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Reverse Homoderivations on (Semi)-prime Rings

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Abstract. In this paper, we explore and examine a new class of maps known as reverse homoderivations. A reverse homoderivation refers to an additive map g defined on a ring T that satisfies the condition, $g(\vartheta\ell) = g(\ell)g(\vartheta) + g(\ell)\vartheta + \ell g(\vartheta)$ for all $\vartheta, \ell \in T$. We present various results that enhance our understanding of reverse homoderivations, including their existence in (semi)-prime rings and the behavior of rings when they satisfy certain functional identities. Some examples are provided to demonstrate the necessity of the constraints, while additional examples are given to clarify the concept of reverse homoderivations.

Key words and Phrases: prime ring, semiprime ring, homoderivation, reverse homoderivation.

1. INTRODUCTION

Throughout this paper, T represents a ring and $Z(T)$ represents the center of T. The investigation of rings with maps started in the latter half of the previous century. Researchers concentrated on some classical types of maps defined on rings, such as, homomorphisms and derivations, where homomorphism f on a ring T is defined as an additive map satisfying $f(\vartheta\ell) = f(\vartheta)f(\ell)$ for all $\vartheta, \ell \in T$ and by a derivation we mean an additive map β such that $\beta(\vartheta\ell) = \beta(\vartheta)\ell + \vartheta\beta(\ell)$ for all $\vartheta, \ell \in T$. A quick recap of some important notations is now essential. In case of a prime ring, whenever $\vartheta T\ell = \{0\}$, we either have $\vartheta = 0$ or $\ell = 0$. Similarly, a semiprime ring has $\vartheta = 0$ whenever, $\vartheta T\vartheta = \{0\}$. On a subset A of T, we say

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that a map α is centralizing on A, if the commutator of $\alpha(\vartheta)$ and ϑ is in $Z(T)$ for all $\vartheta \in A$, while as in case of a commuting map on A we have $[\alpha(\vartheta), \vartheta] = 0$ for all $\vartheta \in T$. An additive map α defined on a ring T is termed as a derivation if $\alpha(\vartheta\ell) = \alpha(\vartheta)\ell + \vartheta\alpha(\ell)$ for all $\vartheta, \ell \in T$.

Numerous studies examined the characteristics of prime and semiprime rings with derivations, including the commutativity of rings. This line of inquiry motivated many scholars to extend this concept in various directions, exploring the commutativity and other properties of rings. Concurrently, the properties of rings with homomorphisms were also examined, focusing on similar characteristics as in the former case. An additive map g defined on a ring T forms a homoderivation if $g(\vartheta\ell) = g(\vartheta)g(\ell) + g(\vartheta)\ell + \vartheta g(\ell)$ for all $\vartheta, \ell \in T$. Following [1], an additive map $g : T \to T$ such that $g(\vartheta^2) = g(\vartheta)g(\vartheta) + g(\vartheta)\vartheta + \vartheta g(\vartheta)$ for all $\vartheta \in T$ is known as a Jordan homoderivation.

With the advent of the concept of homoderivations given by El-Sofy[**?**], a new idea of amalgamation of derivations and that of homomorphisms came into being. Observe that from an endomorphism γ on T we can easily construct homoderivations as, $g(\vartheta) = \gamma(\vartheta) - \vartheta$ for all $\vartheta \in T$. With time the idea of homoderivation has been of immense interest to several authors. Homoderivations were defined to create a more versatile and generalized mapping for investigating algebraic structures, particularly in settings where classical derivations are insufficient or too restrictive. This has led to significant advancements in the study of non-commutative rings, semiprime rings, and related algebraic topics. In 2016, Melaibari et al. [2] demonstrated that a prime ring T with a non-zero homoderivation g satisfying certain identities, such as, (i) $g([\vartheta, \ell]) = 0 (\in Z(T))$, (ii) $[g(\vartheta), g(\ell)] = [\vartheta, \ell]$ for all $\vartheta, \ell \in T$ is commutative.

Alharfie et al. [3] explored homoderivations with involution, they established the commutativity of a prime $*$-ring with various functional identities on $*$ ideals. Recently, Rehman et al. [4] worked with the idea of generalized homoderivations and provided the form of such homoderivations. Precisely, they proved that for a prime ring T, a generalized homoderivation ϕ associated with a homoderivation g satisfying any one of the following conditions: (i) $\phi(\vartheta)\phi(\ell) - \vartheta\ell \in Z(T)$, (ii) $\phi(\vartheta)\phi(\ell) + \vartheta\ell \in Z(T)$, for all $\vartheta, \ell \in T$ happens to make the ring T commutative. In addition to this they were able to show their form, i.e., $\phi(\vartheta) = \pm\vartheta$ and $g(\vartheta) = -\vartheta$. An increasing amount of research has been diverted to homoderivations in rings.

In 2023, Belkadi et al. [1] successfully characterized n-Jordan homoderivations by decomposing an n-Jordan homoderivation in terms of a homoderivation and an anti-homomorphism. They showed that, for a unital ring T with identity e, and an n-Jordan homoderivation $g : T \to T$ satisfying $g(e) = 0$, several cases arise. Whenever T is commutative and δ-torsion free, then g is a homoderivation. If T is prime, δ-torsion free, and $g + Id_T$ is an onto map that is not an anti-homomorphism, then g is also a homoderivation. Furthermore, if T is semiprime, δ-torsion free, and $g + Id_T$ is an onto map, then there exists an essential ideal U of T such that the restriction of g to U can be written as a direct sum $g_1 \oplus g_2$, where g_1 is a homoderivation of U into T, and g_2 is an anti-homomorphism of U into T.

In 1957, Herstein [5] presented the idea of reverse derivation, defining it as an additive map α on T that satisfies the condition, $\alpha(\vartheta\ell) = \alpha(\ell)\vartheta + \ell\alpha(\vartheta)$ for all $\vartheta, \ell \in T$. Reverse derivations in the case of prime Lie and prime Malcev algebras were studied by Hopkins[6] and Filippov ([7], [8]). Those papers provided some examples of nonzero reverse derivations for the simple 3-dimensional Lie algebra sl_2 and characterized the prime Lie algebras admitting a nonzero reverse derivation. In particular, Filippov proved that each prime Lie algebra, admitting nonzero reverse derivation is a PI-algebra. Filippov also described all reverse derivations of prime Malcev algebras [9]. Thus, reverse derivations are used to explore specific algebraic properties and functional identities within rings, particularly in the context of non-commutative rings, where the ordering of multiplication matters. These mappings are often studied in relation to functional identities, central ideals, and commutativity conditions in ring theory. Herstein demonstrated that reverse derivations generally do not exist for prime rings.

Brešar and Vukman [10] studied reverse derivations in rings with involution, while Barros et al. [11] demonstrated the additivity of multiplicative *-reverse derivations in alternative algebras and decomposed Jordan *-reverse derivations into a *-reverse derivation and a singular Jordan *-reverse derivation. In 2015, Aboubakr and Gonzalez [12] examined the connection between generalized reverse derivations and generalized derivations on ideals in semiprime rings. More recently, Sogutcu [13] explored multiplicative (generalized) reverse derivations in semiprime rings, presenting significant findings, and addressing continuous reverse derivations in the context of Banach algebras. Motivated by these findings we venture in this direction and present a new concept which is coined as reverse homoderivation. As the name suggests a reverse homoderivation is actually a homoderivation that acts in reverse fashion, integrating the two maps into a single new map known as reverse homoderivation.

2. MAIN RESULTS

In this section, we present several results that are relevant to our study of reverse homoderivations. These results not only establish fundamental properties of reverse homoderivations but also explore their implications in the context of prime and semiprime rings. We will begin by proving the existence of reverse homoderivations in certain types of rings and then proceed to examine their behavior under various functional identities. Formally, we define reverse homoderivation as following.

Definition 2.1. *On a ring T, an additive map g is defined as a reverse homoderivation if it satisfies the following condition for all $\vartheta, \ell \in T$*

$$g(\vartheta\ell) = g(\ell)g(\vartheta) + g(\ell)\vartheta + \ell g(\vartheta).$$

Clearly, every reverse homoderivation forms a Jordan homoderivation but the converse is not true in general. In order to absorb the concept of reverse homoderivation, we provide two examples.

Example 2.2. *Let $\mathbb{R}$ be a ring of real numbers and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(\vartheta) = -\vartheta$ for all $\vartheta \in \mathbb{R}$. Clearly, g so defined forms a reverse homoderivation.*

Example 2.3. *Let* $T = \left\{ \begin{pmatrix} 0 & \vartheta & \ell & \zeta \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b, \vartheta, \ell, \zeta \in \mathbb{R} \right\}$. *Define a map g on T as* $g \begin{pmatrix} 0 & \vartheta & \ell & \zeta \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \ell - a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *It is easy to see that g forms a reverse homoderivation on T.*

Example 2.4. *Let g be a map defined on* $T = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$ *by*

$$g\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

One can easily check that g forms a homoderivation, but g is not a reverse homoderivation.

The following results will prove fruitful while proving our claims.

Lemma 2.5. [14, Lemma 1.1.8] *Let T be a semiprime ring, and suppose that $a \in T$ centralizes all commutators $\vartheta\ell - \ell\vartheta$, $\vartheta, \ell \in T$ then $a \in Z(T)$.*

Lemma 2.6. [15, Corollary 4.16] *For any ring T, T/P is a prime ring if and only if P is a prime ideal of T. Moreover, if T is a semiprime ring, then*

$$\cap\{P : P \ \text{ is a prime ideal}\} = \{0\}.$$

We know that maps like derivations and higher derivations, preserve the center of rings on which these notions are defined. In the case of homoderivations, a map preserves the center only if it is zero power valued. We say that a map f is zero power valued on a subset A of T, if $f(t) \in A$ for all $t \in A$ and $(f(t))^{n(t)} = 0$ for all $t \in A$, where $n(t)$ is a function of t. We demonstrate that a zero power valued reverse homoderivation also preserves the center.

Proposition 2.7. *Let g be a zero power valued reverse homoderivation defined on a ring T. Then, for every $\zeta \in Z(T)$, $g(\zeta) \in Z(T)$, i.e., g preserves the center of T.*

Proof. For any $t \in T$ and $\zeta \in Z(T)$, we have

$$g(t\zeta) = g(\zeta)g(t) + g(\zeta)t + \zeta g(t).$$

Also,

$$g(\zeta t) = g(t)g(\zeta) + g(t)\zeta + tg(\zeta).$$

On comparing the above equations, we get

$$g(\zeta)g(t) + g(\zeta)t = g(t)g(\zeta) + tg(\zeta)$$

for all $t \in T$ and $\zeta \in Z(T)$. This gives,

$$[g(t) + t, g(\zeta)] = 0 \text{ for all } t \in T, \zeta \in Z(T).$$

Since g is a zero power valued map, we obtain

$$[t, g(\zeta)] = 0 \text{ for all } t \in T, \zeta \in Z(T).$$

Thus, $g(\zeta) \in Z(T)$ for all $\zeta \in Z(T)$.

□

Proposition 2.8. *If T is a ring of characteristic 2, then the existence of a reverse homoderivation g implies that g^2 is a homoderivation.*

Proof. On computing $g^2(\vartheta\ell)$ for any $\vartheta, \ell \in T$, we obtain

$$g^2(\vartheta\ell) = g(g(\vartheta\ell)) = g(g(\ell)g(\vartheta) + g(\ell)\vartheta + \ell g(\vartheta))$$

$$= g^2(\vartheta)g^2(\ell) + 2g^2(\vartheta)g(\ell) + 2g(\vartheta)g^2(\ell) + 2g(\vartheta)g(\ell) + \vartheta g^2(\ell) + g^2(\vartheta)\ell,$$

which further implies

$$g^2(\vartheta\ell) = g^2(\vartheta)g^2(\ell) + g^2(\vartheta)\ell + \vartheta g^2(\ell) \textit{ for all } \vartheta, \ell \in T.$$

Hence, g^2 forms a homoderivation on T.

□

Reverse derivations for prime rings has been studied extensively in [5] and the existence of reverse derivations in case of prime rings implies the existence of ordinary derivations, because in case of prime rings reverse derivations either become zero or make the ring commutative, once the ring is commutative the reverse derivation coincides with the ordinary derivation. This idea does translate in case of reverse homoderivations as well. So, one of our initial objective is to demonstrate that the non-zero reverse homoderivations generally do not exist in prime rings. The following result clarifies our claim.

Theorem 2.9. *Let T be a prime ring and g be a non-zero reverse homoderivation on T. Then, T is a commutative integral domain and consequently g is an ordinary homoderivation on T.*

Proof. Since g is a reverse homoderivation, so we can write

$$\begin{aligned} g(\vartheta\ell^2) &= g(\ell^2)g(\vartheta) + g(\ell^2)\vartheta + \ell^2 g(\vartheta) \\ &= (g(\ell))^2 g(\vartheta) + g(\ell)\ell g(\vartheta) + \ell g(\ell)g(\vartheta) + (g(\ell))^2\vartheta + g(\ell)\ell\vartheta \\ &+ \ell g(\ell)\vartheta + \ell^2 g(\vartheta) \end{aligned}$$

for all $\vartheta, \ell \in T$. Also,

$$
\begin{aligned}
g((\vartheta\ell)\ell) &= g(\ell)g(\vartheta\ell) + g(\ell)\vartheta\ell + \ell g(\vartheta\ell) \\
&= g(\ell)g(\ell)g(\vartheta) + g(\ell)g(\ell)\vartheta + g(\ell)\ell g(\vartheta) \\
&+ g(\ell)\vartheta\ell + \ell g(\ell)g(\vartheta) + \ell g(\ell)\vartheta + \ell\ell g(\vartheta) \\
&= (g(\ell))^2 g(\vartheta) + (g(\ell))^2\vartheta + g(\ell)\ell g(\vartheta) + g(\ell)\vartheta\ell \\
&+ \ell g(\ell)g(\vartheta) + \ell g(\ell)\vartheta + \ell^2 g(\vartheta).
\end{aligned}
$$

Comparing the above calculations, we arrive at

$$g(\ell)\ell\vartheta = g(\ell)\vartheta\ell \text{ for all } \vartheta, \ell \in T.$$

This can be written as,

$$g(\ell)[\vartheta, \ell] = 0 \text{ for all } \vartheta, \ell \in T.$$

On replacing ϑ by $t\vartheta$ for any $t \in T$, we get

$$g(\ell)t[\vartheta, \ell] = 0 \text{ for all } \vartheta, \ell, t \in T.$$

Using the primeness of T, we obtain either $g(\ell) = 0$ for all $\ell \in T$ or $[\vartheta, \ell] = 0$ for all $\vartheta, \ell \in T$. g being non-zero implies T is commutative. Thus, T forms a commutative integral domain and hence g becomes ordinary homoderivation once T is commutative.

□

We now turn our attention solely to semiprime rings, aiming to delve into the behavior of such rings whenever a reverse homoderivation exists. We have seen that reverse homoderivations tend to behave like homoderivations in case of prime rings. Does this scenario prevail in case of semiprime rings? To see this, we prove the following result.

Theorem 2.10. *A central homoderivation on a semiprime ring forms a reverse homoderivation and conversely.*

Proof. Suppose g is a central homoderivation defined on a semiprime ring T. We have

$$g(\vartheta\ell) = g(\vartheta)g(\ell) + g(\vartheta)\ell + \vartheta g(\ell) \textit{ for all } \vartheta, \ell \in T.$$

g being central allows us to rearrange the terms, to obtain

$$g(\vartheta\ell) = g(\ell)g(\vartheta) + g(\ell)\vartheta + \ell g(\vartheta) \textit{ for all } \vartheta, \ell \in T.$$

Thus, g forms a reverse homoderivation on T.
Conversely, suppose g is a reverse homoderivation. Consider,

$$g(\vartheta\ell^2) = g(\ell^2)g(\vartheta) + g(\ell^2)\vartheta + \ell^2 g(\vartheta) \textit{ for all } \vartheta, \ell \in T, \tag{1}$$

and

$$g((\vartheta\ell)\ell) = g(\ell)g(\vartheta\ell) + g(\ell)\vartheta\ell + \ell g(\vartheta\ell) \textit{ for all } \vartheta, \ell \in T,$$

which on solving gives,

$$g((\vartheta\ell)\ell) = (g(\ell))^2 g(\vartheta) + (g(\ell))^2\vartheta + g(\ell)\ell g(\vartheta) + g(\ell)\vartheta\ell + \ell g(\ell)g(\vartheta) + \ell g(\ell)\vartheta + \ell^2 g(\vartheta). \tag{2}$$

On comparing ((1)) and ((2)), we obtain

$$g(\ell^2)g(\vartheta) + g(\ell^2)\vartheta = (g(\ell))^2 g(\vartheta) + (g(\ell))^2\vartheta + g(\ell)\ell g(\vartheta) + g(\ell)\vartheta\ell + \ell g(\ell)g(\vartheta) + \ell g(\ell)\vartheta. \tag{3}$$

Computing it further, we get

$$g(\ell)\ell\vartheta = g(\ell)\vartheta\ell,$$

which can be written as $g(\ell)[\ell,\vartheta] = 0$. Replacing ϑ by $u\vartheta$ with $u \in T$ and using the same condition, we obtain

$$g(\ell)u[\ell,\vartheta] = 0 \text{ for all } \vartheta, \ell, u \in T.$$

Substituting $[\ell, x]ug(\ell)$ in place of u in the above equation,

$$g(\ell)[\ell,\vartheta]ug(\ell)[\ell,\vartheta] = 0 \text{ for all } \vartheta, \ell, u \in T.$$

The given ring is semiprime, so we can deduce that

$$g(\ell)[\ell,\vartheta] = 0 \text{ for all } \vartheta, \ell \in T.$$

In view of Lemma 2.5, we can write $g(\ell) \in Z(T)$ for all $\ell \in T$. Therefore, g becomes a central reverse homoderivation and eventually a homoderivation.

□

In 1978, Herstein [16] demonstrated that a prime ring T with a non-zero derivation β satisfying $[\beta(\vartheta), \beta(\ell)] = 0$ for all $\vartheta, \ell \in T$ is a commutative integral domain if $\mathrm{char}(T) \neq 2$. For $\mathrm{char}(T) = 2$, R is either commutative or an order in a simple algebra that is four-dimensional over its center. This result was extended to semiprime rings by Daif [17] in 1998, and later to any two-sided ideal of T. Taking this forward, El-Sofy [**?**] in his thesis while working with homoderivations proved that for a homoderivation g satisfying the similar identity $[g(\vartheta), g(\ell)] = 0$ for all $\vartheta, \ell \in T$, where T is a prime ring with characteristic not equal to 2 implies T is a commutative integral domain. Also, he showed that in case T is a 2-torsion free semiprime ring, I is its ideal, then a non-zero homoderivation which is zero power valued on I satisfying $[g(\vartheta), g(\ell)] = 0$ for all $\vartheta, \ell \in I$ contains a central ideal. Motivated by this, we study the properties of semiprime rings, once such a functional identity is satisfied by any reverse homoderivation.

Theorem 2.11. *Let T be a semiprime ring and g be a non-zero reverse homoderivation on T satisfying the condition $[g(\vartheta), g(\ell)] = 0$ for all $\vartheta, \ell \in T$. If characteristic of T is not equal to 2, then T contains a non-zero central ideal.*

Proof. Let S be a subring of T that is generated by all the elements of the form $g(\vartheta)$ where $\vartheta \in T$. For any $s \in S$, $\vartheta \in T$, we have $g(s\vartheta) \in S$. By the given hypothesis, S so defined forms a commutative subring and the element $g(s\vartheta)$ can be seen as a centralizer of the set S. For any $b \in S$, we write

$$[b, g(s\vartheta)] = 0$$

giving us

$$g(\vartheta)[b, g(s)] + [b, g(\vartheta)]g(s) + [b, g(\vartheta)]s + g(\vartheta)[b, s] + \vartheta[b, g(s)] + [b, \vartheta]g(s) = 0.$$

From the given criteria, the above equation boils down to

$$[b, \vartheta]g(s) = 0 \text{ for all } b, s \in S, \vartheta \in T.$$

Substituting ϑt in place of ϑ, we obtain

$$[b, \vartheta]tg(s) = 0 \text{ for all } b, s \in S, \vartheta, t \in T.$$

Using s in place of b, so that

$$[s, \vartheta]tg(s) = 0 \text{ for all } s \in S, \vartheta, t \in T.$$

The semiprime character of ring T enables us to obtain a family of prime ideals $\mathfrak{P} = \{P_\nu : \nu \in \kappa\}$ such that their intersection is zero, i.e., $\cap P_\nu = \{0\}$. So, we can deduce that a member P of $\mathfrak{P}$ and for any $s \in T$,

$$\text{either } [s, T] \subseteq P \text{ or } g(s) \subseteq P.$$

Let us define two sets, $M = \{s \in S : [s, T] \subseteq P\}$ and $N = \{s \in S : g(s) \subseteq P\}$. Clearly, sets M and N are additive subgroups of the set S with $M \cup N = S$. That cannot be true as being a subring, S is a group in the first place. Therefore, either $S = M$ or $S = N$. Let us suppose first $S = N$. This implies

$$g(s) \subseteq P \text{ for any } P \in \mathfrak{P}.$$

Since $\cap P_\nu = \{0\}$, so for all $s \in S$ we have $g(s) = 0$, which gives

$$g^2(T) = g(g(T)) \subset g(S) = 0.$$

Now consider, $0 = g^2(\vartheta\ell) = g(g(\vartheta\ell))$, therefore using the characteristic restriction we have

$$g(\vartheta)g(\ell) = 0 \text{ for all } \vartheta, \ell \in T. \tag{4}$$

Substituting ℓ by $u\ell$, $u \in T$ in (4), we get

$$g(\vartheta)g(u\ell) = 0,$$

$$g(\vartheta)g(\ell)g(u) + g(\vartheta)g(\ell)u + g(\vartheta)\ell g(u) = 0,$$

which implies $g(\vartheta)\ell g(u) = 0$ for all $\vartheta, \ell, u \in T$. Replacing u by ϑ, we obtain $g(\vartheta)\ell g(\vartheta) = 0$ for all $\vartheta, \ell \in T$. Thus, semiprimeness of T helps us to conclude that $g \equiv 0$, a contradiction. This gives $g(S) \neq \{0\}$. Therefore, $S = M$, i.e., $[s, T] \subseteq P$ for all $s \in S$ and for any $P \in \mathfrak{P}$. Using similar arguments as above, we conclude

$$[s, T] = \{0\} \text{ for all } s \in S.$$

Consider, $I = Tg(S)T$ an ideal of the ring T. Clearly, I forms a non-zero ideal of T. Furthermore, if we compute $[I, T]T[I, T]$, we get

$$[I, T]T[I, T] = [Tg(S)T, T]T[Tg(S)T, T] \subseteq [TST, T]T[TST, T] \subseteq T[S, T]T[S, T] = \{0\}.$$

The above calculation provides us with a non-zero central ideal I. This proves our result. $\square$

Theorem 2.12. *Let T be a semiprime ring, I be a non-zero ideal of T and $0 \neq g$ be a reverse homoderivation satisfying the conditions $g([\vartheta, \ell]) = 0$ for all $\vartheta, \ell \in T$ and $g(I)I \neq \{0\}$. Then, T contains a non-zero central ideal.*

Proof. We are given with

$$g([\vartheta, \ell]) = 0 \text{ for all } \vartheta, \ell \in I.$$

On replacing ℓ by $\ell\vartheta$ in the above equation, we obtain

$$g(\vartheta)[\vartheta, \ell] = 0 \text{ for all } \vartheta, \ell \in I.$$

Substituting ℓ by ℓt where $t \in T$, we get

$$g(\vartheta)I[\vartheta, t] = \{0\} \text{ for all } \vartheta \in I, t \in T.$$

The above equation can be written as

$$g(\vartheta)IT[\vartheta, T] = \{0\} \text{ for all } \vartheta \in I.$$

By the semiprimeness of T we obtain a family of prime ideals $\mathfrak{P} = \{P_\nu : \nu \in \kappa\}$ such that their intersection is zero, i.e., $\cap P_\nu = \{0\}$. So, for any prime ideal P of $\mathfrak{P}$ and for any $\vartheta \in I$,

$$\text{either } g(\vartheta) \subseteq P \text{ or } [x, T] \subseteq P.$$

We have,

$$\text{either } g(I) \subseteq P \text{ or } [I, T] \subseteq P.$$

In either of the case we get,

$$g(I)I[I, T] \subseteq P,$$

for any $P \in \mathfrak{P}$. This implies,

$$g(I)I[I, T] \subseteq \cap P_\nu = \{0\}.$$

As g is a non-zero reverse homoderivation, so we have $[\vartheta, T] = 0$ for all $\vartheta \in I$. Consider, $J = Tg(I)IT$ an ideal of the ring T. Clearly, J forms a non-zero ideal of T. Further, compute $[J, T]T[J, T]$, we get

$$\begin{aligned}[J, T]T[J, T] &= [Tg(I)IT, T]T[Tg(I)IT, T] \subseteq Tg(I)IT - Tg(I)I \quad \subseteq Tg(I)IT - Tg(I)TI \\ &= Tg(I)I[I, T] = \{0\}.\end{aligned}$$

Thus, $J = Tg(I)IT$ forms a non-zero central ideal of T. $\square$

The following example demonstrates that the condition of semiprimeness in the hypothesis of Theorem 2.11 and Theorem 2.12 are crucial.

Example 2.13. *Let $T = \left\{ \begin{pmatrix} \vartheta & \ell \\ 0 & 0 \end{pmatrix} : \vartheta, \ell \in \mathbb{R} \right\}$. Define $g : T \to T$ by*

$$g\begin{pmatrix} \vartheta & \ell \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \vartheta \\ 0 & 0 \end{pmatrix}.$$

g so defined forms a reverse homoderivation. Also g satisfies both the conditions (i) $[g(\vartheta), g(\ell)] = 0$, *and (ii)* $g([\vartheta, \ell]) = 0$ *for all* $\vartheta, \ell \in T$. *In view of Theorems 2.11 and 2.12, T should contain a non-zero central ideal. But the center of the ring is* $\{0\}$. *Thus, T does not have a non-zero central ideal. This is because the ring under consideration is not semiprime. Hence, in Theorems 2.11 and 2.12, the hypothesis of semiprimeness is essential.*

A commuting map is always a centralizing map but the converse need not be true always. In [18], Brešar proved that if an additive map $g : T \to T$ is centralizing on a prime ring T, then it is commuting as well. In addition to this, he showed the form of such additive maps, i.e., $g(\vartheta) = c\vartheta + f(\vartheta)$, where $c \in C$, the extended centroid of T and $f : T \to C$ an additive map. Further, he extended the result for semiprime rings as well (see [19] for details). On similar lines, Ali and Dar [20] described the structure of $*$- commuting maps in case of semiprime rings. Motivated by this we prove that every centralizing reverse homoderivation forms a commuting homoderivation under certain constraints.

Theorem 2.14. *Let* T *be a semiprime ring with* $char(T) \neq 2$ *and* $\{0\} \neq I$ *be an ideal of* T. *If* g *is a centralizing reverse homoderivation on* I, *then* g *is commuting on* I.

Proof. For any $\vartheta \in I$, we have

$$[g(\vartheta), \vartheta] \in Z(T).$$

Consider the expression,

$$[g(\vartheta^2), \vartheta^2] \in Z(T),$$

we obtain

$$[(g(\vartheta))^2 + g(\vartheta)\vartheta + \vartheta g(\vartheta), \vartheta^2] \in Z(T).$$

This gives,

$$[g(\vartheta), \vartheta^2]g(\vartheta) + g(\vartheta)[g(\vartheta), \vartheta^2] + [g(\vartheta), \vartheta^2]\vartheta + \vartheta[g(\vartheta), \vartheta^2] \in Z(T).$$

On further solving, we get

$$\begin{aligned}&[g(\vartheta), \vartheta]\vartheta g(\vartheta) + \vartheta[g(\vartheta), \vartheta]g(\vartheta) + g(\vartheta)\vartheta[g(\vartheta), \vartheta] + g(\vartheta)[g(\vartheta), \vartheta]\vartheta \\ &+ \vartheta[g(\vartheta), \vartheta]\vartheta + [g(\vartheta), \vartheta]\vartheta^2 + \vartheta^2[g(\vartheta), \vartheta] + \vartheta[g(\vartheta), \vartheta] \in Z(T).\end{aligned}$$

As the map g is centralizing, therefore we can write

$$2[g(\vartheta), \vartheta]\vartheta g(\vartheta) + 2g(\vartheta)\vartheta[g(\vartheta), \vartheta] + 4\vartheta^2[g(\vartheta), \vartheta] \in Z(T).$$

Commuting the above expression with $g(i)$, we get

$$2[[g(\vartheta), \vartheta]\vartheta g(\vartheta), g(\vartheta)] + 2[g(\vartheta)\vartheta[g(\vartheta), \vartheta], g(\vartheta)] + 4[\vartheta^2[g(\vartheta), \vartheta], g(\vartheta)] = 0,$$

further simplification yields

$$2[g(\vartheta), \vartheta][\vartheta g(\vartheta), g(\vartheta)] + 2[g(\vartheta)\vartheta, g(\vartheta)][g(\vartheta), \vartheta] + 4[\vartheta^2, g(\vartheta)][g(\vartheta), \vartheta] = 0.$$

Rearranging the terms using properties of Lie bracket we arrive at,

$$2[g(\vartheta),\vartheta]\vartheta[g(\vartheta),g(\vartheta)]+2[g(\vartheta),\vartheta][\vartheta,g(\vartheta)]g(\vartheta)+2g(\vartheta)[\vartheta,g(\vartheta)][g(\vartheta),\vartheta]$$
$$+2[g(\vartheta),g(\vartheta)]\vartheta[g(\vartheta),\vartheta]+4\vartheta[\vartheta,g(\vartheta)][g(\vartheta),\vartheta]+4[\vartheta,g(\vartheta)]\vartheta[g(\vartheta),\vartheta]=0,$$

which implies

$$2[g(\vartheta),\vartheta][\vartheta,g(\vartheta)]g(\vartheta)+2g(\vartheta)[\vartheta,g(\vartheta)][g(\vartheta),\vartheta]+4\vartheta[\vartheta,g(\vartheta)][g(\vartheta),\vartheta]$$
$$+4[\vartheta,g(\vartheta)]\vartheta[g(\vartheta),\vartheta]=0.$$

This can be written as

$$2[g(\vartheta),\vartheta][g(\vartheta),\vartheta]g(\vartheta)+2g(\vartheta)[g(\vartheta),\vartheta][g(\vartheta),\vartheta]+$$
$$4\vartheta[g(\vartheta),\vartheta][g(\vartheta),\vartheta]+4[g(\vartheta),\vartheta]\vartheta[g(\vartheta),\vartheta]=0$$

$$2[g(\vartheta),\vartheta]^2g(\vartheta)+2g(\vartheta)[g(\vartheta),\vartheta]^2+4\vartheta[g(\vartheta),\vartheta]^2+4[g(\vartheta),\vartheta]^2\vartheta=0.$$

Therefore, we conclude

$$8\vartheta[g(\vartheta),\vartheta]^2+4g(\vartheta)[g(\vartheta),\vartheta]^2=0 \text{ for all } \vartheta\in I.$$

This can be written as, $4(2\vartheta+g(\vartheta))[g(\vartheta),\vartheta]^2=0$ for all $\vartheta\in I$. Let's compute

$$\begin{aligned}8[g(\vartheta),\vartheta]^3 &= 8(g(\vartheta)\vartheta-\vartheta g(\vartheta))[g(\vartheta),\vartheta]^2\\ &= 8g(\vartheta)\vartheta[g(\vartheta),\vartheta]^2-8\vartheta g(\vartheta)[g(\vartheta),\vartheta]^2\\ &= 8g(\vartheta)\vartheta[g(\vartheta),\vartheta]^2+4g(\vartheta)[g(\vartheta),\vartheta]^2g(\vartheta)\\ &= 8g(\vartheta)\vartheta[g(\vartheta),\vartheta]^2+4(g(\vartheta))^2[g(\vartheta),\vartheta]^2\\ &= g(\vartheta)\{4(2\vartheta+g(\vartheta))\}[g(\vartheta),\vartheta]^2=0.\end{aligned}$$

Thus, $(2[g(\vartheta),\vartheta])^3=0$ for all $\vartheta\in I$. The ring T into consideration, is a semiprime ring with characteristic not 2, therefore, $[g(\vartheta),\vartheta]=0$ for all $\vartheta\in I$. Hence, g is commuting on the non-zero ideal I of T.

□

3. CONCLUDING REMARKS

Many concepts related to reverse homoderivations can be investigated for semiprime rings or even for arbitrary rings. Additionally, we suggest that reverse homoderivations are closely connected to various other special functions, such as anti-homomorphisms and reverse derivations. Future researchers should aim to characterize reverse homoderivations in terms of simpler functions. Lastly, we suggest some open problems that should be tried to further understand the behavior of reverse homoderivations.

Problem 1: Let T be a ring and $Z(T)$ be its center. If T admits reverse homoderivations g_1 and g_2 such that $g_1(\vartheta)\circ g_2(\vartheta)\in Z(T)$ for all $\vartheta\in T$, then what we can say about the structure of T, and behaviour of g_1 and g_2?

Problem 2: Let T be a ring and $\mathfrak{P}$ be a semi(prime) ideal of T. If T admits reverse homoderivations g_1 and g_2 such that $[g_1(\vartheta), g_2(\vartheta)] \in \mathfrak{P}$ for all $\vartheta \in T$, then what we can say about the structure of T, and behaviour of g_1 and g_2?

Problem 3: Let i and j be positive integers, T be a ring, and $\mathfrak{P}$ be a semi(prime) ideal of T. If T admits reverse homoderivations g_1 and g_2 such that $[g_1(\vartheta)^i, g_2(\vartheta)^j] \in \mathfrak{P}$ for all $\vartheta \in T$, then what we can say about the structure of T, and behaviour of g_1 and g_2?

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