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On Sombor Energy of the Nilpotent Graph of the Ring of Integers Modulo ε

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Abstract. In chemical graph theory, chemical compounds are represented as graphs where atoms are represented as vertices, and the bonds connecting the atoms are represented as edges. In 2021, Gowtham and Swamy discovered another type of graph energy, called the Sombor energy. This discovery was motivated by Gutman's introduction of the Sombor index in the same year. In the field of abstract algebra, rings can also be represented as graphs. In this article, we aim to explore the Sombor energy of some nilpotent graphs of rings, particularly the ring of integers modulo ε.

Key words and Phrases: Sombor matrix, characteristic polynomial, energy, graph, ring of integer modulo.

1. INTRODUCTION

Mathematical graph theory is used to represent chemical compounds, namely chemical graph theory. In chemical graph theory, chemical compounds are modelled as graphs where the atoms that make up the compound are represented as vertices. Each vertex corresponds to an atom. The bonds, which are the connections between atoms, are represented as edges. These representations help in visualizing the structure of molecules, analyzing their properties, and predicting the behaviour of chemical compounds. By understanding the graph representation, chemists can gain insights into the molecular structure and interactions within compounds, facilitating the study of their reactivity, stability, and various physical and chemical properties. This formalism provides a systematic and mathematical framework for

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the depiction and analysis of chemical structures, facilitating a comprehensive understanding of the relationships between atoms and bonds within a given compound [1].

Research on graphs representing groups or rings is an emerging trend that bridges algebraic structures with graph theory. This interdisciplinary field explores various types of graphs associated with algebraic entities and their applications. Among these, the coprime graph of a group is a notable structure, where vertices represent group elements, and an edge connects two vertices if the orders of the elements are coprime [2]. Conversely, the noncoprime graph focuses on elements whose orders are not coprime [3]. This field also includes studies on numerical invariants and connectivity indices. Researchers study topological indices as numerical invariants that characterize graph structures, such as the distance-based indices like Wiener index (sum of vertex pair distances) and degree-based indices like the Randić and harmonic indices, which highlight degree distributions. These indices reveal connectivity and complexity, offering insights into algebraic structures while enabling classification and analysis of groups and rings. Applications span cryptography, chemistry, and combinatorics, making this a dynamic research field [4], [5].

In this paper, we contribute to this area by exploring another related topic: the energy of a simple graph. The energy of a simple graph is a numerical invariant derived from the eigenvalues of its matrix representation for example adjacent matrix. It is defined as the sum of the absolute values of all its eigenvalues [6]. Graph energy has applications in various fields, including chemistry, where it is used to model molecular stability, and in network analysis for assessing structural properties and connectivity [7] [8]. Specifically, we give some properties of the such type of graph representation of the ring of integers modulo ε, namely nilpotent graph which is denoted as $\mathbb{Z}_\varepsilon$, concerning its Sombor energy. A nilpotent graph of a ring is a simple graph where the vertices are all the elements of the ring, and two elements are said to be adjacent if and only if the multiplications is a nilpotent element [9]. Previous studies give some important properties of the nilpotent graph for the ring $\mathbb{Z}_\varepsilon$ where ε is a prime number or a power of 2.

Lemma 1.1. [9] *The nilpotent graph of the ring of integer modulo $\mathbb{Z}_\varepsilon$ where ε is a prime number is a star graph $K_{1,\varepsilon-1}$.*

And for the case with a different order of elements or operations, the result can be seen in the following Lemma.

Lemma 1.2. [9] *If $\mathbb{Z}_\varepsilon$ is the ring of integers modulo ε where $\varepsilon = 2^k$ for some $k \in \mathbb{N}$, the nilpotent graph of Z_ε, denoted as $\Gamma_{\mathbb{Z}_\varepsilon}$, is contains 2^{k-1} star subgraphs $K_{1,2^{k-1}}$.*

These two lemmas have nilpotent vertices as the center of every star graph [9] and are important to find the degree of the vertices in the graph. The degree of vertices is used to formulate the Sombor matrix and the energy graph of the nilpotent graph of the ring.

Definition 1.3 (Degree of a Vertex)**.** *The degree of $v \in V(\Gamma)$ denoted by d_v is defined as the number of vertices adjacent to v where Γ is a simple graph and $V(\Gamma)$ is the set of all its vertices.*

Based on Lemma 1.1 and 1.2 the vertices in the graph have the degree determined by the structural properties of the graph, as described in detail in the following theorem.

Theorem 1.4. *If $\Gamma_{\mathbb{Z}_\varepsilon}$ is the nilpotent graph of $\mathbb{Z}_\varepsilon$ where ε is a prime number, then $d_0 = \varepsilon - 1$ and $d_v = 1$ for all $v = 1, 2, \ldots, \varepsilon - 1$.*

Proof. By Lemma 1.1, the graph $\Gamma_{\mathbb{Z}_\varepsilon}$ is a star graph. Clearly, the vertex 0 is adjacent to all other elements of $\mathbb{Z}_\varepsilon$. Hence, the number of vertices adjacent with 0 is $\varepsilon - 1$. On the other hand, since a nonzero element v of $\mathbb{Z}_\varepsilon$ is adjacent to zero alone, the degree of v is 1. □

For the case with a different order of the ring, the degree of the elements can be seen below.

Theorem 1.5. *If $\Gamma_{\mathbb{Z}_\varepsilon}$ is the nilpotent graph of $\mathbb{Z}_\varepsilon$ where $\varepsilon = 2^k$ for some $k \in \mathbb{N}$, then $d_x = \frac{1}{2}\varepsilon$ for $x \in \{1, 3, \ldots, 2^k - 1\}$ and $d_y = \varepsilon - 1$ for $y \in \{0, 2, \ldots, 2^k - 2\}$.*

Proof. Suppose that $S_1 = \{1, 3, \ldots, 2^k - 1\}$ and $S_2 = \{0, 2, \ldots, 2^k - 2\}$. Note that S_2 is the set of nilpotent elements of $\mathbb{Z}_n$ so that all elements of S_2 are adjacent to all elements of $\mathbb{Z}_\varepsilon$. On the other hand, the elements of S_1 which are non-nilpotent only neighbour nilpotent elements. So, the number of vertices adjacent to $x \in S_1$ is $d_x = |S_2| = \frac{1}{2}\varepsilon$. Since $y \in S_2$ is adjacent to all of the other vertices except itself, $d_y = |S_1| + |S_2| - 1 = \varepsilon - 1$. □

Based on the results obtained in previous studies, we formulate the Sombor energy of the given graph. In this article, we explore the definition and properties of Sombor energy, building on previous work to offer a clearer understanding of its significance in graph theory.

2. MAIN RESULTS

2.1. **Sombor Energy of Prime Number Order.**

In 2021, Gowtham and Swamy [10] discovered another type of graph energy, the Sombor energy. It was motivated by Gutman's discovery of the Sombor index in 2021 [**?**]. The following is the definition of the Sombor matrix.

Definition 2.1. [10] *The Sombor matrix of graph Γ is defined as matrix $SO(\Gamma) = [so_{ij}]$ with*

$$so_{ij} = \begin{cases} \sqrt{d_{v_i}^2 + d_{v_j}^2} & ;\ \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & ;\ \text{else,} \end{cases} \tag{1}$$

where d_{v_i} is the degree of the vertex v_i.

The Sombor matrix is used to find the eigenvalues to define the Sombor energy of a graph [7].

Definition 2.2. [6, 11] *If $SO(\Gamma)$ is Sombor matrix of a graph Γ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_\varepsilon$, then Sombor energy of the graph Γ is*

$$E_{SO}(\Gamma) = \sum_{i=1}^{\varepsilon} |\lambda_i|. \tag{2}$$

The size of the Sombor matrix is usually very large it is very difficult to calculate its determinant manually. The lemma below will help us determine the determinant of a matrix that has a certain form.

Lemma 2.3. [12] *Let r,s,t, u be real numbers and η is a complex number, the determinant of the block matrix*

$$\begin{vmatrix} (\eta + r)I_{\varepsilon_1} - rJ_{\varepsilon_1} & -tJ_{\varepsilon_1 \times \varepsilon_2} \\ -uJ_{\varepsilon_2 \times \varepsilon_1} & (\eta + s)I_{\varepsilon_2} - sJ_{\varepsilon_2} \end{vmatrix}$$

of order $\varepsilon_1 + \varepsilon_2$ can be expressed in the simplified form as

$$(\eta + r)^{\varepsilon_1 - 1}(\eta + s)^{\varepsilon_2 - 1}[(\eta - (\varepsilon_1 - 1)r)(\eta - (\varepsilon_2 - 1)s) - \varepsilon_1\varepsilon_2 tu], \tag{3}$$

where J is a rectangular matrix with all its entries 1 and I is the identity matrix.

From Definition 2.1, we can determine the Sombor matrix and the characteristic polynomial of the nilpotent graph $\Gamma_{\mathbb{Z}_\varepsilon}$. Thus, it is necessary to find the determinant of $SO(\Gamma_{\mathbb{Z}_\varepsilon})$.

Theorem 2.4. *The characteristic polynomial of the matrix $SO(\Gamma_{\mathbb{Z}_\varepsilon})$ where ε is a prime number is*

$$P_{\Gamma_{\mathbb{Z}_\varepsilon}}(\lambda) = \lambda^{\varepsilon-2}(\lambda^2 - (\varepsilon - 1)(\varepsilon^2 - 2\varepsilon + 2)). \tag{4}$$

Proof. By Definition 2.1 and Lemma 1.1, we construct the Sombor matrix of the nilpotent graph of $\mathbb{Z}_\varepsilon$ with prime number order. That is

$$SO(\Gamma_{\mathbb{Z}_\varepsilon}) = \begin{bmatrix} 0 & \sqrt{\varepsilon^2 - 2\varepsilon + 2} & \sqrt{\varepsilon^2 - 2\varepsilon + 2} & \cdots & \sqrt{\varepsilon^2 - 2\varepsilon + 2} \\ \sqrt{\varepsilon^2 - 2\varepsilon + 2} & 0 & 0 & \cdots & 0 \\ \sqrt{\varepsilon^2 - 2\varepsilon + 2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\varepsilon^2 - 2\varepsilon + 2} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then we will find the eigenvalues of that matrix by using $|SO(\Gamma_{\mathbb{Z}_\varepsilon}) - \lambda I| = 0$. Since $|SO(\Gamma) - \lambda I| = |\lambda I - SO(\Gamma)|$, the determinant of $SO(\Gamma) - \lambda I$ is given by

$$\begin{vmatrix} \lambda & -\sqrt{\varepsilon^2 - 2\varepsilon + 2} & -\sqrt{\varepsilon^2 - 2\varepsilon + 2} & \cdots & -\sqrt{\varepsilon^2 - 2\varepsilon + 2} \\ -\sqrt{\varepsilon^2 - 2\varepsilon + 2} & \lambda & 0 & \cdots & 0 \\ -\sqrt{\varepsilon^2 - 2\varepsilon + 2} & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\varepsilon^2 - 2\varepsilon + 2} & 0 & 0 & \cdots & \lambda \end{vmatrix}.$$

We can rewrite this matrix into

$$\begin{vmatrix} \lambda I_1 & -\sqrt{\varepsilon^2-2\varepsilon+2}J_{1\times\varepsilon-1} \\ -\sqrt{\varepsilon^2-2\varepsilon+2}J_{\varepsilon-1\times 1} & \lambda I_{\varepsilon-1} \end{vmatrix}$$

Using Lemma 2.3, the characteristic polynomial of the matrix $SO(\Gamma_{\mathbb{Z}_\varepsilon})$, where ε is a prime number, is

$$\begin{aligned} P_{\Gamma_{\mathbb{Z}_\varepsilon}}(\lambda) =&(\lambda+0)^{1-1}(\lambda+0)^{\varepsilon-1-1}[(\lambda-(1-1)0)(\lambda-(\varepsilon-1-1)0) \\ &-(1)(\varepsilon-1)(-\sqrt{\varepsilon^2-2\varepsilon+2})(-\sqrt{\varepsilon^2-2\varepsilon+2})] \\ =&\lambda^{\varepsilon-2}(\lambda^2-(\varepsilon-1)(\varepsilon^2-2\varepsilon+2)). \end{aligned}$$

□

Based on the Theorem 2.4, we can easily formulate the general formula for Sombor energy.

Theorem 2.5. *The Sombor energy of* $\Gamma_{\mathbb{Z}_\varepsilon}$ *where* ε *is a prime number is*

$$E_{SO}(\Gamma_{\mathbb{Z}_\varepsilon}) = 2\sqrt{(\varepsilon-1)(\varepsilon^2-2\varepsilon+2)}. \tag{5}$$

Proof. Based on Theorem 2.4, the eigenvalues of $SO(\Gamma_{\mathbb{Z}_\varepsilon})$ are $\lambda = 0$ with multiplicity $\varepsilon-2$ and $\lambda = \pm\sqrt{(\varepsilon-1)(\varepsilon^2-2\varepsilon+2)}$ each with multiplicity 1. Hence, the Sombor energy of $\Gamma_{\mathbb{Z}_\varepsilon}$ where ε is a prime number is

$$\begin{aligned} E_{SO}(\Gamma_{\mathbb{Z}_\varepsilon}) =&(\varepsilon-2)|0| + |\sqrt{(\varepsilon-1)(\varepsilon^2-2\varepsilon+2)}| + |-\sqrt{(\varepsilon-1)(\varepsilon^2-2\varepsilon+2)}| \\ =&\sqrt{(\varepsilon-1)(\varepsilon^2-2\varepsilon+2)} + \sqrt{(\varepsilon-1)(\varepsilon^2-2\varepsilon+2)} \\ =&2\sqrt{(\varepsilon-1)(\varepsilon^2-2\varepsilon+2)}. \end{aligned}$$

□

2.2. **Sombor Energy when ε is a Power of Two.**

In this section, the Sombor energy for the order of squared prime numbers is presented. The derivation involves constructing the associated matrix and determining its eigenvalues.

Theorem 2.6. *The characteristic polynomial of the matrix* $SO(\Gamma_{\mathbb{Z}_\varepsilon})$ *where* $\varepsilon = 2^k$ *for some* $k \in \mathbb{N}$ *is*

$$\begin{aligned} P_{\Gamma_{\mathbb{Z}_\varepsilon}}(\lambda) =&\lambda^{\frac{\varepsilon}{2}-1}(\lambda+(\varepsilon-1)\sqrt{2})^{\frac{\varepsilon}{2}-1}\left(\lambda^2-(\varepsilon-1)\left(\frac{\varepsilon}{2}-1\right)\sqrt{2}\lambda-\frac{\varepsilon^2}{16}\right. \\ &\left.(5\varepsilon^2-8\varepsilon+4)\right). \end{aligned} \tag{6}$$

Proof. By Definition 2.1 and Lemma 1.2, we construct the Sombor matrix of the nilpotent graph of $\mathbb{Z}_\varepsilon$ with the order a power of two. That is

$$SO(\Gamma_{\mathbb{Z}_\varepsilon}) = \begin{bmatrix} 0 & (\varepsilon-1)\sqrt{2} & \cdots & (\varepsilon-1)\sqrt{2} & \delta & \delta & \cdots & \delta \\ (\varepsilon-1)\sqrt{2} & 0 & \cdots & (\varepsilon-1)\sqrt{2} & \delta & \delta & \cdots & \delta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (\varepsilon-1)\sqrt{2} & (\varepsilon-1)\sqrt{2} & \cdots & 0 & \delta & \delta & \cdots & \delta \\ \delta & \delta & \cdots & \delta & 0 & 0 & \cdots & 0 \\ \delta & \delta & \cdots & \delta & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta & \delta & \cdots & \delta & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $\delta = \frac{1}{2}\sqrt{5\varepsilon^2 - 8\varepsilon + 4}$. Then we will find the eigenvalues of that matrix using $|SO(\Gamma_{\mathbb{Z}_\varepsilon}) - \lambda I| = 0$. Thus,

$$|\lambda I - SO(\Gamma_{\mathbb{Z}_\varepsilon})| = \begin{vmatrix} \lambda & -(\varepsilon-1)\sqrt{2} & \cdots & -(\varepsilon-1)\sqrt{2} & -\delta & -\delta & \cdots & -\delta \\ -(\varepsilon-1)\sqrt{2} & \lambda & \cdots & -(\varepsilon-1)\sqrt{2} & -\delta & -\delta & \cdots & -\delta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(\varepsilon-1)\sqrt{2} & -(\varepsilon-1)\sqrt{2} & \cdots & \lambda & -\delta & -\delta & \cdots & -\delta \\ -\delta & -\delta & \cdots & -\delta & \lambda & 0 & \cdots & 0 \\ -\delta & -\delta & \cdots & -\delta & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\delta & -\delta & \cdots & -\delta & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

We can rewrite this matrix into

$$\begin{vmatrix} (\lambda + (\varepsilon-1)\sqrt{2})I_{\frac{\varepsilon}{2}} - (\varepsilon-1)\sqrt{2}J_{\frac{\varepsilon}{2}} & -\frac{1}{2}\sqrt{5\varepsilon^2 - 8\varepsilon + 4}J_{\frac{\varepsilon}{2}} \\ -\frac{1}{2}\sqrt{5\varepsilon^2 - 8\varepsilon + 4}J_{\frac{\varepsilon}{2}} & \lambda I_{\frac{\varepsilon}{2}} \end{vmatrix}.$$

Using Lemma 2.3, the characteristic polynomial of the matrix $SO(\Gamma_{\mathbb{Z}_\varepsilon})$, where $\varepsilon = 2^k$ for some $k \in \mathbb{N}$ is

$$\begin{aligned} P_{\Gamma_{\mathbb{Z}_\varepsilon}}(\lambda) =& (\lambda + (\varepsilon-1)\sqrt{2})^{\frac{\varepsilon}{2}-1}(\lambda+0)^{\frac{\varepsilon}{2}-1}\left[\left(\lambda - \left(\frac{\varepsilon}{2}-1\right)(\varepsilon-1)\sqrt{2}\right)\left(\lambda - \left(\frac{\varepsilon}{2}-1\right)0\right)\right. \\ &\left. - \frac{\varepsilon}{2}\frac{\varepsilon}{2}\left(-\frac{1}{2}\sqrt{5\varepsilon^2-8\varepsilon+4}\right)\left(-\frac{1}{2}\sqrt{5\varepsilon^2-8\varepsilon+4}\right)\right] \\ =& \lambda^{\frac{\varepsilon}{2}-1}(\lambda + (\varepsilon-1)\sqrt{2})^{\frac{\varepsilon}{2}-1}\left(\lambda^2 - (\varepsilon-1)\left(\frac{\varepsilon}{2}-1\right)\sqrt{2}\lambda - \frac{\varepsilon^2}{16}\right. \\ &\left.(5\varepsilon^2 - 8\varepsilon + 4)\right). \end{aligned}$$

□

Building on the theorem above regarding the characteristic polynomial, we can readily derive the general formula for Sombor energy.

Theorem 2.7. *The Sombor energy of* $\Gamma_{\mathbb{Z}_\varepsilon}$ *where* $\varepsilon = 2^k$ *for some* $k \in \mathbb{N}$ *is*

$$E_{SO}(\Gamma_{\mathbb{Z}_\varepsilon}) = \left(\frac{\varepsilon}{2}-1\right)(\varepsilon-1)\sqrt{2} + \sqrt{2(\varepsilon-1)^2\left(\frac{\varepsilon}{2}-1\right)^2 + \frac{\varepsilon^2}{4}(5\varepsilon^2-8\varepsilon+4)}. \quad (7)$$

Proof. Based on Theorem 2.6, the eigenvalues of the matrix $SO(\Gamma_{\mathbb{Z}_\varepsilon})$ are $\lambda = 0$ with multiplicity $\frac{\varepsilon}{2}-1$, $\quad \lambda = -(\varepsilon-1)\sqrt{2}$ with multiplicity $\frac{\varepsilon}{2}-1$, and

$$\begin{aligned}\lambda =& \frac{1}{2}\left((\varepsilon-1)\left(\frac{\varepsilon}{2}-1\right)\sqrt{2} \pm \sqrt{\left(-(\varepsilon-1)\left(\frac{\varepsilon}{2}-1\right)\sqrt{2}\right)^2 - 4\left(-\frac{\varepsilon^2}{16}(5\varepsilon^2-8\varepsilon+4)\right)}\right)\\ =& \frac{1}{2}\left((\varepsilon-1)\left(\frac{\varepsilon}{2}-1\right)\sqrt{2} \pm \sqrt{2(\varepsilon-1)^2\left(\frac{\varepsilon}{2}-1\right)^2 + \frac{\varepsilon^2}{4}(5\varepsilon^2-8\varepsilon+4)}\right)\end{aligned}$$

each with multiplicity 1. Hence, the Sombor energy of $\Gamma_{\mathbb{Z}_\varepsilon}$ with $\varepsilon = 2^k$ for some $k \in \mathbb{N}$ is

$$\begin{aligned}E_{SO}(\Gamma_{\mathbb{Z}_\varepsilon}) =& \left(\frac{\varepsilon}{2}-1\right)|0| + \left(\frac{\varepsilon}{2}-1\right)|-(\varepsilon-1)\sqrt{2}|\\ &+ \left|\frac{1}{2}\left((\varepsilon-1)\left(\frac{\varepsilon}{2}-1\right)\sqrt{2} + \sqrt{2(\varepsilon-1)^2\left(\frac{\varepsilon}{2}-1\right)^2 + \frac{\varepsilon^2}{4}(5\varepsilon^2-8\varepsilon+4)}\right)\right|\\ &+ \left|\frac{1}{2}\left((\varepsilon-1)\left(\frac{\varepsilon}{2}-1\right)\sqrt{2} - \sqrt{2(\varepsilon-1)^2\left(\frac{\varepsilon}{2}-1\right)^2 + \frac{\varepsilon^2}{4}(5\varepsilon^2-8\varepsilon+4)}\right)\right|\\ =& \left(\frac{\varepsilon}{2}-1\right)(\varepsilon-1)\sqrt{2} + \sqrt{2(\varepsilon-1)^2\left(\frac{\varepsilon}{2}-1\right)^2 + \frac{\varepsilon^2}{4}(5\varepsilon^2-8\varepsilon+4)}.\end{aligned}$$

□

In conclusion, we have highlighted the relationship between the graph representation of a ring and its Sombor energy. By examining the properties of the graph, we have provided insights into how the graph's structure influences its Sombor energy. This work extends previous studies and contributes to a deeper understanding of the connection between algebraic structures and graph invariants.

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References

[1] R. García-Domenech, J. Gálvez, J. V. de Julián-Ortiz, and L. Pogliani, "Some new trends in chemical graph theory," *Chemical Reviews*, vol. 108, no. 3, pp. 1127–1169, 2008. https://doi.org/10.1021/cr0780006.

[2] N. Nurhabibah, I. G. A. W. Wardhana, and N. W. Switrayni, "Numerical invariants of coprime graph of a generalized quaternion group," *Journal of the Indonesian Mathematical Society*, pp. 36–44, 2023. https://doi.org/10.22342/jims.29.1.1245.36-44.

[3] M. Afdhaluzzikri, I. G. A. W. Wardhana, F. Maulana, and H. R. Biswas, "The non-coprime graphs of upper unitriangular matrix groups over the ring of integer modulo with prime order and their topological indices," *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 19, no. 1, pp. 547–556, 2025. https://doi.org/10.30598/barekengvol19iss1pp547-556.

[4] S. Zahidah, D. M. Mahanani, and K. L. Oktaviana, "Connectivity indices of coprime graph of generalized quaternion group," *J. Indones. Math. Soc*, vol. 27, no. 3, pp. 285–296, 2021. https://doi.org/10.22342/jims.27.3.1043.285-296.

[5] J. Yang, M. H. Muhammad, M. K. Siddiqui, M. F. Hanif, M. Nasir, S. Ali, and J.-B. Liu, "Topological co-indices of hydroxyethyl starch conjugated with hydroxychloroquine used for covid-19 treatment," *Polycyclic Aromatic Compounds*, vol. 42, no. 10, pp. 7130–7142, 2022. https://doi.org/10.1080/10406638.2021.1996407.

[6] Gutman and I, "The energy of graph," *Ber. Math.Stat. Sekt. Forschungszent. Graz*, pp. 1–22, 1978.

[7] R. Balakrishnan, "The energy of a graph," *Linear Algebra and its Applications*, vol. 387, pp. 287–295, 2004. https://doi.org/10.1016/j.laa.2004.02.038.

[8] G. Y. Karang, I. G. A. W. Wardhana, N. I. Alimon, and N. H. Sarmin, "Energy and degree sum energy of non-coprime graphs on dihedral groups," *Journal of the Indonesian Mathematical Society*, vol. 31, no. 1, pp. 1900–1900, 2025. https://doi.org/10.22342/jims.v31i1.1900.

[9] D. P. Malik, M. N. Husni, M. Miftahurrahman, I. G. A. W. Wardhana, G. Semil, *et al.*, "the chemical topological graph associated with the nilpotent graph of a modulo ring of prime power order," *Journal of Fundamental Mathematics and Applications (JFMA)*, vol. 7, no. 1, pp. 1–9, 2024. https://doi.org/10.14710/jfma.v0i0.20269.

[10] K. J. Gowtham and S. N. Narasimha, "On sombor energy of graphs," *Наносистемы: физика, химия, математика*, vol. 12, no. 4, pp. 411–417, 2021. https://doi.org/10.17586/2220-8054-2021-12-4-411-417.

[11] I. Gutman, I. Redžepović, and J. Rada, "Relating energy and sombor energy," *Contrib. Math*, vol. 4, pp. 41–44, 2021. https://doi.org/10.47443/cm.2021.0054.

[12] H. S. Ramane and S. S. Shinde, "Degree exponent polynomial of graphs obtained by some graph operations," *Electronic Notes in Discrete Mathematics*, vol. 63, pp. 161–168, 2017. https://doi.org/10.1016/j.endm.2017.11.010.