; \\ i + k(n + j - 2) + 1, & \text{if } i \in [k+1, 2k] \text{ and } j \in [1, n]; \end{cases}$$

and $f(z) = kn + 1$. Notice then that

$$\{f(y_{i,j}) | \, i \in [1, k] \text{ and } j \in [1, n]\} \cup \{f(z)\} = [1, kn+1],$$

$$\{f(y_{i,j}) | \, i \in [k+1, 2k] \text{ and } j \in [1, n]\} = [kn+2, 2kn+1],$$

and

$$\{f(x_i) | \, i \in [1, 2k]\} = [2kn+2, 2kn+2k+1],$$

which implies that f is a bijective function. Notice also that

$$\begin{aligned} & \{f(x_i) + f(y_{i,j}) | \, i \in [1, k] \text{ and } j \in [1, n]\} \\ = \; & [2kn + k + 3, 3kn + k + 2] \end{aligned}$$

and

$$\begin{aligned} & \{f(x_i) + f(y_{i,j}) | \, i \in [k+1, 2k] \text{ and } j \in [1, n]\} \\ = \; & [3kn + k + 3, 4kn + k + 2], \end{aligned}$$

implying that

$$\{f(u) + f(v) | \, uv \in E(F)\} = [2kn + k + 3, 4kn + k + 2]$$

is a set of $2kn$ consecutive integers. Thus, by Lemma 1.1, f extends to a super edge-magic labeling of F with valence $6kn + 3k + 4$.

Therefore, we conclude that $\mu_s(mK_{1,n}) \leq 1$ for all positive integers m and n such that m is even. □

Ivančo and Lučkaničová [18] proved that the forest $K_{1,m} \cup K_{1,n}$ is super edge-magic if and only if either m is a multiple of $n+1$ or n is a multiple of $m+1$. Thus, $\mu_s(2K_{1,n}) \geq 1$ for every positive integer n. Combining this with Theorem 3.2, we obtain the following result.

Corollary 3.3. *For every positive integer n,*

$$\mu_s(2K_{1,n}) = 1.$$

The previous result supports the validity of the conjecture of Figueroa-Centeno et al. [9] that if F is a forest with two components, then $\mu_s(F) \leq 1$.

Ringel and Lladó [22] proved that a graph of order p and size q is not edge-magic if q is even, $p+q \equiv 2 \pmod 4$ and each vertex has odd degree. This together with Theorem 3.2 leads us to conclude the following result.

Corollary 3.4. *For all positive integers m and n such that $m \equiv 2 \pmod 4$ and n is odd,*

$$\mu_s(mK_{1,n}) = 1.$$

Our final result on the super edge-magic deficiency of forests concerns $mK_{1,3}$.

Corollary 3.5. *For every positive integer m,*

$$\mu_s(mK_{1,3}) = \begin{cases} 0, & \text{if } m \equiv 4 \pmod 8 \text{ or } m \text{ is odd;} \\ 1, & \text{if } m \equiv 2 \pmod 4. \end{cases}.$$

Proof. Define the forest $4K_{1,3}$ with

$$V(4K_{1,3}) = \{x_i | \, i \in [1,4]\} \cup \{y_{i,j} | \, i \in [1,4] \text{ and } j \in [1,3]\}$$

and

$$E(4K_{1,3}) = \{x_i y_{i,j} | \, i \in [1,4] \text{ and } j \in [1,3]\}.$$

Then the vertex labeling $f : V(4K_{1,3}) \to [1,16]$ such that

$$(f(x_i))_{i=1}^{4} = (13,12,10,8);$$

$$(f(y_{1,j}))_{j=1}^{3} = (1,2,7); \qquad (f(y_{2,j}))_{j=1}^{3} = (4,5,6);$$
$$(f(y_{3,j}))_{j=1}^{3} = (3,9,11); \qquad (f(y_{4,j}))_{j=1}^{3} = (14,15,16)$$

induces a super edge-magic labeling of $4K_{1,3}$ with valence 41. Now, recall the result presented in [8] that if G is a (super) edge-magic bipartite or tripartite graph and m is odd, then mG is (super) edge-magic. Since the forests $K_{1,3}$ and $4K_{1,3}$ are super edge-magic bipartite graphs, it follows from the mentioned result that $\mu_s(mK_{1,3}) = 0$ when $m \equiv 4 \pmod 8$ or m is odd. The remaining case is an immediate consequence of Corollary 3.4. □

The preceding results in this section motivate us to propose the following problem.

Problem 1. *For even $m \geq 4$ and $n \geq 3$, determine the exact value of $\mu_s(mK_{1,n})$.*

We now direct our attention briefly to the super edge-magic deficiency of the 2-regular graph $mK_{2,2}$. For every positive integer m, Ngurah et al. [21] proved that if m is odd, then $\mu_s(mK_{2,2}) \leq m$ while if m is even, then $\mu_s(mK_{2,2}) \leq m-1$. They also posed the problem of finding a better upper bound for $\mu_s(mK_{2,2})$. However, with the aid of Corollary 2.2, we are able to provide the exact value of $\mu_s(mK_{2,2})$ which we determine to be 1.

Corollary 3.6. *For every positive integer m,*

$$\mu_s(mK_{2,2}) = 1.$$

Proof. As we mentioned in the proof of Theorem 3.1, the 2-regular graph $3C_4 \cong 3K_{2,2}$ does not admit an α-valuation. Also, Gnanajothi [13] has shown that the 2-regular graph mC_n is sequential if and only if m and n are odd. By adding these facts to Corollary 2.2, we obtain that $\mu_s(mK_{2,2}) = 1$ except for $m = 3$, and $\mu_s(3K_{2,2}) \geq 1$. However, the graph $3K_{2,2} \cup K_1$ is super edge-magic by labeling the vertices in its cycles with $1-8-3-9-1$, $2-6-7-12-2$, and $4-11-5-13-4$, and its isolated vertex with 10 to obtain a valence of 33, which implies that $\mu_s(3K_{2,2}) \leq 1$. Consequently, $\mu_s(mK_{2,2}) = 1$ for every positive integer m. □

The previous result adds credence to the conjecture of Figueroa-Centeno et al. [9] that for all integers $m \geq 1$ and $n \geq 3$, $\mu_s(mC_n) = 1$, if $mn \equiv 0 \pmod 4$.

The final result of this section concerns an upper bound for $\mu_s(mK_{s,t})$. For all integers m, s and t with $m \geq 1$, $s \geq 4$ and $t \geq 4$, Ngurah et al. [21] discovered an upper bound for $\mu_s(mK_{s,t})$, namely, $\mu_s(mK_{s,t}) \leq m(st - s - t) + 1$. Actually, the conditions on s and t in their result can be relaxed as we will see next.

Corollary 3.7. *For all integers m, s and t with $m \geq 1$, $s \geq 2$ and $t \geq 2$,*

$$\mu_s(mK_{s,t}) \leq m(st - s - t) + 1.$$

Proof. It has already been verified in the proof of Corollary 3.6 that $\mu_s(3K_{2,2}) \leq 1$. Thus, the desired result readily follows from this, and Theorems 2.1 and 3.1. □

By Corollaries 1.3, 3.6 and 3.7, we suspect the following conjecture to be true.

Conjecture 1. *For all integers m, s and t with $m \geq 1$, $s \geq 2$ and $t \geq 2$,*

$$\mu_s(mK_{s,t}) = m(st - s - t) + 1.$$

Of course, if it is true that the graph $mK_{s,t}$ is not sequential for all integers m, s and t with $m \geq 1$ and $s \geq 2$ and $t \geq 2$, so is Conjecture 1 by Corollary 2.2 and Theorem 3.1. However, we do not know whether or not the mentioned statement is true. Thus, we propose the following problem.

Problem 2. *For all integers m, s and t with $m \geq 1$, $s \geq 2$ and $t \geq 2$, determine whether or not the graph $mK_{s,t}$ is sequential.*

4. Concluding Remarks

We conclude this paper with some remarks on bounds for the super edge-magic deficiency of bipartite graphs and open problems.

Figueroa-Centeno et al. [9] have shown that if G is a bipartite or tripartite graph and m is odd, then $\mu_s(mG) \leq m\mu_s(G)$. Unfortunately, this bound is not sharp. For instance, we can easily see that $\mu_s(K_{2,2}) = 1$, which implies that $\mu_s(3K_{2,2}) \leq 3$; however, we know by Corollary 3.6 that $\mu_s(3K_{2,2}) = 1$. Also, the same bound does not hold for even m, since we know that $\mu_s(K_{1,n}) = 0$ (see [5]) and $\mu_s(2K_{1,n}) = 1$ (see Corollary 3.3). On the other hand, by Corollaries 1.3 and 3.7, we obtain that $\mu_s(mG) \leq m\mu_s(G) - m + 1$ when $G \cong K_{s,t}$. This leads us to ask in the next problem whether a similar upper bound is obtained for any bipartite graph.

Problem 3. *Given a bipartite graph G and an integer $m \geq 2$, find a good upper bound for $\mu_s(mG)$ in terms of m and $\mu_s(G)$.*

To proceed further, another definition is required here. For two graphs G_1 and G_2 with disjoint vertex sets, the *cartesian product* $G \cong G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. An important class of graphs is defined in terms of cartesian product. The *n-dimensional cube* Q_n is the graph K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as $Q_n \times K_2$. It is easily observed that Q_n is an n-regular bipartite graph of order 2^n and size $n2^{n-1}$.

We now discuss briefly lower and upper bounds for $\mu_s(Q_n)$. Figueroa-Centeno et al. [6] pointed out that Q_n is super edge-magic if and only if $n = 1$. Kotzig [19] has shown that Q_n has an α-valuation for all n, whereas the authors proved that Q_n is sequential for $n \geq 4$ (see [17]). Combining these with Corollary 2.2 and Theorem 2.1, we obtain exact values $\mu_s(Q_1) = 0$, $\mu_s(Q_2) = 1$ and $\mu_s(Q_3) = 5$, and the upper bound $\mu_s(Q_n) \leq (n-2)\,2^{n-1} + 1$ for $n \geq 4$. It is now important to mention that the largest vertex labeling of the sequential labeling found in [17] is $n2^{n-1} - 5$, which implies that $\sigma(Q_n) \leq n2^{n-1} - 5$. This together with Theorem 2.1 gives us the upper bound $\mu_s(Q_n) \leq (n-2)\,2^{n-1} - 4$ for $n \geq 4$. This bound is certainly better than the above bound obtained by applying an α-valuation of Q_n provided in [19] to Theorem 2.1. Figueroa-Centeno et al. [8] found an upper bound for the size of a super edge-magic triangle-free graph of order $p \geq 4$ and size q, namely, $q \leq 2p - 5$. By utilizing this, we obtain the lower bound $\mu_s(Q_n) \geq (n-4)\,2^{n-2} + 3$ for $n \geq 2$. In the light of the mentioned bounds and exact values for $\mu_s(Q_n)$, we finally propose the following two problems.

Problem 4. *For every integer $n \geq 4$, find better lower and upper bounds for $\mu_s(Q_n)$.*

Problem 5. *For every integer $n \geq 4$, determine the exact value of $\mu_s(Q_n)$.*

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