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Solutions of a Generalization of Linear Volterra Integro-Differential Equations

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Abstract. In this paper, we combine linear Volterra integro-differential equations of first and second kinds to be a generalization. Then, we use Laplace transform to solve an analytical solution on a convolution kernel and apply Laguerre polynomials to approximate a solution on a non-convolution kernel of this generalization.

Key words and Phrases: Volterra integro-differential equation, Laplace transform, Laguerre polynomial.

1. INTRODUCTION

In this study, we consider the following linear *Volterra integro-differential equations* (or only VIDEs) with initial conditions. The linear VIDEs of first kind are given by

$$\int_0^x k_1(x,t)u^{(n)}(t)dt = f(x)u(x) + g(x) + \int_0^x k_2(x,t)u(t)dt, \quad x \in [0,T],$$
$$u(0) = a_0, u^{'}(0) = a_1, ..., u^{(n-1)}(0) = a_{n-1}, \tag{1}$$

where $f(x), g(x), k_1(x,t)$ and $k_2(x,t)$ are sufficiently smooth functions. The linear VIDEs of second kind are expressed by

$$u^{(n)}(x) = h(x)u(x) + k(x) + \int_0^x k_3(x,t)u(t)dt, \quad x \in [0,T],$$
$$u(0) = b_0, u^{'}(0) = b_1, ..., u^{(n-1)}(0) = b_{n-1}, \tag{2}$$

where $h(x), k(x)$ and $k_3(x,t)$ are sufficiently smooth functions. The functions $k_1(x,t), k_2(x,t)$ and $k_3(x,t)$ are called *kernels* of the linear VIDEs.

Let $F(x)$ be defined and piecewise continuous function for all positive values of x and be of exponential order. The *Laplace transform* of $F(x)$ is defined by

$$L\Big\{F(x)\Big\} = \int_0^\infty F(x)e^{-sx}dx = G(s),$$

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where $F(x)$ is said to be the *inverse Laplace transform* of $G(s)$, denoted by $F(x) = L^{-1}\Big\{G(s)\Big\}$. Let us recall some useful results on the Laplace transform that shall be used in the next hereinafter.

(1) The Laplace transform of some functions:

$L\Big\{1\Big\} = \frac{1}{s}$, $s > 0$, $L\Big\{x^n\Big\} = \frac{n!}{s^{n+1}}$, $s > 0$,

$L\Big\{e^{ax}\Big\} = \frac{1}{s-a}$, $s > a$, $L\Big\{\sin ax\Big\} = \frac{a}{s^2+a^2}$, $s > 0$,

$L\Big\{\cos ax\Big\} = \frac{s}{s^2+a^2}$, $s > 0$, $L\Big\{\sinh ax\Big\} = \frac{a}{s^2-a^2}$, $s > |a|$,

$L\Big\{\cosh ax\Big\} = \frac{s}{s^2-a^2}$, $s > |a|$.

(2) If $L\Big\{F(x)\Big\} = G(s)$ then $L\Big\{F'(x)\Big\} = sG(s) - F(0)$.

(3) If $L\Big\{F(x)\Big\} = G(s)$ then $L\Big\{F^{(n)}(x)\Big\} = s^nG(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0)$.

(4) If $L\Big\{F(x)\Big\} = G(s)$ then $L\Big\{x^nF(x)\Big\} = (-1)^n\frac{d^n}{ds^n}\Big[G(s)\Big]$.

(5) The *convolution* of two functions $F(x)$ and $H(x)$, denoted by $F(x) * H(x)$, is defined by $F(x) * H(x) = \int_0^x F(t)H(x-t)dt = \int_0^x F(x-t)H(t)dt$. Let $L\Big\{F(x)\Big\} = G(s)$ and $L\Big\{H(x)\Big\} = I(s)$. Then the *convolution theorem* says that $L\Big\{F(x) * H(x)\Big\} = L\Big\{F(x)\Big\}L\Big\{H(x)\Big\} = G(s)I(s)$.

(6) The inverse Laplace transform of some functions:

$L^{-1}\Big\{\frac{1}{s}\Big\} = 1$, $L^{-1}\Big\{\frac{1}{s^{n+1}}\Big\} = \frac{1}{n!}x^n$,

$L^{-1}\Big\{\frac{1}{s-a}\Big\} = e^{ax}$, $L^{-1}\Big\{\frac{1}{s^2+a^2}\Big\} = \frac{\sin ax}{a}$,

$L^{-1}\Big\{\frac{s}{s^2+a^2}\Big\} = \cos ax$, $L^{-1}\Big\{\frac{1}{s^2-a^2}\Big\} = \frac{\sinh ax}{a}$,

$L^{-1}\Big\{\frac{s}{s^2-a^2}\Big\} = \cosh ax$.

The *Laguerre polynomial* is a polynomial function given by

$$L_\chi(x) = \sum_{k=0}^{\chi}\binom{\chi}{k}\frac{(-1)^kx^k}{k!}, \quad \binom{\chi}{k} = \frac{\chi!}{k!(\chi-k)!},$$

where χ and k are called the *degree* and the *index* of the Laguerre polynomial, respectively. Some important results on the Laguerre polynomials that shall be referred in the next as follows:

(1) $L'_\chi(x) = \frac{d}{dx}\Big[\sum_{k=0}^{\chi}\binom{\chi}{k}\frac{(-1)^kx^k}{k!}\Big] = \sum_{k=1}^{\chi}\binom{\chi}{k}\frac{(-1)^kx^{k-1}}{(k-1)!}$,

(2) $L''_\chi(x) = \frac{d^2}{dx^2}\Big[\sum_{k=0}^{\chi}\binom{\chi}{k}\frac{(-1)^kx^k}{k!}\Big] = \sum_{k=2}^{\chi}\binom{\chi}{k}\frac{(-1)^kx^{k-2}}{(k-2)!}$,

(3) $L_{\chi}^{'''}(x) = \dfrac{d^3}{dx^3}\Big[\sum_{k=0}^{\chi}\binom{\chi}{k}\dfrac{(-1)^k x^k}{k!}\Big] = \sum_{k=3}^{\chi}\binom{\chi}{k}\dfrac{(-1)^k x^{k-3}}{(k-3)!},$

(4) $L_{\chi}^{(n)}(x) = \dfrac{d^n}{dx^n}\Big[\sum_{k=0}^{\chi}\binom{\chi}{k}\dfrac{(-1)^k x^k}{k!}\Big] = \sum_{k=n}^{\chi}\binom{\chi}{k}\dfrac{(-1)^k x^{k-n}}{(k-n)!}, n \leq \chi.$

The Volterra integro-differential equations are typically mathematical models in many areas of science and engineering. Solutions of these equations play vital roles in a number of processes and phenomena such as nuclear reactors, circuit analyses, wave propagation, glass forming processes, nano-hydrodynamics, visco elasticity, biological populations, etc. Therefore, there are many researchers who have been interested in the VIDEs and founded numerous methods to solve the analytical and numerical solutions of VIDEs up to the present as follows. Estimated solutions of nonlinear VIDEs of a fractional order were investigated applying the Laplace transform and the Adomian polynomials by Yang and Hou [1]. Moreover, the Legendre polynomial approximation was used to find numerical solutions of nonlinear VIDEs of second kind by Gachpazan, Erfanian and Beiglo [2]. In addition, analytical solutions of linear VIDEs of second kind were solved using the Kamal transform by Aggarwal and Gupta [3]. The modified Adomian decomposition method was utilized to explain exact solutions of linear VIDEs of second kind by Okai, Ilejimi and Ibrahim [4]. Furthermore, approximate solutions of nonlinear VIDEs involving delay were found taking a new higher order method by Jhinga, Patade and Gejji [5]. Other than those findings, the Sadik transform was applied to figure out exact solutions of first kind VIDEs on convolution type kernels by Aggarwal, Vyas and Sharma [6]. Numerical solutions of linear VIDEs were estimated using the Laguerre and Touchard polynomials by Abdullah and Ali [7]. So far, some asymptotic behavior of exact solutions of nonlinear VIDEs has been studied by Cakir, Gunes and Duru [8]. The quasilinearization technique to difference scheme also has been applied to solve estimated solutions of VIDEs in [8]. Recently, the asymptotic behavior of the analytical solutions of the singularly perturbed nonlinear VIDEs has been established by Cakir, Cakir and Cakir [9]. The uniform difference scheme on the Bakhvalov-Shishkin mesh points according to the boundary layer conditions has been introduced to find numerical solutions of VIDEs as well in [9]. Exact solutions of the Faltung type VIDEs for first kind have been solved applying the Kushare transform by Patil, Nikam and Shinde [10].

In this research, we compound linear Volterra integro-differential equations of first and second kinds to be a general form. Then, we take the Laplace transform to find an exact solution on a convolution type and utilize the Laguerre polynomials to estimate a solution on a non-convolution type of this generalization.

2. MAIN RESULTS

For this section, we assume that $p, m \leq n$ and start to introduce a generalization of linear Volterra integro-differential equations expressed by

$$\begin{aligned}
&\alpha u^{(n)}(x) + \beta \int_0^x K_1(x,t) u^{(m)}(t)dt \\
&= A(x)u(x) + B(x) + \int_0^x K_2(x,t) u^{(p)}(t)dt, \quad x \in [0,T], \\
&u(0) = c_0, u^{'}(0) = c_1, ..., u^{(n-1)}(0) = c_{n-1},
\end{aligned} \tag{3}$$

where $u(x)$ is an exponentially bounded and smooth function, $A(x), B(x), K_1(x,t)$ and $K_2(x,t)$ are exponentially bounded and sufficiently smooth functions and α, β are real numbers. If $\alpha = 0$ and $m = 0$ or $\alpha = 0$ and $p = 0$, then the equation (3) is the linear VIDE of first kind and we can see [6] and [10] for more vital results. If $\beta = 0$ and $p = 0$, then the equation (3) is the linear VIDE of second kind and we can see [1], [2], [3], [4], [5], [7], [8] and [9] for more comprehensive findings.

Now, we will focus on a solution of this generalization on a convolution type with a constant function A, that is, we then consider the following initial-value problem as follows:

$$\begin{aligned}
&\alpha u^{(n)}(x) + \beta \int_0^x K_1(x-t) u^{(m)}(t)dt \\
&= Au(x) + B(x) + \int_0^x K_2(x-t) u^{(p)}(t)dt, \quad x \in [0,T], \\
&u(0) = c_0, u^{'}(0) = c_1, ..., u^{(n-1)}(0) = c_{n-1}.
\end{aligned} \tag{4}$$

We will utilize the Laplace transform to solve the problem as the following steps: At the beginning, applying the Laplace transform to (4), we get

$$\begin{aligned}
&\alpha L\Big\{u^{(n)}(x)\Big\} + \beta L\Big\{\int_0^x K_1(x-t) u^{(m)}(t)dt\Big\} \\
&= AL\Big\{u(x)\Big\} + L\Big\{B(x)\Big\} + L\Big\{\int_0^x K_2(x-t) u^{(p)}(t)dt\Big\}.
\end{aligned} \tag{5}$$

After, using the convolution theorem to (5), we then obtain

$$\begin{aligned}
&\alpha L\Big\{u^{(n)}(x)\Big\} + \beta L\Big\{K_1(x)\Big\} L\Big\{u^{(m)}(x)\Big\} \\
&= AL\Big\{u(x)\Big\} + L\Big\{B(x)\Big\} + L\Big\{K_2(x)\Big\} L\Big\{u^{(p)}(x)\Big\}.
\end{aligned} \tag{6}$$

Then, taking the Laplace transform of derivatives on (6) with initial conditions, we have

$$\begin{aligned}
&\alpha\Big[s^n L\Big\{u(x)\Big\} - s^{n-1}c_0 - s^{n-2}c_1 - \cdots - c_{n-1}\Big] \\
&+\beta L\Big\{K_1(x)\Big\}\Big[s^m L\Big\{u(x)\Big\} - s^{m-1}c_0 - s^{m-2}c_1 - \cdots - c_{m-1}\Big] = AL\Big\{u(x)\Big\} \\
&+L\Big\{B(x)\Big\} + L\Big\{K_2(x)\Big\}\Big[s^p L\Big\{u(x)\Big\} - s^{p-1}c_0 - s^{p-2}c_1 - \cdots - c_{p-1}\Big]
\end{aligned}$$

and we also obtain

$$\Big[\alpha s^n+\beta L\Big\{K_1(x)\Big\}s^m-L\Big\{K_2(x)\Big\}s^p-A\Big]L\Big\{u(x)\Big\}$$
$$=\alpha\Big(s^{n-1}c_0+s^{n-2}c_1+\cdots+c_{n-1}\Big)$$
$$+\beta L\Big\{K_1(x)\Big\}\Big(s^{m-1}c_0+s^{m-2}c_1+\cdots+c_{m-1}\Big)$$
$$+L\Big\{B(x)\Big\}-L\Big\{K_2(x)\Big\}\Big(s^{p-1}c_0+s^{p-2}c_1+\cdots+c_{p-1}\Big). \tag{7}$$

At last, operating the inverse Laplace transform on (7), we receive the solution of initial-value problem (4) as follows.

$$u(x)=L^{-1}\left\{\frac{\alpha\Big(s^{n-1}c_0+s^{n-2}c_1+\cdots+c_{n-1}\Big)}{\alpha s^n+\beta L\Big\{K_1(x)\Big\}s^m-L\Big\{K_2(x)\Big\}s^p-A}\right\}$$
$$+L^{-1}\left\{\frac{\beta L\Big\{K_1(x)\Big\}\Big(s^{m-1}c_0+s^{m-2}c_1+\cdots+c_{m-1}\Big)}{\alpha s^n+\beta L\Big\{K_1(x)\Big\}s^m-L\Big\{K_2(x)\Big\}s^p-A}\right\}$$
$$+L^{-1}\left\{\frac{L\Big\{B(x)\Big\}-L\Big\{K_2(x)\Big\}\Big(s^{p-1}c_0+s^{p-2}c_1+\cdots+c_{p-1}\Big)}{\alpha s^n+\beta L\Big\{K_1(x)\Big\}s^m-L\Big\{K_2(x)\Big\}s^p-A}\right\}.$$

Example 2.1. *Solve the Volterra integro-differential problem:*
$u^{(4)}(x)+\int_0^x \sin(x-t)u^{''}(t)dt=u(x)-\frac{1}{2}e^x-\frac{1}{2}\cos x-\frac{1}{2}\sin x-x^2+x-1+\int_0^x (x-t)u(t)dt, u(0)=3, u^{'}(0)=1, u^{''}(0)=1, u^{'''}(0)=1.$

Solution. Firstly, applying Laplace transform to the problem, we have

$$L\Big\{u^{(4)}(x)\Big\}+L\Big\{\int_0^x \sin(x-t)u^{''}(t)dt\Big\}=L\Big\{u(x)\Big\}$$
$$+L\Big\{-\frac{1}{2}e^x-\frac{1}{2}\cos x-\frac{1}{2}\sin x-x^2+x-1\Big\}+L\Big\{\int_0^x (x-t)u(t)dt\Big\}.$$

Secondly, using the convolution theorem, we get

$$L\Big\{u^{(4)}(x)\Big\}+L\Big\{\sin x\Big\}L\Big\{u^{''}(x)\Big\}=L\Big\{u(x)\Big\}$$
$$+L\Big\{-\frac{1}{2}e^x-\frac{1}{2}\cos x-\frac{1}{2}\sin x-x^2+x-1\Big\}+L\Big\{x\Big\}L\Big\{u(x)\Big\}.$$

Thirdly, taking the Laplace transform of derivatives, we have

$$s^4L\Big\{u(x)\Big\}-s^3u(0)-s^2u^{'}(0)-su^{''}(0)-u^{'''}(0)$$
$$+\frac{1}{s^2+1}\Big[s^2L\Big\{u(x)\Big\}-su(0)-u^{'}(0)\Big]=L\Big\{u(x)\Big\}$$
$$-\frac{1}{2(s-1)}-\frac{s}{2(s^2+1)}-\frac{1}{2(s^2+1)}-\frac{2}{s^3}+\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s^2}L\Big\{u(x)\Big\}$$

and using initial conditions, we also obtain

$$s^4L\Big\{u(x)\Big\} - 3s^3 - s^2 - s - 1 + \frac{1}{s^2+1}\Big[s^2L\Big\{u(x)\Big\} - 3s - 1\Big]$$
$$= L\Big\{u(x)\Big\} - \frac{1}{2(s-1)} - \frac{s}{2(s^2+1)} - \frac{1}{2(s^2+1)}$$
$$-\frac{2}{s^3} + \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s^2}L\Big\{u(x)\Big\}.$$

Fourthly, rearranging the equation, we certainly receive

$$\Big(s^4 + \frac{s^2}{s^2+1} - \frac{1}{s^2} - 1\Big)L\Big\{u(x)\Big\} = 3s^3 + s^2 + s + 1 + \frac{3s}{s^2+1}$$
$$+\frac{1}{s^2+1} - \frac{1}{2(s-1)} - \frac{s}{2(s^2+1)} - \frac{1}{2(s^2+1)} - \frac{2}{s^3} + \frac{1}{s^2} - \frac{1}{s}$$

or

$$\Big[\frac{s^8+s^6-2s^2-1}{s^2(s^2+1)}\Big]L\Big\{u(x)\Big\} = \frac{6s^9 - 4s^8 + 6s^7 - 4s^6 - 12s^3 + 8s^2 - 6s + 4}{2s^3(s-1)(s^2+1)}.$$

Thus, we get

$$L\Big\{u(x)\Big\} = \frac{2(3s-2)(s^8+s^6-2s^2-1)}{2s(s-1)(s^8+s^6-2s^2-1)} = \frac{3s-2}{s(s-1)}.$$

Finally, taking the inverse Laplace transform of the equation, we suddenly have an analytical solution

$$u(x) = L^{-1}\Big\{\frac{3s-2}{s(s-1)}\Big\} = L^{-1}\Big\{\frac{s}{s(s-1)}\Big\} + L^{-1}\Big\{\frac{2s-2}{s(s-1)}\Big\}$$
$$= L^{-1}\Big\{\frac{1}{s-1}\Big\} + 2L^{-1}\Big\{\frac{1}{s}\Big\} = e^x + 2.$$

Example 2.2. *Solve the Volterra integro-differential problem:*
$u^{'''}(x) + \int_0^x (x-t)^2u^{''}(t)dt = u(x) + \frac{9}{2}\sinh(x) - \frac{1}{2}x\cosh(x) - \frac{1}{2}x\sinh(x) - x - 1 + \int_0^x e^{x-t}u^{'}(t)dt, u(0) = 1, u^{'}(0) = -1, u^{''}(0) = 1.$

Solution. The first one, applying Laplace transform to the problem, we have

$$L\Big\{u^{'''}(x)\Big\} + L\Big\{\int_0^x (x-t)^2u^{''}(t)dt\Big\} = L\Big\{u(x)\Big\}$$
$$+L\Big\{\frac{9}{2}\sinh(x) - \frac{1}{2}x\cosh(x) - \frac{1}{2}x\sinh(x) - x - 1\Big\} + L\Big\{\int_0^x e^{x-t}u^{'}(t)dt\Big\}.$$

The second one, using the convolution theorem, we get

$$L\Big\{u^{'''}(x)\Big\} + L\Big\{x^2\Big\}L\Big\{u^{''}(x)\Big\} = L\Big\{u(x)\Big\}$$
$$= L\Big\{\frac{9}{2}\sinh(x) - \frac{1}{2}x\cosh(x) - \frac{1}{2}x\sinh(x) - x - 1\Big\} + L\Big\{e^x\Big\}L\Big\{u^{'}(x)\Big\}.$$

The third one, taking the Laplace transform of derivatives, we have

$$
\begin{aligned}
&s^3L\Big\{u(x)\Big\} - s^2u(0) - su^{'}(0) - u^{''}(0) + \frac{2}{s^3}\Big[s^2L\Big\{u(x)\Big\} - su(0) - u^{'}(0)\Big]\\
&= L\Big\{u(x)\Big\} + \frac{9}{2(s^2-1)} - \frac{(s^2+1)}{2(s^2-1)^2} - \frac{s}{(s^2-1)^2} - \frac{1}{s^2} - \frac{1}{s}\\
&+\frac{1}{s-1}\Big[sL\Big\{u(x)\Big\} - u(0)\Big]
\end{aligned}
$$

and using initial conditions, we also obtain

$$
\begin{aligned}
&s^3L\Big\{u(x)\Big\} - s^2 + s - 1 + \frac{2}{s^3}\Big[s^2L\Big\{u(x)\Big\} - s + 1\Big]\\
&= L\Big\{u(x)\Big\} + \frac{9}{2(s^2-1)} - \frac{(s^2+1)}{2(s^2-1)^2} - \frac{s}{(s^2-1)^2} - \frac{1}{s^2} - \frac{1}{s}\\
&+\frac{1}{s-1}\Big[sL\Big\{u(x)\Big\} - 1\Big].
\end{aligned}
$$

The fourth one, rearranging the equation, we certainly receive

$$
\begin{aligned}
&\Big(s^3 + \frac{2}{s} - \frac{s}{s-1} - 1\Big)L\Big\{u(x)\Big\} = s^2 - s + 1 + \frac{2}{s^2} - \frac{2}{s^3}\\
&+\frac{9}{2(s^2-1)} - \frac{(s^2+1)}{2(s^2-1)^2} - \frac{s}{(s^2-1)^2} - \frac{1}{s^2} - \frac{1}{s} - \frac{1}{s-1}
\end{aligned}
$$

or

$$
\begin{aligned}
&\Big[\frac{s^5 - s^4 - 2s^2 + 3s - 2}{s(s-1)}\Big]L\Big\{u(x)\Big\}\\
&= \frac{2s^{10} - 4s^9 + 2s^7 + 6s^6 - 8s^5 - 8s^4 + 16s^3 - 4s^2 - 6s + 4}{2s^3(s-1)(s^2-1)^2}.
\end{aligned}
$$

Hence, we get

$$
L\Big\{u(x)\Big\} = \frac{2(s^2-1)(s^3-s^2+1)(s^5-s^4-2s^2+3s-2)}{2s^2(s^2-1)^2(s^5-s^4-2s^2+3s-2)} = \frac{s^3-s^2+1}{s^2(s^2-1)}.
$$

The last one, taking the inverse Laplace transform of the equation, we suddenly have an exact solution

$$
\begin{aligned}
u(x) &= L^{-1}\Big\{\frac{s^3-s^2+1}{s^2(s^2-1)}\Big\} = L^{-1}\Big\{\frac{s^3}{s^2(s^2-1)}\Big\} - L^{-1}\Big\{\frac{s^2-1}{s^2(s^2-1)}\Big\}\\
&= L^{-1}\Big\{\frac{s}{s^2-1}\Big\} - L^{-1}\Big\{\frac{1}{s^2}\Big\} = \cosh x - x.
\end{aligned}
$$

Here, we will emphasize on an approximate solution of this generalization for a non-convolution type given by

$$
\begin{aligned}
&\alpha u^{(n)}(x) + \beta\int_0^x K_1(x,t)u^{(m)}(t)dt\\
&= A(x)u(x) + B(x) + \int_0^x K_2(x,t)u^{(p)}(t)dt, \quad x \in [0,T],\\
&u(0) = c_0, u^{'}(0) = c_1, ..., u^{(n-1)}(0) = c_{n-1}. \qquad (8)
\end{aligned}
$$

The approximation using the Laguerre polynomials is below: To start with Supposing that the function $u_\chi(x)$ is an estimated solution of the equation (8) defined by

$$u_\chi(x) = \sum_{k=0}^{\chi} d_k L_k(x) = d_0 L_0(x) + d_1 L_1(x) + d_2 L_2(x) + \cdots + d_\chi L_\chi(x), \tag{9}$$

where $m, n, p \leq \chi$, $L_k(x)$ are the Laguerre polynomials and d_k are unknown constants, $k = 0, 1, ..., \chi$. Then, writing equation (9) as a dot product, we have

$$u_\chi(x) = \begin{bmatrix} L_0(x) & L_1(x) & L_2(x) & ... & L_\chi(x) \end{bmatrix} \cdot \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_\chi \end{bmatrix}. \tag{10}$$

Next, rearranging the equation (10) in a matrix formula, we also have

$$u_\chi(x) = \begin{bmatrix} 1 & x & x^2 & x^3 & ... & x^\chi \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\chi} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\chi} \\ 0 & 0 & b_{22} & \cdots & b_{2\chi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\chi\chi} \end{bmatrix} \cdot \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_\chi \end{bmatrix},$$

where b_{ij} are known constants. After that, finding the derivatives of $u_\chi(x)$, we have as follows:

$$u_\chi^{'}(x) = \begin{bmatrix} 0 & 1! & 2x & 3x^2 & ... & \chi x^{\chi-1} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\chi} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\chi} \\ 0 & 0 & b_{22} & \cdots & b_{2\chi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\chi\chi} \end{bmatrix} \cdot \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_\chi \end{bmatrix},$$

$$u_\chi^{''}(x) = \begin{bmatrix} 0 & 0 & 2! & 6x & ... & \chi(\chi-1)x^{\chi-2} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\chi} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\chi} \\ 0 & 0 & b_{22} & \cdots & b_{2\chi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\chi\chi} \end{bmatrix} \cdot \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_\chi \end{bmatrix},$$

$$u_\chi^{'''}(x) = \begin{bmatrix} 0 & 0 & 0 & 3! & ... & \chi(\chi-1)(\chi-2)x^{\chi-3} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\chi} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\chi} \\ 0 & 0 & b_{22} & \cdots & b_{2\chi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\chi\chi} \end{bmatrix} \cdot \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_\chi \end{bmatrix}$$

and

$$u_{\chi}^{(n)}(x) = \begin{bmatrix} 0 & 0 & 0 & ... & n! & ... & \chi(\chi-1)...(\chi-n+1)x^{\chi-n} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\chi} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\chi} \\ 0 & 0 & b_{22} & \cdots & b_{2\chi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\chi\chi} \end{bmatrix} . \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{\chi} \end{bmatrix}. \tag{11}$$

Then, substituting the equation (11) into the equation (8), we receive

$$\begin{aligned} &\alpha\begin{bmatrix} 0 & 0 & 0 & ... & n! & ... & \chi(\chi-1)...(\chi-n+1)x^{\chi-n} \end{bmatrix} \\ &\begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\chi} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\chi} \\ 0 & 0 & b_{22} & \cdots & b_{2\chi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\chi\chi} \end{bmatrix} . \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{\chi} \end{bmatrix} \\ &+\beta\int_0^x K_1(x,t)\Big\{\begin{bmatrix} 0 & 0 & 0 & ... & m! & ... & \chi(\chi-1)...(\chi-m+1)t^{\chi-m} \end{bmatrix} \\ &\begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\chi} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\chi} \\ 0 & 0 & b_{22} & \cdots & b_{2\chi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\chi\chi} \end{bmatrix} . \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{\chi} \end{bmatrix}\Big\}dt \\ &= A(x)\begin{bmatrix} 1 & x & x^2 & x^3 & ... & x^{\chi} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\chi} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\chi} \\ 0 & 0 & b_{22} & \cdots & b_{2\chi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\chi\chi} \end{bmatrix} . \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{\chi} \end{bmatrix} \\ &+B(x) + \int_0^x K_2(x,t)\Big\{\begin{bmatrix} 0 & 0 & 0 & ... & p! & ... & \chi(\chi-1)...(\chi-p+1)t^{\chi-p} \end{bmatrix} \\ &\begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0\chi} \\ 0 & b_{11} & b_{12} & \cdots & b_{1\chi} \\ 0 & 0 & b_{22} & \cdots & b_{2\chi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{\chi\chi} \end{bmatrix} . \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{\chi} \end{bmatrix}\Big\}dt. \end{aligned} \tag{12}$$

Simplifying and integrating the equation (12), we then have the new equation with unknown constants $d_0, d_1, ..., d_{\chi}$. In order to determine $d_0, d_1, ..., d_{\chi}$, using n initial conditions and selecting $x_i \in [0, T], i = 1, 2, ..., \chi-n+1$, with substituting in the new equation, we get a system of linear algrbraic equations of $\chi+1$ unknown constants. Solving this system by a program, we have the values of the unknown constants, that is, the numerical solution of the initial-value problem (8) is obtained.

In order to guarantee the convergence of this method, we will verify as follows. Let $u(x)$ be an analytical solution of initial-value problem (8) that has derivatives of all orders at $x=0$. Then, the Taylor series of $u(x)$ at $x=0$ is defined by

$$u(x) = u(0) + u^{'}(0)x + \frac{1}{2!}u^{''}(0)x^2 + \cdots + \frac{1}{\chi!}u^{(\chi)}(0)x^{\chi} + \cdots .$$

Thus, by the definition and process to find $u_\chi(x)$, we obtain that

$$|u(x) - u_\chi(x)| \leq |\frac{1}{(\chi+1)!}u^{(\chi+1)}(0)x^{\chi+1}| + |\frac{1}{(\chi+2)!}u^{(\chi+2)}(0)x^{\chi+2}| + \cdots .$$

Here, it is sufficient to show that $\frac{1}{\chi!}x^\chi$ converges to 0 as $\chi \to \infty$ to confirm that $|u(x) - u_\chi(x)|$ converges to 0 as $\chi \to \infty$. Since $e\big(\frac{\chi}{e}\big)^\chi \leq \chi! \leq e\big(\frac{\chi+1}{e}\big)^{\chi+1}$, we get $\frac{1}{\chi+1}\big(\frac{ex}{\chi+1}\big)^\chi \leq \frac{1}{\chi!}x^\chi \leq \frac{1}{e}\big(\frac{ex}{\chi}\big)^\chi$. It is easy to determine that $\frac{1}{\chi+1}\big(\frac{ex}{\chi+1}\big)^\chi$ and $\frac{1}{e}\big(\frac{ex}{\chi}\big)^\chi$ converge to 0 as as $\chi \to \infty$. This means that $\frac{1}{\chi!}x^\chi$ converges to 0 as $\chi \to \infty$ to confirm the convergence.

Example 2.3. *Estimate a solution of the linear Volterra integro-differential problem using* $u_6(x)$*:* $u^{(5)}(x) + \int_0^x xtu^{'''}(t)dt = xu(x) - \frac{3}{4}x^5 - x^4 + \frac{9}{2}x^3 + 3x^2 - 2xe^x + 2e^x + \int_0^x xtu^{'}(t)dt, u(0) = 2, u^{'}(0) = -1, u^{''}(0) = 2, u^{'''}(0) = 8, u^{(4)}(0) = 2, 0 \leq x \leq 2$. *An exact solution is* $u(x) = x^3 - 3x + 2e^x$.

Solution. First, suppose that a function $u_6(x)$ is an approximate solution of this problem, that is,

$$\begin{aligned}
&u_6(x) = d_0L_0(x) + d_1L_1(x) + d_2L_2(x) + \cdots + d_6L_6(x)\\
&= d_0(1) + d_1(-x+1) + d_2\big[\frac{1}{2}(x^2 - 4x + 2)\big] + d_3\big[\frac{1}{6}(-x^3 + 9x^2 - 18x + 6)\big]\\
&+ d_4\big[\frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)\big]\\
&+ d_5\big[\frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)\big]\\
&+ d_6\big[\frac{1}{720}(x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720)\big].
\end{aligned}$$

Second, finding derivatives of $u_6(x)$, we have as follows:

$$\begin{aligned}
&u_6^{'}(x) = d_1(-1) + d_2\big[\frac{1}{2}(2x - 4)\big] + d_3\big[\frac{1}{6}(-3x^2 + 18x - 18)\big]\\
&+ d_4\big[\frac{1}{24}(4x^3 - 48x^2 + 144x - 96)\big]\\
&+ d_5\big[\frac{1}{120}(-5x^4 + 100x^3 - 600x^2 + 1200x - 600)\big]\\
&+ d_6\big[\frac{1}{720}(6x^5 - 180x^4 + 1800x^3 - 7200x^2 + 10800x - 4320)\big],
\end{aligned}$$

$$u_6^{''}(x) = d_2\big[\frac{1}{2}(2)\big] + d_3\big[\frac{1}{6}(-6x+18)\big] + d_4\big[\frac{1}{24}(12x^2-96x+144)\big]$$
$$+d_5\big[\frac{1}{120}(-20x^3+300x^2-1200x+1200)\big]$$
$$+d_6\big[\frac{1}{720}(30x^4-720x^3+5400x^2-14400x+10800)\big],$$

$$u_6^{'''}(x) = d_3\big[\frac{1}{6}(-6)\big] + d_4\big[\frac{1}{24}(24x-96)\big]$$
$$+d_5\big[\frac{1}{120}(-60x^2+600x-1200)\big]$$
$$+d_6\big[\frac{1}{720}(120x^3-2160x^2+10800x-14400)\big],$$

$$u_6^{(4)}(x) = d_4\big[\frac{1}{24}(24)\big] + d_5\big[\frac{1}{120}(-120x+600)\big]$$
$$+d_6\big[\frac{1}{720}(360x^2-4320x+10800)\big]$$

and

$$u_6^{(5)}(x) = d_5\big[\frac{1}{120}(-120)\big] + d_6\big[\frac{1}{720}(720x-4320)\big].$$

Third, substituting the derivatives into the problem, we receive

$$d_5\big[\frac{1}{120}(-120)\big] + d_6\big[\frac{1}{720}(720x-4320)\big]$$
$$+\int_0^x xt\Big\{d_3\big[\frac{1}{6}(-6)\big] + d_4\big[\frac{1}{24}(24t-96)\big] + d_5\big[\frac{1}{120}(-60t^2+600t-1200)\big]$$
$$+d_6\big[\frac{1}{720}(120t^3-2160t^2+10800t-14400)\big]\Big\}dt$$
$$= x\Big\{d_0(1) + d_1(-x+1) + d_2\big[\frac{1}{2}(x^2-4x+2)\big]$$
$$+d_3\big[\frac{1}{6}(-x^3+9x^2-18x+6)\big] + d_4\big[\frac{1}{24}(x^4-16x^3+72x^2-96x+24)\big]$$
$$+d_5\big[\frac{1}{120}(-x^5+25x^4-200x^3+600x^2-600x+120)\big]$$
$$+d_6\big[\frac{1}{720}(x^6-36x^5+450x^4-2400x^3+5400x^2-4320x+720)\big]\Big\}$$
$$-\frac{3}{4}x^5 - x^4 + \frac{9}{2}x^3 + 3x^2 - 2xe^x + 2e^x$$

$$+\int_0^x xt\Big\{d_1(-1)+d_2\big[\frac{1}{2}(2t-4)\big]+d_3\big[\frac{1}{6}(-3t^2+18t-18)\big]$$
$$+d_4\big[\frac{1}{24}(4t^3-48t^2+144t-96)\big]$$
$$+d_5\big[\frac{1}{120}(-5t^4+100t^3-600t^2+1200t-600)\big]$$
$$+d_6\big[\frac{1}{720}(6t^5-180t^4+1800t^3-7200t^2+10800t-4320)\big]\Big\}dt.$$

Fourth, simplifying and integrating the equation, we obtain the new equation. Selecting $x_1=0.25$ and $x_2=0.5$ to substitute in the new equation with using 5 initial conditions, we get the following system:

$$\begin{aligned}
&d_0+d_1+d_2+d_3+d_4+d_5+d_6=2,\\
&-d_1-2d_2-3d_3-4d_4-5d_5-6d_6=-1,\\
&d_2+3d_3+6d_4+10d_5+15d_6=2,\\
&-d_3-4d_4-10d_5-20d_6=8,\\
&d_4+5d_5+15d_6=2,\\
&-0.25d_0-0.179687d_1-0.118489d_2-0.073445d_3-0.050341d_4\\
&-1.053831d_5-5.837542d_6=2.179211,\\
&-0.5d_0-0.1875d_1+0.041666d_2+0.139322d_3+0.076302d_4\\
&-1.161273d_5-6.075211d_6=2.875283.
\end{aligned}$$

At last, solving the system by a program, we have

$$\begin{aligned}
&d_0=17.757015, d_1=-61.703708, d_2=100.163313, d_3=-92.756475,\\
&d_4=54.971399, d_5=-19.350177, d_6=2.918632.
\end{aligned}$$

Therefore, the numerical solution is

$$u_6(x)=17.757015-61.703708(-x+1)+100.163313\big[\frac{1}{2}(x^2-4x+2)\big]$$
$$-92.756475\big[\frac{1}{6}(-x^3+9x^2-18x+6)\big]$$
$$+54.971399\big[\frac{1}{24}(x^4-16x^3+72x^2-96x+24)\big]$$
$$-19.350177\big[\frac{1}{120}(-x^5+25x^4-200x^3+600x^2-600x+120)\big]$$
$$+2.918632\big[\frac{1}{720}(x^6-36x^5+450x^4-2400x^3+5400x^2-4320x+720)\big].$$

TABLE 1. Values of exact and approximate $u_6(x)$ solutions for example 2.3.

x	Exact solution	Approx. solution	Absolute error
0.00	2.000000	1.999999	0.000001
0.20	1.850806	1.850805	0.000001
0.40	1.847649	1.847640	0.000009
0.60	2.060238	2.060181	0.000057
0.80	2.563082	2.562883	0.000199
1.00	3.436564	3.436040	0.000523
1.20	4.768234	4.767025	0.001209
1.40	6.654400	6.651716	0.002684
1.60	9.202065	9.196116	0.005949
1.80	12.531295	12.518153	0.013142
2.00	16.778112	16.749669	0.028443

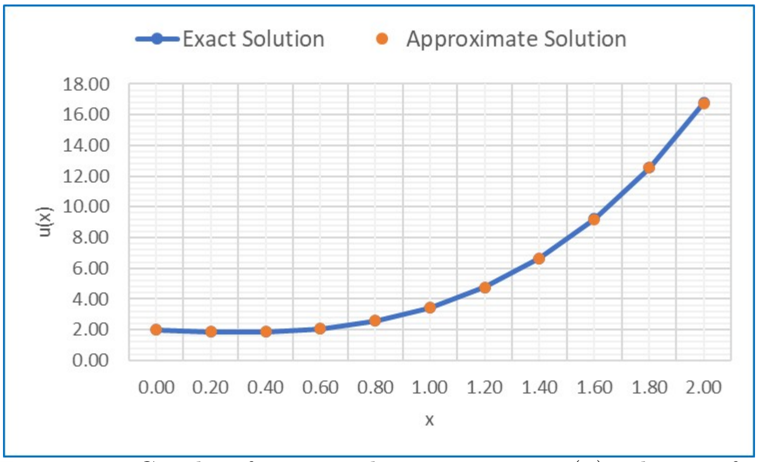

FIGURE 1. Graphs of exact and approximate $u_6(x)$ solutions for example 2.3.

Example 2.4. *Approximate a solution of the linear Volterra integro-differential problem using* $u_4(x)$ *and* $u_5(x)$*:* $u^{'''}(x) + \int_0^x x\sin(t)u^{'}(t)dt = -2x^2u(x) - \cos(x) - 4x\cos(x) - x\sin(x) + 4x^3 + 2x^2 + 4x + \int_0^x x\cos(t)u(t)dt, u(0) = 1, u^{'}(0) = 3, u^{''}(0) = 0, 0 \leq x \leq \pi$. *An exact solution is* $u(x) = \sin(x) + 2x + 1$.

Solution. Firstly, let $u_4(x)$ is an approximate solution of this problem, that is,

$$u_4(x) = d_0L_0(x) + d_1L_1(x) + d_2L_2(x) + d_3L_3(x) + d_4L_4(x)$$
$$= d_0(1) + d_1(-x+1) + d_2\big[\frac{1}{2}(x^2 - 4x + 2)\big] + d_3\big[\frac{1}{6}(-x^3 + 9x^2 - 18x + 6)\big]$$
$$+d_4\big[\frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)\big].$$

Secondly, finding derivatives of $u_4(x)$, we have as follows:

$$u_4^{'}(x) = d_1(-1) + d_2\big[\frac{1}{2}(2x-4)\big] + d_3\big[\frac{1}{6}(-3x^2 + 18x - 18)\big]$$
$$+d_4\big[\frac{1}{24}(4x^3 - 48x^2 + 144x - 96)\big],$$

$$u_4^{''}(x) = d_2\big[\frac{1}{2}(2)\big] + d_3\big[\frac{1}{6}(-6x+18)\big] + d_4\big[\frac{1}{24}(12x^2 - 96x + 144)\big]$$

and

$$u_4^{'''}(x) = d_3\big[\frac{1}{6}(-6)\big] + d_4\big[\frac{1}{24}(24x - 96)\big].$$

Thirdly, substituting the derivatives into the problem, we receive

$$d_3\big[\frac{1}{6}(-6)\big] + d_4\big[\frac{1}{24}(24x-96)\big]$$
$$+\int_0^x x\sin(t)\Big\{d_1(-1) + d_2\big[\frac{1}{2}(2t-4)\big] + d_3\big[\frac{1}{6}(-3t^2 + 18t - 18)\big]$$
$$+d_4\big[\frac{1}{24}(4t^3 - 48t^2 + 144t - 96)\big]\Big\}dt$$
$$= -2x^2\Big\{d_0(1) + d_1(-x+1) + d_2\big[\frac{1}{2}(x^2 - 4x + 2)\big]$$
$$+d_3\big[\frac{1}{6}(-x^3 + 9x^2 - 18x + 6)\big] + d_4\big[\frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)\big]\Big\}$$
$$-\cos(x) - 4x\cos(x) - x\sin(x) + 4x^3 + 2x^2 + 4x$$
$$+\int_0^x x\cos(t)\Big\{d_0(1) + d_1(-t+1) + d_2\big[\frac{1}{2}(t^2 - 4t + 2)\big]$$
$$+d_3\big[\frac{1}{6}(-t^3 + 9t^2 - 18t + 6)\big] + d_4\big[\frac{1}{24}(t^4 - 16t^3 + 72t^2 - 96t + 24)\big]\Big\}dt.$$

Fourthly, simplifying and integrating the equation, we obtain the new equation. Selecting $x_1 = 0.25$ and $x_2 = 0.5$ to substitute in the new equation with using 3 initial conditions, we get the following system:

$$
\begin{aligned}
&d_0 + d_1 + d_2 + d_3 + d_4 = 1,\\
&-d_1 - 2d_2 - 3d_3 - 4d_4 = 3,\\
&d_2 + 3d_3 + 6d_4 = 0,\\
&0.063149d_0 + 0.031817d_1 + 0.005048d_2 - 1.017567d_3\\
&-3.786408d_4 = -0.812175,\\
&0.260287d_0 + 0.007726d_1 - 0.171664d_2 - 1.290905d_3\\
&-3.861327d_4 = 0.127539.
\end{aligned}
$$

Finally, solving the system by a program, we get

$$d_0 = 3.306071, d_1 = -1.284838, d_2 = -0.981913, d_3 = -0.405944, d_4 = 0.366624.$$

Therefore, the numerical solution is

$$
\begin{aligned}
&u_4(x) = 3.306071 - 1.284838(-x+1) - 0.981913\big[\frac{1}{2}(x^2 - 4x + 2)\big]\\
&-0.405944\big[\frac{1}{6}(-x^3 + 9x^2 - 18x + 6)\big]\\
&+0.366624\big[\frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)\big].
\end{aligned}
$$

TABLE 2. Values of exact and approximate $u_4(x)$ solutions for example 2.4.

x	Exact solution	Approx. solution	Absolute error
0.00	1.000000	1.000000	0.000000
0.31	1.937336	1.937146	0.000190
0.63	2.844422	2.843491	0.000931
0.94	3.693973	3.691509	0.002464
1.26	4.464331	4.457244	0.007087
1.57	5.141593	5.120311	0.021282
1.88	5.720968	5.663897	0.057071
2.20	6.207247	6.074761	0.132486
2.51	6.614333	6.343231	0.271102
2.83	6.963884	6.463209	0.500675
3.14	7.283185	6.432166	0.851019

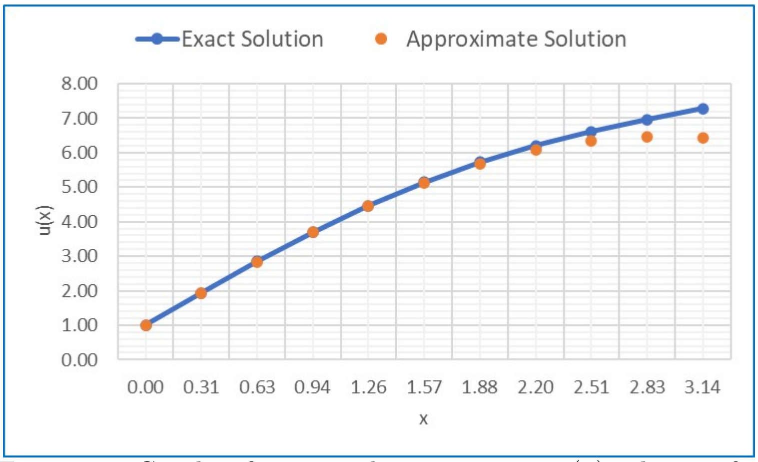

FIGURE 2. Graphs of exact and approximate $u_4(x)$ solutions for example 2.4.

For the approximate solution $u_5(x)$, the first one, let $u_5(x)$ be an approximate solution of this problem, that is,

$$u_5(x) = d_0L_0(x) + d_1L_1(x) + d_2L_2(x) + d_3L_3(x) + d_4L_4(x) + d_5L_5(x)$$
$$= d_0(1) + d_1(-x+1) + d_2\big[\frac{1}{2}(x^2-4x+2)\big] + d_3\big[\frac{1}{6}(-x^3+9x^2-18x+6)\big]$$
$$+d_4\big[\frac{1}{24}(x^4-16x^3+72x^2-96x+24)\big]$$
$$+d_5\big[\frac{1}{120}(-x^5+25x^4-200x^3+600x^2-600x+120)\big].$$

The second one, finding derivatives of $u_5(x)$, we have as follows:

$$u_5^{'}(x) = d_1(-1) + d_2\big[\frac{1}{2}(2x-4)\big] + d_3\big[\frac{1}{6}(-3x^2+18x-18)\big]$$
$$+d_4\big[\frac{1}{24}(4x^3-48x^2+144x-96)\big]$$
$$+d_5\big[\frac{1}{120}(-5x^4+100x^3-600x^2+1200x-600)\big],$$

$$u_5^{''}(x) = d_2\big[\frac{1}{2}(2)\big] + d_3\big[\frac{1}{6}(-6x+18)\big] + d_4\big[\frac{1}{24}(12x^2-96x+144)\big]$$
$$+d_5\big[\frac{1}{120}(-20x^3+300x^2-1200x+1200)\big]$$

and

$$u_5^{'''}(x) = d_3\big[\frac{1}{6}(-6)\big] + d_4\big[\frac{1}{24}(24x-96)\big] + d_5\big[\frac{1}{120}(-60x^2+600x-1200)\big].$$

The third one, substituting the derivatives into the problem, we receive

$$d_3\big[\frac{1}{6}(-6)\big] + d_4\big[\frac{1}{24}(24x-96)\big] + d_5\big[\frac{1}{120}(-60x^2+600x-1200)\big]$$
$$+\int_0^x x\sin(t)\Big\{d_1(-1) + d_2\big[\frac{1}{2}(2t-4)\big] + d_3\big[\frac{1}{6}(-3t^2+18t-18)\big]$$
$$+d_4\big[\frac{1}{24}(4t^3-48t^2+144t-96)\big]$$
$$+d_5\big[\frac{1}{120}(-5t^4+100t^3-600t^2+1200t-600)\big]\Big\}dt$$
$$= -2x^2\Big\{d_0(1) + d_1(-x+1) + d_2\big[\frac{1}{2}(x^2-4x+2)\big]$$
$$+d_3\big[\frac{1}{6}(-x^3+9x^2-18x+6)\big] + d_4\big[\frac{1}{24}(x^4-16x^3+72x^2-96x+24)\big]$$
$$+d_5\big[\frac{1}{120}(-x^5+25x^4-200x^3+600x^2-600x+120)\big]\Big\}$$
$$-\cos(x) - 4x\cos(x) - x\sin(x) + 4x^3 + 2x^2 + 4x$$
$$+\int_0^x x\cos(t)\Big\{d_0(1) + d_1(-t+1) + d_2\big[\frac{1}{2}(t^2-4t+2)\big]$$
$$+d_3\big[\frac{1}{6}(-t^3+9t^2-18t+6)\big] + d_4\big[\frac{1}{24}(t^4-16t^3+72t^2-96t+24)\big]$$
$$+d_5\big[\frac{1}{120}(-t^5+25t^4-200t^3+600t^2-600t+120)\big]\Big\}dt.$$

The fourth one, simplifying and integrating the equation, we obtain the new equation. Selecting $x_1 = 0.25, x_2 = 0.5$ and $x_3 = 0.75$ to substitute in the new equation with using 3 initial conditions, we get the following system:

$$d_0 + d_1 + d_2 + d_3 + d_4 + d_5 = 1,$$
$$-d_1 - 2d_2 - 3d_3 - 4d_4 - 5d_5 = 3,$$
$$d_2 + 3d_3 + 6d_4 + 10d_5 = 0,$$
$$0.063149d_0 + 0.031817d_1 + 0.005048d_2 - 1.017567d_3$$
$$-3.786408d_4 - 8.833081d_5 = -0.812175,$$
$$0.260287d_0 + 0.007726d_1 - 0.171664d_2 - 1.290905d_3$$
$$-3.861327d_4 - 8.017746d_5 = 0.127539,$$
$$0.613771d_0 - 0.249023d_1 - 0.739887d_2 - 1.957671d_3$$
$$-4.232047d_4 - 7.407697d_5 = 2.374515.$$

The last one, solving the system by a program, we get

$$d_0 = 3.906579, d_1 = -4.506704, d_2 = 5.955424, d_3 = -7.891840,$$
$$d_4 = 4.411328, d_5 = -0.874787.$$

Therefore, the numerical solution is

$$u_5(x) = 3.906579 - 4.506704(-x+1) + 5.955424\Big[\frac{1}{2}(x^2 - 4x + 2)\Big]$$
$$-7.891840\Big[\frac{1}{6}(-x^3 + 9x^2 - 18x + 6)\Big]$$
$$+4.411328\Big[\frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)\Big]$$
$$-0.874787\Big[\frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)\Big].$$

TABLE 3. Values of exact and approximate $u_5(x)$ solutions for the example 2.4.

x	Exact solution	Approx. solution	Absolute error
0.00	1.000000	1.000000	0.000000
0.31	1.937336	1.937318	0.000017
0.63	2.844422	2.844339	0.000084
0.94	3.693973	3.693774	0.000199
1.26	4.464331	4.464054	0.000276
1.57	5.141593	5.142007	0.000414
1.88	5.720968	5.725529	0.004562
2.20	6.207247	6.226270	0.019024
2.51	6.614333	6.672304	0.057971
2.83	6.963884	7.110808	0.146924
3.14	7.283185	7.610740	0.327555

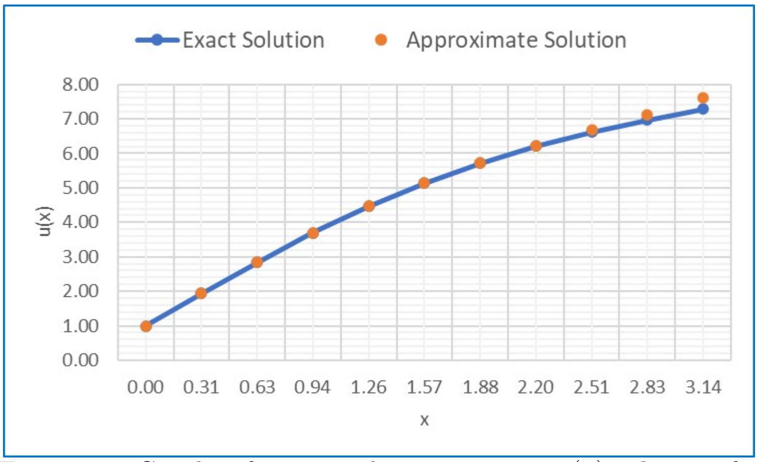

FIGURE 3. Graphs of exact and approximate $u_5(x)$ solutions for example 2.4.

To demonstrate the practical applicability of the proposed method, we consider a classical population model with memory effects, where the current growth rate depends not only on the present population but also on its past values. Let $u(t)$ denote the population size at time $t \geq 0$. The population growth with memory is defined by

$$u'(t) = ru(t) + \int_0^t K(t-s)u(s)ds, u(0) = u_0,$$

where $r \in \mathbb{R}$ is the intrinsic growth rate and K is a memory kernel. For the growth model of *Drosophila*, we get $r = 0.25, u_0 = 50$ and the exponentially decaying kernel $K(\tau) = 0.15e^{-0.1\tau}$ and see more in [11]. Thus, the growth model is in the form

$$u'(t) = 0.25u(t) + \int_0^t 0.15e^{-0.1(t-s)}u(s)ds, u(0) = 50.$$

This kernel is smooth, exponentially bounded and therefore satisfies the Laplace-transform conditions and other regularity assumptions used in our analysis. By Laplace transform, the exact solution of the problem is $u(t) = \dfrac{600}{17}e^{0.5t} + \dfrac{250}{17}e^{-0.35t}$. Then, applying the proposed Laguerre method with $u_4(t)$ and $u_6(t)$ on $t \in [0, 2]$, we obatian the results as follows.

$$\begin{aligned}
&u_4(t) = 80.06930788 - 58.02243169(-t+1) + 41.82881774\big[\frac{1}{2}(t^2 - 4t + 2)\big] \\
&-17.37952146\big[\frac{1}{6}(-t^3 + 9t^2 - 18t + 6)\big] \\
&+3.50382753\big[\frac{1}{24}(t^4 - 16t^3 + 72t^2 - 96t + 24)\big]
\end{aligned}$$

and

$$\begin{aligned}
&u_6(t) = 80.81383638 - 62.43684455(-t+1) + 52.39097279\big[\frac{1}{2}(t^2 - 4t + 2)\big] \\
&-30.15647576\big[\frac{1}{6}(-t^3 + 9t^2 - 18t + 6)\big] \\
&+11.40683809\big[\frac{1}{24}(t^4 - 16t^3 + 72t^2 - 96t + 24)\big] \\
&-2.10565106\big[\frac{1}{120}(-t^5 + 25t^4 - 200t^3 + 600t^2 - 600t + 120)\big] \\
&+0.08732412\big[\frac{1}{720}(t^6 - 36t^5 + 450t^4 - 2400t^3 + 5400t^2 - 4320t + 720)\big].
\end{aligned}$$

TABLE 4. Values of exact and approximate $u_4(t)$ solutions with absolute errors.

t	Exact solution	Approx. solution	Absolute error
0.00	50.000000	50.000000	0.000000
0.25	53.467282	53.466132	0.001150
0.50	57.663500	57.662390	0.001110
0.75	62.663320	62.661870	0.001450
1.00	68.553223	68.551354	0.001868
1.25	75.432924	75.431314	0.001610
1.50	83.416991	83.415904	0.001087
1.75	92.636690	92.632969	0.003721
2.00	103.242084	103.224038	0.018046

TABLE 5. Values of exact and approximate $u_6(t)$ solutions with absolute errors.

t	Exact solution	Approx. solution	Absolute error
0.00	50.000000	50.000000	0.000000
0.25	53.467282	53.467436	0.000154
0.50	57.663500	57.663728	0.000228
0.75	62.663320	62.663634	0.000314
1.00	68.553223	68.553593	0.000370
1.25	75.432924	75.433329	0.000405
1.50	83.416991	83.417464	0.000473
1.75	92.636690	92.637166	0.000476
2.00	103.242084	103.241802	0.000287

3. CONCLUDING REMARKS

In this paper, a generalization of linear VIDEs has been introduced already. In general, all results show that the Laplace transform has been effective to solve analytical solutions of the generalization on convolution type kernels repeatedly and the Laguerre polynomials have been successful to figure out numerical solutions of the generalization on non-convolution type kernels several times. However, the Kushare transform, Sadik transform and Kamal transform are other methods that can be analytically solved on convolution types of this generalization similarly. Moreover, the main advantage of this analytical method is the fact that it gives the exact solutions in just few processes and uses very less computational work. We also suggest that this numerical method can be applicable to singularly perturbed linear VIDEs, which are one case of this generalization, to obtain accurately approximate solutions.

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