

Secondary Flow for Slow Rotation of a Pervious Sphere with Source at its Centre in a Viscous Fluid

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Abstract. In this paper, the problem of secondary motion of fluid in region between two concentric spheres are studied. Navier Stokes equations are employed to obtain the flow field, and the components of velocity for primary and secondary flow, stream function are obtained. Further, two particular cases are deduced and discussed for the radius of cavity. Finally, Equi-pressure lines, streamlines, and vortex lines are constructed to visually demonstrate the impact of different factors on the flow pattern.

Key words and Phrases: Primary flow, secondary flow, spherical cavity, pervious sphere, source, streamlines

1. INTRODUCTION

Datta [1] investigated the steady slow viscous flow past a pervious sphere with a source at its center. The source is of such a strength that it brings forth a linear part of the inertia terms neglected in Stokes flow for small Reynolds number Re . Using the method of matched asymptotic expansion the problem was extended to include $O(Re)$ terms in the case of a cylinder and a sphere by Sthapit & Datta ([2],[3],[4]). In 2011, Datta and Singhal [5] revisited the problem, incorporating the inner and outer expansions for the stream function as proposed by Proudman and Pearson [6] to account for the effect of a source. Their approach simplified the derivation of higher-order approximations, specifically the $O(Re^2 \log Re)$ term. Subsequently, Datta and Singhal [7] extended their analysis to include the influence

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of slip conditions. They also investigated a related problem involving axial flow along a porous cylinder with a sink at its center (see Datta and Singhal [8]).

Datta and Srivastava [9] investigated the slow rotation of a sphere with a source at its center in a viscous fluid. Bickley [10] analyzed the secondary flow generated by a rotating sphere in a viscous medium. Haberman [11] addressed the secondary flow problem for a rotating sphere enclosed within a coaxially rotating spherical container. Datta [12] studied the steady rotation of a magnetized sphere in a viscous conducting fluid, while Datta [13] explored the secondary flow around a magnetized sphere rotating in such a medium. Ranger [14] examined the axially symmetric flow past a rotating sphere induced by a uniform stream at infinity. He demonstrated that the leading terms of the flow represent a linear superposition of a primary Stokes flow past a non-rotating sphere and an antisymmetric secondary flow in the azimuthal plane caused by the sphere's rotation. In the paper Stokes flow around an axially symmetric rotating pervious body Srivastava [15] has not attempted to satisfy the boundary conditions for the general case; he has only reproduced the case of a sphere already covered in Datta & Srivastava [1]. In recent past years, Some authors have studied and explored this kind of work for various medium by considering different parametrs. Some of them are [16, 17, 18, 19, 20].

In this paper we study the analysis of the secondary flow for the slow rotation of a pervious sphere with a source at its center enclosed in a concentric spherical cavity at rest. The analysis is done through a perturbation expansion in Reynolds number Re . The first term corresponds to the azimuthal flow presented earlier by Datta & Srivastava [9] and the second term provides the streamlines corresponding to the Stokes stream function in meridian flow.

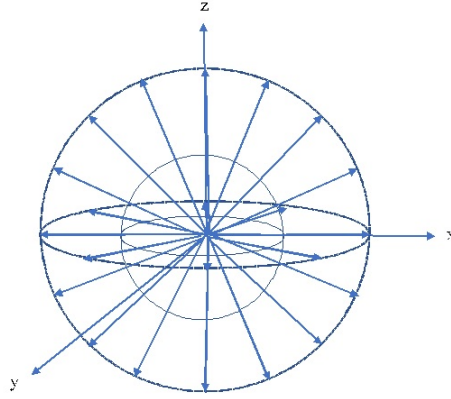


FIGURE 1. Figure for the rotation of pervious sphere with source at its center enclosing in concentric spherical cavity

2. FORMULATION OF THE PROBLEM

Consider the slow steady rotation of a pervious sphere of radius a with constant angular velocity $\mathbf{\Omega}$ carrying a source of strength Q at its center generating a radial flow field $(Q/r'^2)\hat{\mathbf{r}}$ and encompassed by a concentric pervious spherical cavity of radius b . The region ($a \leq r' \leq b$) in between pervious sphere and cavity is filled with an incompressible viscous fluid of density ρ and kinematic viscosity ν , no fluid accumulates. Now it is convenient to introduce dimensionless quantities

$$ra = r', \Omega a \mathbf{q} = \mathbf{q}', \mu \Omega p = p'. \quad (1)$$

Here, ' denotes dimensional quantities, r space coordinate, $\mathbf{q}$ velocity and p pressure.

The equations that govern the motion of an incompressible viscous fluid in the standard notation are:

Equation of continuity

$$\nabla \cdot \mathbf{q} = 0. \quad (2)$$

Navier-Stokes equation

$$Re((\mathbf{q} \cdot \nabla) \mathbf{q}) = -\nabla p + \nabla^2 \mathbf{q}, \quad (3)$$

or using equation (2)

$$Re(((\nabla \times \mathbf{q}) \times \mathbf{q}) + (1/2)\nabla(\mathbf{q} \cdot \mathbf{q})) = -\nabla p - \nabla \times (\nabla \times \mathbf{q}), \quad (4)$$

where $Re = \Omega a^2/\nu$ is Reynolds number.

The equations are to be solved under the following no slip boundary conditions:

$$\mathbf{q} = \sin \theta \hat{\phi} \text{ at } r = 1, \quad (5)$$

$$\mathbf{q} = \mathbf{0} \text{ at } r = 1/\lambda = b/a, \quad (6)$$

To solve the problem, we make use of the following perturbation scheme:

$$\mathbf{q}(r, \theta) = \frac{1}{Re} \frac{s}{r^2} \hat{\mathbf{r}} + w(r, \theta) \hat{\phi} + Re(u(r, \theta) \hat{\mathbf{r}} + v(r, \theta) \hat{\theta}), \quad (7)$$

where $s = Q/a\nu$ is source parameter.

$$p = p_0 + Re p_1. \quad (8)$$

Inserting above representation in equation of motion (4) and equating $O(1)$ and $O(Re)$ terms, we obtain the following equations:

Primary flow:

$$\nabla^2 w - \frac{w}{r^2 \sin^2 \theta} = \frac{s}{r^3} \frac{\partial}{\partial r}(rw), p_0 = \text{constant}. \quad (9)$$

Secondary Flow :

$$\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial(\sin \theta v)}{\partial \theta} - s \frac{\partial}{\partial r} \frac{u}{r^2} = \frac{\partial p_1}{\partial r} - \frac{w^2}{r}, \quad (10)$$

$$\nabla^2 v - \frac{v}{r^2 \sin^2 \theta} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{s}{r^3} \frac{\partial(rv)}{\partial r} = \frac{1}{r} \frac{\partial p_1}{\partial \theta} - \frac{w^2}{r} \cot \theta, \quad (11)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta}. \quad (12)$$

3. PRIMARY FLOW

As in Datta & Srivastva [9] and Devi [21], we consider the primary motion which is induced in the fluid by slow rotation of pervious sphere with a source at its center. The motion is governed by the equation (9).

The boundary conditions (5, 6) for the azimuthal velocity component w are

$$w(1, \theta) = \sin \theta, \quad (13)$$

and

$$w(1/\lambda, \theta) = 0. \quad (14)$$

The boundary condition (13) suggests that solution of equation (9) of the form

$$w = f(r) \sin \theta, \quad (15)$$

where $f(r)$ is derived below and given by equation (17). Now equation (9) provides

$$f''(r) + \left(\frac{2}{r} - \frac{s}{r^2} \right) f'(r) - \left(\frac{2}{r^2} + \frac{s}{r^3} \right) f(r) = 0. \quad (16)$$

Solving the above differential equation, using boundary conditions (13) and (14), we get

$$f(r) = \frac{(2r^2 - 2rs + s^2) - r^2 e^{(\lambda - \frac{1}{r})s} (\lambda^2 s^2 - 2\lambda s + 2)}{r ((s^2 - 2s + 2) - e^{(\lambda - 1)s} (\lambda^2 s^2 - 2\lambda s + 2))}, \quad (17)$$

For the case of the pervious sphere rotating an infinite expanse of fluid ($\lambda \rightarrow 0$), and the result above reduces to

$$f(r) = \frac{2r^2 (1 - e^{-\frac{s}{r}}) - 2rs + s^2}{r (s^2 - 2s + 2 (1 - e^{-s}))}, \quad (18)$$

which conforms to the result obtained by Datta & Srivastava [9].

The couple on the sphere required to maintain the motion is obtained by integration of viscous stress in dimensionless form

$$\begin{aligned} \tau_{r\phi}(r, \theta) &= \frac{\tau'_{r\phi}(r, \theta)}{\mu\Omega} = \left(\frac{\partial w}{\partial r} - \frac{w}{r} \right)_{r=1} = (f'(1) - f(1)) \sin \theta \\ &= \frac{s(e^{(\lambda-1)s}(\lambda^2 s^2 - 2\lambda s + 2) + 2(s-1))}{e^{(\lambda-1)s}(\lambda^2 s^2 - 2\lambda s + 2) - (s^2 - 2s + 2)} \sin \theta. \end{aligned} \quad (19)$$

Thus, the moment of the required couple is given by ([22],[23])

$$M' = -2\pi a^3 \int_0^\pi \tau'_{r\phi} \sin^2 \theta d\theta, \quad (20)$$

where $\tau'_{r,\phi}$ is the tangential stress on the pervious sphere is given by (19). Thus, on evaluating the integral (20), the dimensional moment of the required couple M' is comes out.

$$M' = -8\pi\mu\Omega a^3 \frac{s(e^{(\lambda-1)s}(\lambda^2 s^2 - 2\lambda s + 2) + 2(s-1))}{3(e^{(\lambda-1)s}(\lambda^2 s^2 - 2\lambda s + 2) - (s^2 - 2s + 2))}, \quad (21)$$

The non dimensional moment of the required couple M is

$$M = \frac{M'}{8\pi\mu\Omega a^3}, \quad (22)$$

$$M = -\frac{s(e^{(\lambda-1)s}(\lambda^2 s^2 - 2\lambda s + 2) + 2(s-1))}{3(e^{(\lambda-1)s}(\lambda^2 s^2 - 2\lambda s + 2) - (s^2 - 2s + 2))}. \quad (23)$$

For the case of the pervious sphere rotating an infinite expanse of fluid $\lambda \rightarrow 0$, so, the non dimensional moment of the required couple M^∞ in infinite expanse of fluid is

$$M^\infty = \frac{2((s-1) + e^{-s})s}{3s^2 - 6(s + e^{-s} - 1)}. \quad (24)$$

Also, the result above reduces to the result obtained by Datta & Srivastava [9]. Here $s \rightarrow 0$ in above equation describe the case for rotation of solid sphere in an unbounded fluid.

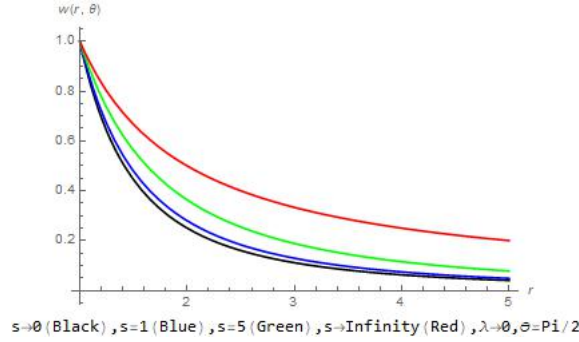


FIGURE 2. Variation azimuthal velocity w with r at $\theta = \pi/2$ for various values of s ($s \rightarrow 0$ (Blue), $s = 1$ (Green), $s = 10$ (Brown), $s = 100$ (Black), $s \rightarrow \infty$ (Red)) when the pervious sphere rotating in an infinite expanse of fluid (i.e. $\lambda \rightarrow 0$).

Now we present few graphs in figures 2 to 5 to show the effects of parameters on the velocity w and moment of the couple M . Figure 2 shows variations in the azimuthal velocity w with r at $\theta = \pi/2$ for various values of source parameter s in an infinite expanse of fluid. In this graph w increases as s increases but in each graph rate of variation of the azimuthal velocity w with r diminishes as r increases. Figure 3 shows variations in the azimuthal velocity w with r for various values of source parameter s when the pervious sphere rotating in a concentric cavity whose radius is 2 times of the pervious sphere. In this graph w increases as s increases

but in each graph rate of variation of w with r diminishes as s increases. Figure 4 shows variations in the couple M with source parameter s for various values of separation parameter λ . In this graph M increases as λ increases but in each graph rate of variation of the couple with s diminishes as s increases. Figure 5 shows variations in the couple M with separation parameter λ for various values of source parameter s . In this graph M increases as s decreases but in each graph rate of variation of the couple increases as λ increases and increases sharply as $\lambda \rightarrow 1$.

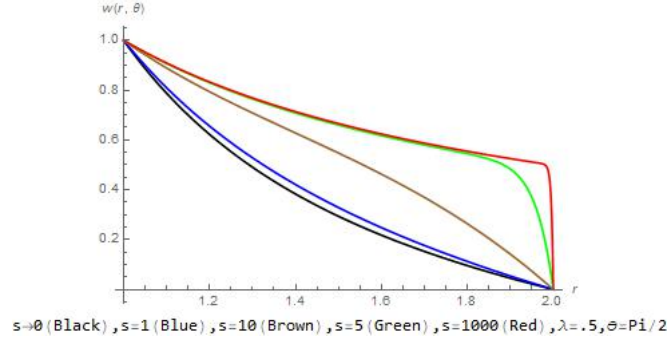


FIGURE 3. Variation azimuthal velocity w with r at $\theta = \pi/2$ for various values of s ($s \rightarrow 0$ (Blue), $s = 1$ (Green), $s = 10$ (Brown), $s = 100$ (Black), $s = 10^7$ (Red)) when the pervious sphere rotating in a concentric cavity whose radius is 2 times of the pervious sphere (i.e. $\lambda = 0.5$).

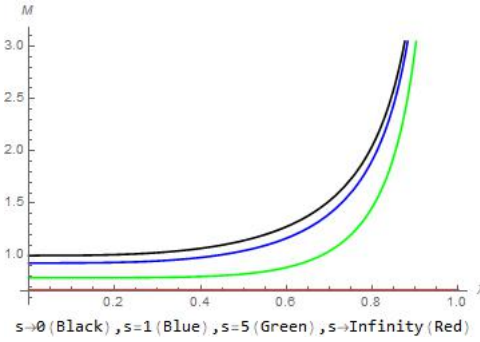


FIGURE 4. Variation of torque M with s for various values of $\lambda = a/b$ ($\lambda = .001$ (Blue), $\lambda = .4$ (Green), $\lambda = .6$ (Brown), $\lambda = .8$ (Black), $\lambda = .9$ (Red)) when the pervious sphere rotating in a concentric cavity.

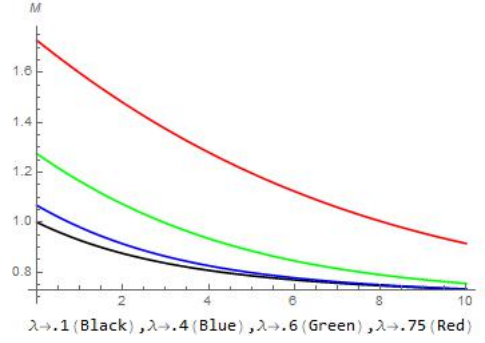


FIGURE 5. Variation of torque M with $\lambda = a/b$ for various values of source parameter s ($s \rightarrow 0$ (Blue), $s = 1$ (Green), $s = 10$ (Brown), $s = 100$ (Black), $s = 1000$ (Red)) when the pervious sphere rotating in a concentric cavity.

4. SECONDARY FLOW

Secondary flow is determined by the equations (10, 11) with w replaced by $f(r) \sin \theta$; thus, we have the equations

$$\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (v \sin \theta) - s \frac{\partial}{\partial r} \left(\frac{u}{r^2} \right) = \frac{\partial p_1}{\partial r} - \frac{f^2(r)}{r} \sin^2 \theta, \quad (25)$$

$$\nabla^2 v - \frac{v}{r^2 \sin^2 \theta} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{s}{r^3} \frac{\partial}{\partial r} (rv) = \frac{1}{r} \frac{\partial p_1}{\partial \theta} - \frac{f^2(r)}{r} \sin \theta \cos \theta, \quad (26)$$

where $f(r)$ is given by equation (17). These equations have to be solved under boundary conditions:

At the common interface $r = 1$

$$u = 0, v = 0, \quad (27)$$

At cavity wall $r = 1/\lambda$

$$u = 0, v = 0. \quad (28)$$

It will be convenient using Stokes stream function ψ to express velocity components

$$u = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, v = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}. \quad (29)$$

Next, eliminating p_1 in between equations (25) and (26) and using (29), we get

$$E^4 \psi - s \frac{\partial}{\partial r} \left(\frac{1}{r^2} E^2 \psi \right) = 2 \left(f(r) f'(r) - \frac{(f(r))^2}{r} \right) \sin^2 \theta \cos \theta, \quad (30)$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right), \quad (31)$$

where using the value of $f(r)$ as given by equation (17), we have

$$2f(r) \left(f'(r) - \frac{f(r)}{r} \right) = (2s(r^3(\lambda^2 s^2 - 2\lambda s + 2)^2 e^{2s(\lambda - (1/r))} + (4r^3 - 8r^2 s + 6rs^2 - 2s^3) - r(s - 2r)^2 (\lambda^2 s^2 - 2\lambda s + 2) e^{s(\lambda - (1/r))})) / (r^3((s^2 - 2s + 2) - e^{(\lambda - 1)s}(\lambda^2 s^2 - 2\lambda s + 2))^2). \quad (32)$$

Now, keeping in view the form of equation (30), we may assume the solution as

$$\psi = G(r) \sin^2 \theta \cos \theta. \quad (33)$$

In terms of $G(r)$ equation (30) written as

$$\left(\left(\frac{d^2}{dr^2} - \frac{6}{r^2} \right) \left(\frac{d^2}{dr^2} - \frac{6}{r^2} \right) - s \frac{d}{dr} \left(\frac{1}{r^2} \left(\frac{d^2}{dr^2} - \frac{6}{r^4} \right) \right) \right) G(r) = 2f(r) \left(f'(r) - \frac{f(r)}{r} \right), \quad (34)$$

on collecting terms of the same order in s

$$r^4 G''''(r) - 12r^2 G''(r) + 24r G'(r) - s(r^2 G'''(r) - 2r G''(r) - 6G'(r) + \frac{24}{r} G(r)) = 2r^4 f(r) \left(f'(r) - \frac{f(r)}{r} \right), \quad (35)$$

where the right hand term is evaluated above in (32).

Since finding the analytical solution of the equation (35) is very tedious, we use the following perturbation expansion for small s

$$G(r) = G_0(r) + sG_1(r) + s^2G_2(r). \quad (36)$$

Also, we need the expansion up to $O(s^2)$ term of the right hand side of equation (35) $2f(r)(f'(r) - (f(r))/r)$ where $f(r)$ given by equation (17); thus we have $2f(r)(f'(r) - (f(r))/r)$

$$\begin{aligned} & 6((\lambda^3 r^3 - 1)/((\lambda^3 - 1)^2 r^5)) + s(((1 - \lambda^3)(\lambda^3 r^3(4\lambda^3 r^3 + 9\lambda r - 20) + 7) \\ & + 6(3\lambda^4 - 4\lambda^3 + 1)r(\lambda^3 r^3 - 1))/(2(1 - \lambda^3)^3 r^6)) + s^2((3(\lambda - 1)^2(\lambda(\lambda(3\lambda(29\lambda \\ & + 36) + 70) + 28) + 7)r^2(\lambda^3 r^3 - 1) - 10(\lambda - 1)(\lambda^2 + \lambda + 1)(\lambda(3\lambda + 2) + 1) \\ & r(\lambda r - 1)^2(\lambda r(\lambda r(4\lambda r(\lambda r + 2) + 21) + 14) + 7) + 4(\lambda^2 + \lambda + 1)^2(\lambda r - 1)^3 \\ & (\lambda r(2\lambda r(5\lambda r(3\lambda r + 5) + 39) + 39) + 13))/40(\lambda - 1)^2(\lambda^2 + \lambda + 1)^4 r^7) + O(s^3). \end{aligned} \quad (37)$$

Substituting $G(r)$ and $f(r)$ from (36) and (37) in (35) and collecting zero order terms in s , we get the equation for $G_0(r)$

$$r^4 G_0^{(4)}(r) - 12r^2 G_0''(r) + 24r G_0'(r) = 6((\lambda^3 r^3 - 1)/((\lambda^3 - 1)^2 r)), \quad (38)$$

with boundary conditions :

$$G_0(1) = G_0'(1) = G_0(1/\lambda) = G_0'(1/\lambda) = 0, \quad (39)$$

we get the solution

$$\begin{aligned}
G_0(r) = & ((1-r)^2(\lambda r - 1)^2(\lambda(\lambda(\lambda(2\lambda - 1) - 16) + \lambda(\lambda(\lambda(2\lambda(\lambda \\
& + 4) + 5) - 2) - 3)r^3 + 2(\lambda + 1)(\lambda(\lambda(2\lambda(\lambda + 4) + 5) - 2) - 3) \\
& r^2 + \lambda(2\lambda(\lambda(4\lambda + 9) + 3) - 17)r - 4(4r + 5)) - 4(r + 2)) - 2)) \\
& / (4(\lambda^3 - 1)^2(\lambda(\lambda(4\lambda(\lambda(\lambda + 4) + 10) + 55) + 40) + 16) + 4)r^2). \quad (40)
\end{aligned}$$

A similar process for $G_1(r)$ yields differential equation:

$$\begin{aligned}
r^4 G_1''''(r) - 12r^2 G_1''(r) + 24r G_1'(r) - (r^2 G_0'''(r) - 2r G_0''(r) - 6G_0'(r) \\
+ (24/r)G_0(r)) = ((1 - \lambda^3)(\lambda^3 r^3(4\lambda^3 r^3 + 9\lambda r - 20) + 7) \\
+ 6(3\lambda^4 - 4\lambda^3 + 1)r(\lambda^3 r^3 - 1)) / (2(1 - \lambda^3)^3 r^6). \quad (41)
\end{aligned}$$

Use $G_0(r)$ from (40) in (41), we get

$$\begin{aligned}
r^4 G_1^{(4)}(r) - 12r^2 G_1''(r) + 24r G_1'(r) = r^4 (((\lambda^3 - 1)^2(4\lambda^6 + 16\lambda^5 + 40\lambda^4 + 55\lambda^3 \\
+ 40\lambda^2 + 16\lambda + 4)((1 - \lambda^3)(\lambda^3 r^3(4\lambda^3 r^3 + 9\lambda r - 20) + 7) + 6(3\lambda^4 - 4\lambda^3 + 1)r(\lambda^3 r^3 \\
- 1)) + (1 - \lambda^3)^3(10(4\lambda^6 + 16\lambda^5 + 40\lambda^4 + 55\lambda^3 + 40\lambda^2 + 16\lambda + 4) + 7\lambda^5(2\lambda^4 + 8\lambda^3 + \\
5\lambda^2 - 2\lambda - 3)r^6 + 4\lambda^3(4\lambda^6 + 16\lambda^5 + 40\lambda^4 + 55\lambda^3 + 40\lambda^2 + 16\lambda + 4)r^3 - 12(10\lambda^7 + \\
25\lambda^6 + 40\lambda^5 + 57\lambda^4 + 48\lambda^3 + 20\lambda^2 + 8\lambda + 2)r)) / (2(1 - \lambda^3)^3(\lambda^3 - 1)^2(4\lambda^6 + 16\lambda^5 + \\
40\lambda^4 + 55\lambda^3 + 40\lambda^2 + 16\lambda + 4)r^6)). \quad (42)
\end{aligned}$$

Boundary conditions

$$G_1(1) = G_1'(1) = G_1(1/\lambda) = G_1'(1/\lambda) = 0. \quad (43)$$

Solution for $G_2(r)$ are form differential equation

$$\begin{aligned}
r^4 G_2^{(4)}(r) - 12r^2 G_2''(r) + 24r G_2'(r) - (r^2 G_1^{(3)}(r) - 2r G_1''(r) - 6G_1'(r) \\
+ (24/r)G_1(r)) = (3(\lambda - 1)^2(\lambda(\lambda(3\lambda(29\lambda + 36) + 70) + 28) + 7)r^2(\lambda^3 r^3 \\
- 1) + 4(\lambda^2 + \lambda + 1)^2(\lambda r(2\lambda r(5\lambda r(3\lambda r + 5) + 39) + 39) + 13)(\lambda r - 1)^3 \\
- 10(\lambda - 1)(\lambda^2 + \lambda + 1)(\lambda(3\lambda + 2) + 1)r(\lambda r(\lambda r(4\lambda r(\lambda r + 2) + 21) + \\
14) + 7)(\lambda r - 1)^2) / (40(\lambda - 1)^2(\lambda^2 + \lambda + 1)^4 r^7). \quad (44)
\end{aligned}$$

Use $G_1(r)$ from (42) in (44), we get

$$\begin{aligned}
r^4 G_2^{(4)}(r) - 12r^2 G_2''(r) + 24r G_2'(r) = & (21(\lambda - 1)^3(3(\lambda - 1)^2(\lambda(\lambda(3\lambda(29\lambda + 36) + \\
& 70) + 28) + 7)r^2(\lambda^3 r^3 - 1) + 4(\lambda^2 + \lambda + 1)^2(\lambda r(2\lambda r(5\lambda r(3\lambda r + 5) + 39) + 39) + 13)(\lambda r \\
& - 1)^3 - 10(\lambda - 1)(\lambda^2 + \lambda + 1)(\lambda(3\lambda + 2) + 1)r(\lambda r(\lambda r(4\lambda r(\lambda r + 2) + 21) + 14) \\
& + 7)(\lambda r - 1)^2) + (2(\lambda^2 + \lambda + 1)(7140(\lambda^2 + \lambda + 1)(\lambda(\lambda(\lambda(4\lambda(\lambda + 4) + 10) \\
& + 55) + 40) + 16) + 4)\lambda^5 r^2 \log(1/\lambda)(\lambda(\lambda(\lambda^2 + \lambda + 1)r^5 + 6) + 6) + (\lambda - 1)(-1530(\lambda \\
& - 1)^2(\lambda^2 + \lambda + 1)(\lambda(\lambda(\lambda(4\lambda(\lambda + 4) + 10) + 55) + 40) + 16) + 4)^2 + 7\lambda^5(\lambda(\lambda(\lambda(\lambda \\
& (\lambda(\lambda(\lambda(\lambda(2\lambda(\lambda(\lambda(35\lambda(2\lambda(4\lambda(\lambda + 7) + 103) + 399) + 15837) + 6489) + 3609) \\
& + 12832) + 50015) + 143002) + 326409) + 423534) + 328887) + 161105) + \\
& 46874) + 5853) - 2247) - 1296)r^7 - 210(\lambda - 1)^2\lambda^3(\lambda(\lambda(3\lambda - 5) - 5) - 2)(\lambda(\lambda(\\
& \lambda(4\lambda(\lambda(\lambda + 4) + 10) + 55) + 40) + 16) + 4)^2 r^4 - 2520(\lambda - 1)^2\lambda^3(\lambda^2 + \lambda + 1)(\lambda(\lambda(\lambda(4\lambda \\
& (\lambda(\lambda + 4) + 10) + 55) + 40) + 16) + 4)^2 r^3 - 6(\lambda(\lambda(\lambda(\lambda(\lambda(\lambda(\lambda(\lambda(\lambda(2\lambda(\lambda(\lambda(\\
& 2\lambda(4\lambda(175\lambda - 557) - 19341) - 139923) - 259503) - 247863) - 19071) + 693727) \\
& + 808123) - 571160) - 2383773) - 2813680) - 1782647) - 553148) + 97413) + \\
& 224049) + 138084) + 48384) + 12096) + 1728)r^2 + 1050(\lambda - 1)^2(\lambda + 1)(\lambda(\lambda(\lambda(4\lambda(\\
& \lambda(\lambda + 4) + 10) + 55) + 40) + 16) + 4)(\lambda(\lambda(\lambda(\lambda(2\lambda(\lambda(4\lambda + 9) + 16) + 107) + 174) + \\
& 163) + 88) + 32) + 8)r)))/((\lambda(\lambda(\lambda(4\lambda(\lambda(\lambda + 4) + 10) + 55) + 40) + 16) + 4 \\
&)^2))/((840(\lambda - 1)^5(\lambda^2 + \lambda + 1)^4 r^7) \quad (45)
\end{aligned}$$

Boundary conditions

$$G_2(1) = 0, G_2'(1) = 0, G_2(1/\lambda) = 0, G_2'(1/\lambda) = 0. \quad (46)$$

Solution $G_2(r)$ is given in appendix of this chapter.

Substituting $G_0(r)$, $G_1(r)$ and $G_2(r)$ in (36), we get $G(r)$ which is given in appendix. We discuss only two cases for separation parameter λ for various flow/physical quantities. One for rotation of pervious sphere in infinite expanse of fluid (i.e. $\lambda \rightarrow 0$) and other for rotation of pervious sphere in concentric spherical cavity which is at rest and of radius two times of sphere (i.e. $\lambda = 0.5$).

Case(i): $\lambda = 0$.

$$\begin{aligned}
G(r) = & -((r - 1)(r^2(485s^2 + 2592s + 5040) - r(2989s^2 + 7488s + 5040) \\
& - 1384s^2) + 144rs(27s + 34) \log(r))/(40320r^3), \quad (47)
\end{aligned}$$

At $s = 0$

$$G(r) = -\frac{(r - 1)^2}{8r^2}, \quad (48)$$

which conforms to the solution of [10], Haberman [11], Collins [24] and Chakraborty and Roy [25].

Case(ii): $\lambda = 0.5$.

$$\begin{aligned}
G(r) = & (-3r^8(619s^2(20063904\log(2) - 14740493) + 2044s(8339520\log(2) \\
& - 5870101) + 1629395040) + 3577r^7s(s(8339520\log(2) - 10985941) - 3679200) \\
& + r^6(s^2(127928580192\log(2) - 85925650073) + 6132s(51705024\log(2) - 37619 \\
& 975) - 57530178720) + 122540355r^5(881s^2 + 2408s + 3136) - 9149679840r^4s(43 \\
& s + 56) - 6r^3(19s^2(5138027328\log(2) - 6273170923) + 12264s(3547337 + 83395 \\
& 20\log(2)) + 357464204160) + 183960r^2(s^2(2398405 + 3335808\log(2)) + 1822430 \\
& 4s + 16711744) + 7162176rs((3577r^5 + 14308r^2 - 167466)s - 208488)\log(r) \\
& - 8r(s^2(43864924147 + 18552562224\log(2)) - 36792s(1181432\log(2) - 905 \\
& 8899) + 156421923840) - 422105229952s^2)/(9415020555360r^3). \quad (49)
\end{aligned}$$

It may be seen that above result at $s = 0$ conforms to the corresponding solution of the flow due to the rotation of a sphere in a Newtonian fluid (as in [21], [11]).

5. DISCUSSION OF THE SECONDARY FLOW SOLUTION

Having obtained $G(r)$ and hence the stream function associated with the secondary flow, we present below certain quantities viz. pressure, streamlines, vortex lines that give more information about of the flow. Graphs have been drawn to depict pressure, streamlines, vortex lines for twelve values (0, 1, 10, 20, 30, 40) of the source parameter s and two values (0, 0.5) of the separation parameter λ . Here it may be noted that $\lambda = 0$ corresponds to an infinite expanse of fluid media outside of the pervious sphere and decreasing values of $\lambda = 0.5$ imply shrinking of the space in-between the pervious sphere and the cavity. Also that a increase in the value of s results in an increase in the effect of source; thus, the value $s=0$ approximates the Source modified Stokes equation to Stokes equation.

5.1. Pressure.

In order to determine the pressure p_1 in x - z plane we need to use the differential equation

$$dp_1 = \frac{\partial p_1}{\partial r} dr + \frac{\partial p_1}{\partial \theta} d\theta. \quad (50)$$

On inserting the values of u and v from equations (29) in the equations (25, 26) with use of (33), we get

$$\begin{aligned}
\frac{\partial p_1}{\partial r} = & \frac{(r^4 f(r)^2 - 3r^3 G''(r) + 3rsG'(r) - 12sG(r) + 18rG(r))}{r^5} \sin^2 \theta \\
& + \frac{2(r^3 G''(r) - rsG'(r) + (4s - 6r)G(r))}{r^5}, \quad (51)
\end{aligned}$$

$$\frac{\partial p_1}{\partial \theta} = \frac{1}{r^3} (r^3 f(r)^2 - r^3 G^{(3)}(r) + rsG''(r) + 6rG'(r) - 12G(r)) \sin \theta \cos \theta. \quad (52)$$

Above equations indicate that p_1 is to be written as

$$p_1(r, \theta) = p_{11}(r) \sin^2 \theta + p_{12}(r). \quad (53)$$

Integrating (52) with respect to θ gives

$$p_1 = \frac{1}{2r^3} (r^3 f(r)^2 - r^3 G^{(3)}(r) + rsG''(r) + 6rG'(r) - 12G(r)) \sin^2 \theta + g(r). \quad (54)$$

Differentiating above equation with respect to r and equating with right hand side of (51) and using the differential equation (35), we get

$$g'(r) = \frac{2(r^3 G''(r) - rsG'(r) + (4s - 6r)G(r))}{r^5}. \quad (55)$$

Since $r \geq 1$ here, so equation (55) provide

$$g(r) = \int_1^r \left(\frac{2(r^3 G''(r) - rsG'(r) + (4s - 6r)G(r))}{r^5} \right) dr. \quad (56)$$

Due to very much larger expression of $G(r)$ in terms of λ . Also, the needed $f(r)$ in form as

$$\begin{aligned} f(r) = & (\lambda^3 r^3 - 1)/((\lambda^3 - 1)r^2) + s(\lambda^2 + \lambda + \lambda^3(2\lambda + 1)^2 r^4 - 4\lambda^3(\lambda^2 + \lambda + 1)r^3 + \\ & (3\lambda^3 - \lambda^2 - \lambda - 1)r + 1)/(4(\lambda - 1)(\lambda^2 + \lambda + 1)^2 r^3) - (s^2(4(\lambda^2 + \lambda + 1)^2 - \lambda^3(40\lambda^4 + \\ & 101\lambda^3 + 57\lambda^2 + 17\lambda + 1)r^5 + 20\lambda^3(2\lambda + 1)^2(\lambda^2 + \lambda + 1)r^4 - 40\lambda^3(\lambda^2 + \lambda + 1)^2 r^3 + \\ & (\lambda - 1)^2(21\lambda^4 + 24\lambda^3 + 10\lambda^2 + 4\lambda + 1)r^2 + 5(3\lambda^5 + 2\lambda^4 + \lambda^3 - 3\lambda^2 - 2\lambda - 1)r)) \\ & / (80((\lambda - 1)(\lambda^2 + \lambda + 1)^3 r^4)). \end{aligned} \quad (57)$$

Using $g(r)$ from (56) and $f(r)$ from (57) in (54), we get p_1 completely known from. Since expression of p_1 in terms of λ is very huge, therefore, we discuss for two values 0 and 0.5 of λ .

Expression for pressure for $\lambda = 0$

$$\begin{aligned} p_1 = & ((3r - 4)/(4r^4) + s((108r^3 - 350r^2 + 340r - 105)/(280r^6)) + s^2((485r^3 \\ & - 3180r^2 + 7320r - 2448 \log(r) - 4618)/(6720r^6))) \sin^2 \theta - ((r - 1)/(2r^4)) - s((36 \\ & r^3 - 105r^2 + 104r - 35)/(140r^6)) + s^2((-485r^3 + 3033r^2 - 6984r + 2448 \log(r) \\ & + 4436)/(10080r^6)). \end{aligned} \quad (58)$$

Expression for pressure for $\lambda = 0.5$

$$\begin{aligned}
p_1 = & ((1057r^6 - 8176r^3 + 68448r - 65408)/(50078r^4) + s((21r^8(2779840 \log(2) \\
& - 1672487) - 8809640r^7 - 73543120r^5 + 208896800r^4 + 24r^3(3547337 + 8339520 \\
& \log(2)) - 962151680r^2 + 811713280r - 204072960)/(511797160r^6)) + s^2((21r^8 \\
& (12419556576 \log(2) - 8142486259) - 3066r^7(75055680 \log(2) - 40893949) + \\
& 16377509120r^6 + 73584r^5(4308752 \log(2) - 5091185) + 983590582800r^4 + 4r^3(\\
& 292867557696 \log(2) - 442967754451) - 29278975488(7r^3 + 51) \log(r) - 98112r^2 \\
& (7769681 + 12509280 \log(2)) + 4712998295040r + 65408(5316444 \log(2) - 4891 \\
& 5751))/(3138340185120r^6))) \sin^2 \theta - 2((91r^6 + 765r^4 - 4088r^3 + 11408r - 8176)/ \\
& (25039r^4)) - s(((r^2 - 3r + 2)(7r^6(8339520 \log(2) - 5870101) + 21r^5(8339520 \log(\\
& 2) - 5294861) + 6r^4(25018560 \log(2) - 10549159) + 180r^3(555968 \log(2) - 94092 \\
& 3) - 68163312r^2 + 219378432r - 102036480))/(767695740r^6)) - s^2((12999r^8(\\
& 20063904 \log(2) - 14740493) - 9198r^7(25018560 \log(2) - 29547263) - r^6(63964 \\
& 2900960 \log(2) - 390848648413) + 12264r^5(25852512 \log(2) - 54021355) + 1475 \\
& 385874200r^4 + 12r^3(97622519232 \log(2) - 170428454641) - 1876392r^2(374249 \\
& + 654080 \log(2)) - 3659871936(35r^6 + 56r^3 + 408) \log(r) + 4551963929856r + 32 \\
& 704(10632888 \log(2) - 94436929))/(4707510277680r^6)). \tag{59}
\end{aligned}$$

Now we discuss variation of pressure flow pattern; showing in figures 6 to 17 are discussed here. This is done for two values of separation parameter $\lambda = a/b$ with various values of source parameter s as depicted on the diagrams. Equi-pressure surfaces are obtained by setting surfaces are given by setting the pressure function $p_1 = c$ for different values of the constant c . Since the flow is axial symmetric about the z -axis, their intersection with the x - z plane may be termed as streamlines. Thus they are given by equation (33) using above expressions; we present below the graphs clearly depicting the variation in the pressure in Figure 6-17.

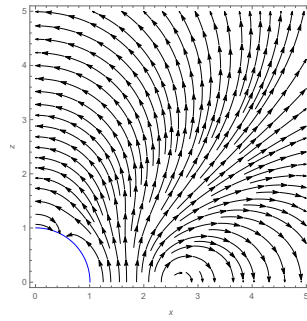
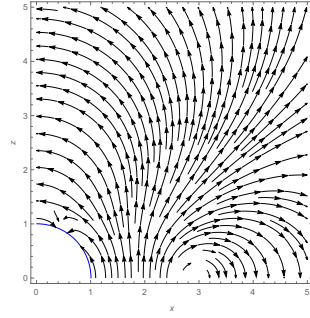
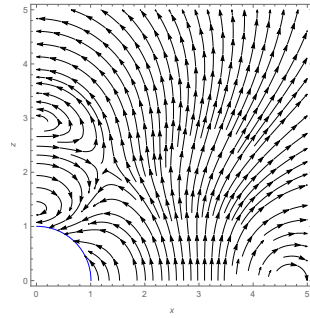
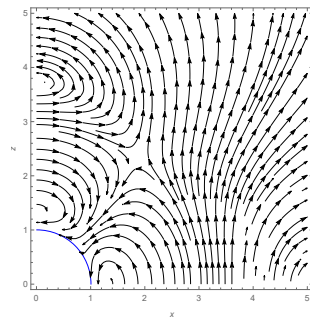
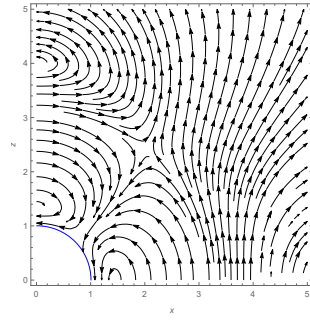
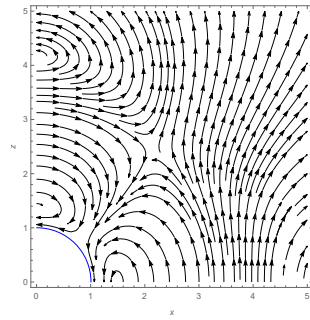
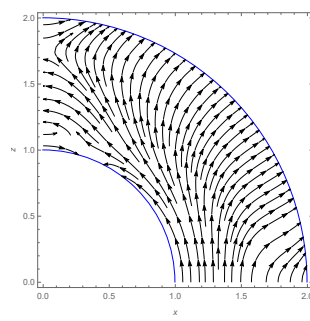
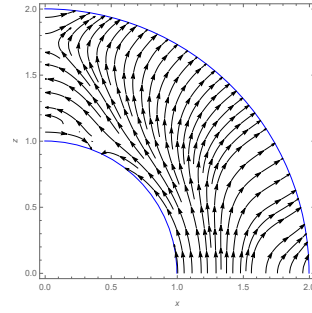
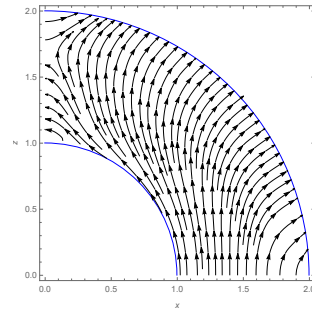
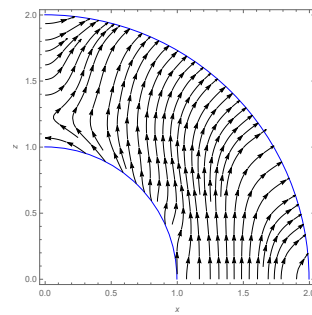
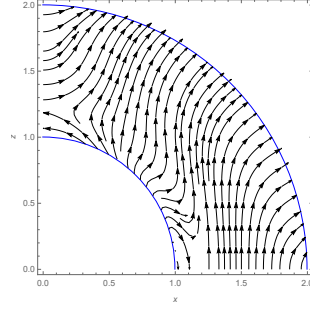
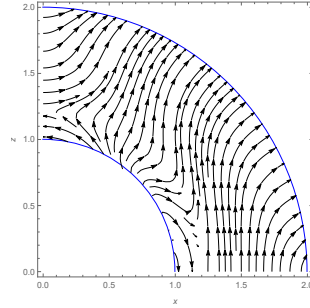


FIGURE 6. $\lambda \rightarrow 0$ and $s = 0$

FIGURE 7. $\lambda \rightarrow 0$ and $s = 1$ FIGURE 8. $\lambda \rightarrow 0$ and $s = 10$.FIGURE 9. $\lambda \rightarrow 0$ and $s = 20$.

FIGURE 10. $\lambda \rightarrow 0$ and $s = 30$.FIGURE 11. $\lambda \rightarrow 0$ and $s = 40$.FIGURE 12. $\lambda = 0.5$ and $s = 0$

FIGURE 13. $\lambda = 0.5$ and $s = 1$ FIGURE 14. $\lambda = 0.5$ and $s = 10$ FIGURE 15. $\lambda = 0.5$ and $s = 20$.

FIGURE 16. $\lambda = 0.5$ and $s = 30$.FIGURE 17. $\lambda = 0.5$ and $s = 40$.

5.2. Streamlines.

Stream surfaces are given by setting the Stokes stream function $\psi = c$ for different values of the constant c . Since the flow is axial symmetric, about the z -axis, their intersection the x - z may be termed as streamlines. Thus they are given by equation (33)

$$\psi = G(r) \sin^2 \theta \cos \theta = \frac{G(r)x^2 z}{r^3}. \quad (60)$$

This shows that the flow pattern is symmetric about z -axis and anti symmetric about x -axis. Therefore, it suffices to draw the streamline pattern graphically only in the first quadrant of x - z plane. This is done for two values of separation parameter $\lambda (= a/b)$; $\lambda = 0$ for the pervious sphere rotating in an infinite expanse of fluid (Figs. 18-23) and $\lambda = 0.5$ for the case when the fluid is confined to a cavity has radius 2 (Figs. 24-29). In both cases the streamlines are depicted for 6 values of the source parameter viz. $s = 0, 1, 10, 20, 30, 40$. First we observe and discuss the disturbance induced by the presence of source in the two cases. The stream lines only refer to the exterior flow and hence do not appear in the interior of the sphere. Case (i): The fluid extends to infinity, $\lambda = 0$. It may be noted that $s = 0$ corresponds to the case when source is absent. It is observed that in Figs.18, 19, 20

($s : 0, 1, 10$), the stream lines are in the anticlockwise direction appearing to diverge to infinity with hardly any difference in the flow pattern. It is also noteworthy that these secondary vorticities are in clockwise direction as depicted in Figs. 20, 21, 22, 23.

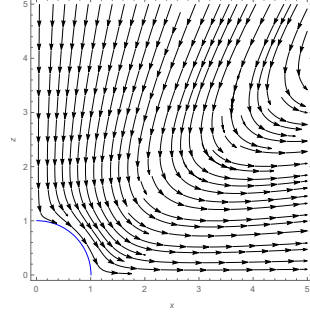


FIGURE 18. $\lambda \rightarrow 0$ and $s = 0$.

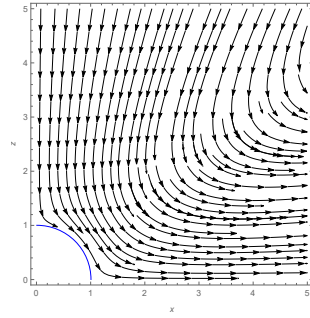


FIGURE 19. $\lambda \rightarrow 0$ and $s = 1$.

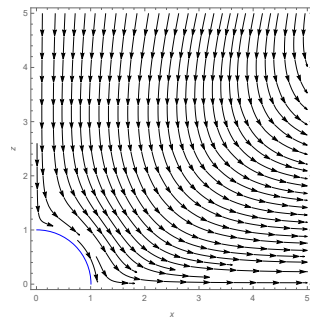
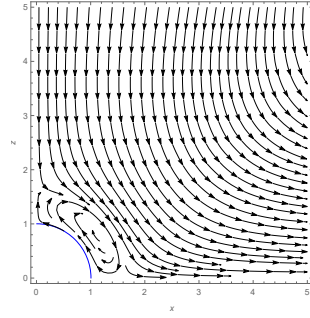
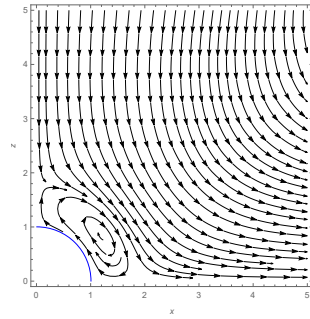
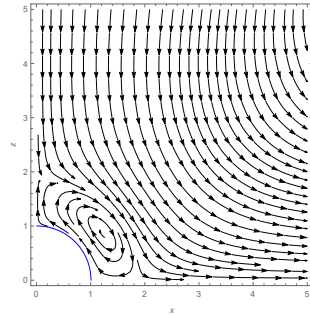


FIGURE 20. $\lambda \rightarrow 0$ and $s = 10$.

FIGURE 21. $\lambda \rightarrow 0$ and $s = 20$.FIGURE 22. $\lambda \rightarrow 0$ and $s = 30$.FIGURE 23. $\lambda \rightarrow 0$ and $s = 40$.

Case (ii): The fluid is confined in the region $1 < r < 2$, $\lambda = 0.5$. Since fluid remains confined to a closed region, the flow is markedly different from the $\lambda = 0$ case. For s from the value 0 to the value 10 (Figs. 24, 25, 26) there is only a single eddy rotating anticlockwise but streamlines became more compact as s increases.

The base flow here is the primary flow when $s = 0$, i.e. when the source is absent and a solid sphere rotating steadily. The flow lines consist of circles round the axis of rotation that we have taken as x -axis. Then centrifugal force as in Fig.18 throws the fluid outwards in the radial direction and on account of continuity equation the fluid returns down ward in axial direction as shown in Fig. 18 for the infinite region case and Fig. 24 in the confined region case. It is seen that in the latter case the streamlines form closed curves, while in the former case they are closed at infinity. It may also be inferred from the expression (29) of velocity that as we move away from the equatorial plane towards the pole above, it decreases and so does the circular streamline there. Thus, velocity is maximum at the equators and tends to zero at the pole. Then the secondary flow pattern emerges and is modified by the source flow which is in the spheres radial r -direction, and wanes as it moves away from the surface of the sphere and it remains the same for all the emerging points. It may be seen that at points near the pole outgoing source flow meets the incoming secondary flow and the former gets, for sufficient larger values of s , stronger than the latter appoints and reverse happens at points near the equator; this process goes to generate eddies (see Figs. 21, 22, 23 and 27, 28, 29) near the inner sphere.

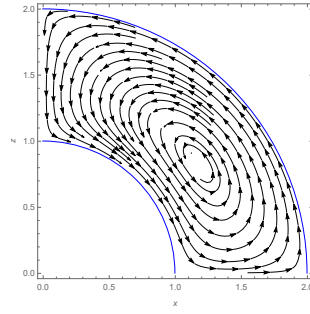


FIGURE 24. $\lambda = 0.5$ and $s = 0$

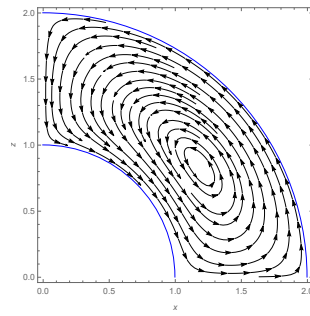
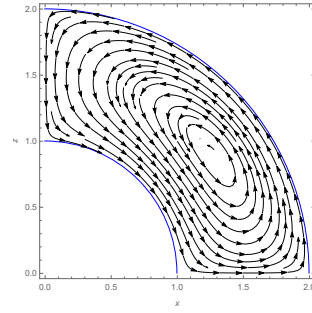
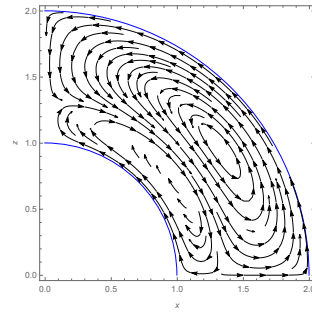
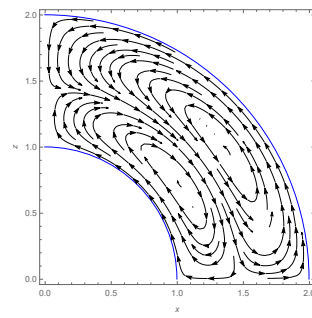
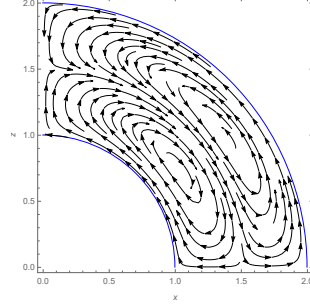


FIGURE 25. $\lambda = 0.5$ and $s = 1$

FIGURE 26. $\lambda = 0.5$ and $s = 10$ FIGURE 27. $\lambda = 0.5$ and $s = 20$.FIGURE 28. $\lambda = 0.5$ and $s = 30$.

FIGURE 29. $\lambda = 0.5$ and $s = 40$.

5.3. Vorticity.

The vorticity ζ is in the azimuthal direction and is given by

$$\zeta = \frac{1}{r} \left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \theta} \right)$$

or

$$\zeta = \frac{1}{r \sin \theta} E^2 \psi$$

or

$$\zeta = \frac{1}{r} \left(G''(r) - \frac{6G(r)}{r^2} \right) \sin \theta \cos \theta. \quad (61)$$

Now we present ζ graphically (30 to 41) for two values of λ one for $\lambda \rightarrow 0$ and other for $\lambda = 0.5$. As for streamlines.

In both cases the streamlines are depicted for 6 values of the source parameter viz. $s = 0, 1, 10, 20, 30, 40$. First we observe and discuss the disturbance induced by the presence of source in the two cases. The graphical results are depicted below. In both the cases inside the sphere there are no vortices as source flow itself is irrotational.

Case (i). The fluid extends to infinity, $\lambda = 0$. Fig. 30 provides the base state when $s = 0$; it is seen that there are two sections of vortices, a small section, in clockwise direction, near the surface of the inner sphere and a large section, in anticlockwise direction, away from the sphere and extending to infinity. With increase in the value of s upto 10 (Fig. 32), the same pattern persists except that the inner section becomes more and more prominent and the centers of the two sections shifting away from the surface of the sphere. Then with further increase of s , a third section, again in anticlockwise direction, starts appearing and becoming prominent (Figs.33-35).

The primary circulatory motion around the sphere inhibited by its presence of the sphere generate the vortices in the secondary flow and to preserve the circulation the next vortex appears circulating in opposite direction. Interaction with the source flow enhances this phenomena; this causes the appearance of the third section. Case (ii) The fluid is confined in the region $1 < r < 2$, $\lambda = 0.5$. In this case the 3 sections are present from the very beginning and circulating in the same sense as

in case (i); noticeable difference being that the central portions get elongated as s increases. Vortices are generated by the presence of a solid boundary. There are two in this case at $r = 1$ and at $r = 1/\lambda$ and that explains the presence of third region from the very start.

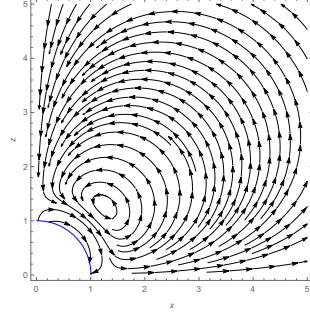


FIGURE 30. $\lambda \rightarrow 0$ and $s = 0$.

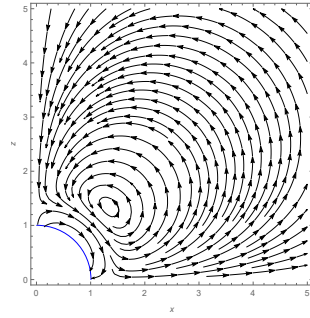


FIGURE 31. $\lambda \rightarrow 0$ and $s = 1$.

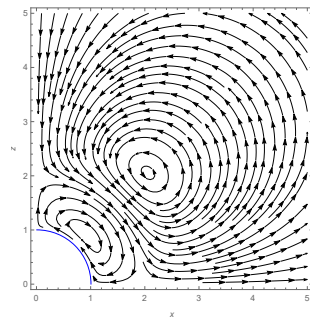
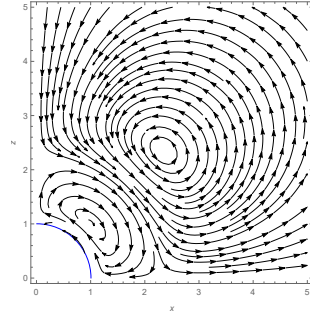
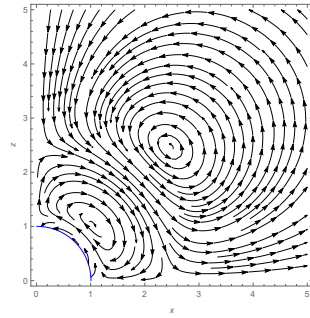
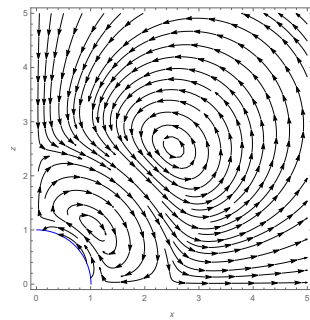
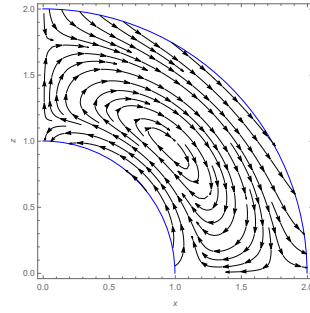
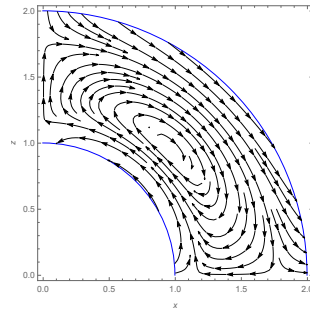
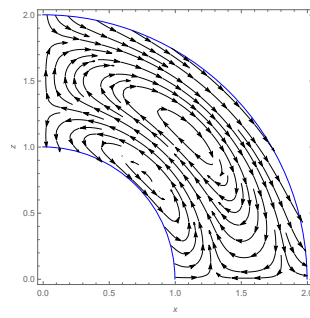
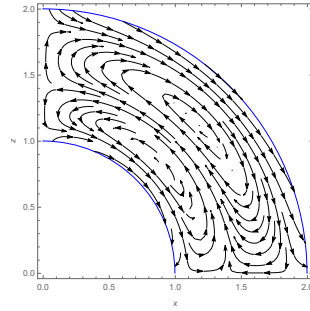
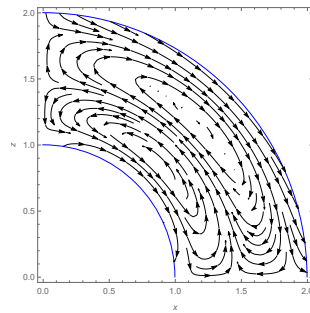
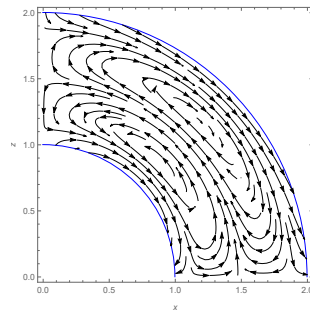


FIGURE 32. $\lambda \rightarrow 0$ and $s = 10$.

FIGURE 33. $\lambda \rightarrow 0$ and $s = 20$.FIGURE 34. $\lambda \rightarrow 0$ and $s = 30$.FIGURE 35. $\lambda \rightarrow 0$ and $s = 40$.

FIGURE 36. $\lambda = 0.5$ and $s = 0$.FIGURE 37. $\lambda = 0.5$ and $s = 1$.FIGURE 38. $\lambda = 0.5$ and $s = 10$.

FIGURE 39. $\lambda = 0.5$ and $s = 20$.FIGURE 40. $\lambda = 0.5$ and $s = 30$.FIGURE 41. $\lambda = 0.5$ and $s = 40$.

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