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Nonlocal-Adjacency Metric Dimension of Graphs

Rinurwati[1*], Syaima Shafa Az Zahra[1], Falda Isna Andini[1], Soleha[1], Dian Winda Setyawati[1], Komar Baihaqi[1], and Iis Herisman[1]

[1]Department of Mathematics, Institut Teknologi Sepuluh Nopember, Indonesia

Abstract. Let $T = \{t_1, t_2, \dots, t_k\} \subseteq V(G)$ be an ordered subset of the vertex set of a graph G, and let $u \in V(G)$ be a vertex in G. The adjacency metric representation of vertex u with respect to the set T is the k-vector $r_A(u \mid T) = (d_A(u,t_1), d_A(u,t_2), \dots, d_A(u,t_k))$. The set T is called a nonlocal-adjacency metric resolving set of the graph G if $r_A(u \mid T) \neq r_A(w \mid T)$ for every pair of vertices $u, v \in G$ with u not adjacent to v. The minimum cardinality of a nonlocal-adjacency metric resolving set of G is called the nonlocal-adjacency metric dimension of G, denoted by $dim_{Anl}(G)$. In this paper, we present graphs obtained from the degree corona product of two graphs. The degree corona product of graphs G and H, denoted by $G \odot_{\deg} H$, is the graph constructed by taking a graph G and $\sum_{i=1}^{|V(G)|} \deg(v_i)$ copies H_{ij} of graph H, and then connecting every vertex $v_i \in V(G)$ to all vertices in H_{ij} for every $j \in \{1, 2, \dots, \deg(v_i)\}$ and $i \in \{1, 2, \dots, |V(G)|\}$. Furthermore, we determine and analyze the nonlocal-adjacency metric dimension of basic graphs $G_b \in \{P_n, C_n\}$, centered graphs $G_c \in \{K_n, S_n, K_1 + P_n, K_1 + C_n, K_m + \overline{K_n}\}$, and the degree corona product graphs $G_c \odot_{\deg} K_1$. In addition, we provide upper bounds, characterizations of the nonlocal-adjacency metric dimension of graphs, and examples of applications of this concept.

Key words and Phrases: adjacency metric representation, nonlocal-adjacency metric resolving set, nonlocal-adjacency metric dimension, degree corona product.

1. INTRODUCTION

The metric dimension concept of graphs was first introduced by Harary and Malter in 1976 in their book titled *Distance in Graphs* [1]. In this book, Harary and Melter explained that "The metric dimension of graphs is the cardinality of the metric basis of graphs" [1]. Shortly afterward, Slater, Harary, and Melter developed

*Corresponding author: rinur@matematika.its.ac.id
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an algorithm to determine the metric dimension of tree graphs, demonstrating that every tree graph has a fixed metric basis, which consists of terminal vertices [1].

Over time, numerous researchers have applied and further developed the theory of the metric dimensions of graphs. In 1996, this concept was used in robotic navigation modeling by Khuller et al. [2]. In 2004, Sebo and Tannier applied this concept to solve combinatorial optimization problems [3]. Besides conceptual advancements, the metric dimension has also undergone development in various graph operations. In 1970, Roberto Frucht and Frank Harary [4] introduced the corona operation of graphs. In 2010, Hou and Shiu applied and developed the corona operation to obtain the spectrum of the edge corona graphs [5].

In 2011, Iswadi et al. applied the concept of the metric dimension to determine this parameter for corona graphs [6]. In the same year, Yero et al. successfully determined the metric dimension of the recursive corona graphs [7]. Gopalapillai also advanced the corona operation, naming it the neighborhood corona operation, and applied it to obtain the spectrum of the circular corona graphs [8]. In 2017, Rinurwati et al. successfully extended the corona operation into the edge corona operation, enabling the determination of the metric dimension of the edge corona graphs [9]. Additionally, Rinurwati et al. also identified the local metric dimension of m-pendant vertex graphs [10]. In 2021, R.E. Nabila and Rinurwati further developed the corona operation by introducing bobble-neighborhood-corona graph, and studied its metric and edge-metric dimensions [11].

The concept of the nonlocal metric dimension was first introduced in 2022 by Sandi Klavžar and Dorota Kuziak. The nonlocal metric dimension of graphs is the cardinality of the smallest nonlocal resolving set, which represents every pairs of non-adjacent vertices in graphs [12]. Sandi KlavZar and Dorota Kuziak successfully determined the nonlocal metric dimension of block graphs, wheel graphs, and corona vertex graphs [12]. Rinurwati et al. determined the nonlocal edge metric dimension of graphs in 2024 [13].

The notion of adjacency in metric dimensions was initially introduced by Jannesari and Omoomi [14], and has since gained significant attention from various researchers. One notable application of this concept is the study of the local adjacency metric dimension in generalized wheel graphs featuring m-pendant vertices, as previously investigated by Rinurwati et al. [15]. In an effort to deepen the exploration of adjacency-based metric dimensions, including their nonlocal variants, this paper focuses on local adjacency metric dimension in generalized wheel structures and the enhancement of corona operations, particularly in the context of degree corona graphs.

2. Preliminaries

All graphs $G = (V(G), E(G))$ (basic and operating) used in this study are connected and simple. The operation graphs explained here are corona and joint.

The concepts that will be developed are adjacency metric resolving set and nonlocal property.

2.1. **Basic Graphs G_b.**

Various types of basic graphs $G_b \in \{P_n, C_n, S_n, K_n\}$, which are commonly utilized in the construction of new graphs through graph operations, are described in this part. The definitions in this subsection are referenced from [16].

Definition 2.1. [16] *A path graph P_n, is a graph with order n and size $n-1$. The set of vertices in P_n is $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ where $n \geq 1$ and the set of edges $E(P_n) = \{v_iv_{i+1} \mid i \in \{1, 2, \ldots, n-1\}\}$.*

Figure 1 shows the path graph with n vertices, commonly denoted as P_n.

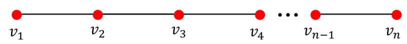

FIGURE 1. Path Graph P_n

Definition 2.2. [16] *A cycle graph C_n is a graph with order n and size n, where $n \geq 3$, with the vertex set $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(C_n) = \{v_iv_{i+1} \mid i \in \{1, 2, \ldots, n-1\}\} \cup \{v_nv_1\}$.*

Figure 2 presents a cycle graph with order n, C_n.

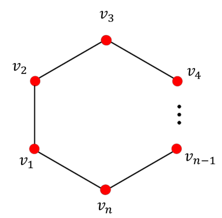

FIGURE 2. Cycle Graph C_n

Definition 2.3. [17] *A star graph is a tree consisting of n vertices, in which a single vertex has degree $n-1$, while each of the remaining $n-1$ vertices has degree one.*

Figure 3 presents a star graph with order n, S_n.

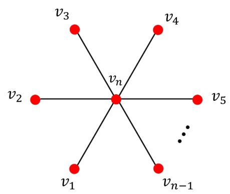

FIGURE 3. Star Graph S_n

Definition 2.4. [16] *A complete graph K_n is a graph that has n vertices, where every vertex forms an edge with each other vertex. A Complete graph K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.*

When $n = 1$, the complete graph $K_n = K_1$ is referred to a trivial graph, and $\overline{K_n}$ is also known as a null graph or an empty graph.

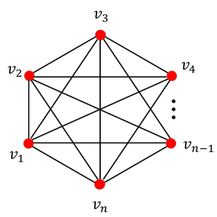

FIGURE 4. Complete Graph K_n

2.2. Some Graphs Operation.

In this part, the corona and joint product between graphs G and H are discussed. The corona operation was originally introduced in [4], and its construction can be found in Definition 2.5. The joint operation was introduced in [17] and described in Definition 2.6.

Definition 2.5. [4] *Let G and H be two graphs. The corona product of G and H, denoted by $G \odot H$, is a graph obtained by taking one copy of G and creating $|V(G)|$ copies of H, denoted as H_i for each $i \in \{1, 2, \ldots, |V(G)|\}$, and then joininh every vertex in H_i to the i^{th} vertex of G. The resulting graph is referred to the corona graph.*

Figure 5 illustrates the graph obtained from the corona operation between P_n, whose vertices are colored red, and K_2, whose vertices are colored blue.

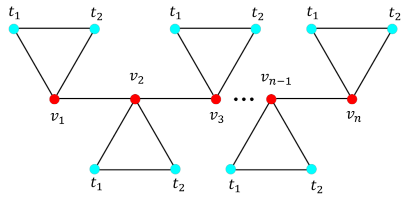

FIGURE 5. $P_n \odot K_2$

Definition 2.6. [18] *The joint operation of G and H, written as $G + H$, is the graph formed by taking G and H, and adding an edge between every vertex in G and every vertex in H. This resulting graph is referred to the joint graph.*

Figure 6 illustrates the graph obtained from the joint operation between P_n, whose vertices are colored red, and K_1, with its single vertex colored blue.

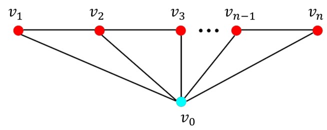

FIGURE 6. $K_1 + P_n$

2.3. Centered Graphs G_c.

A vertex v in graph G is referred to a center vertex of G if it is adjacent to each other vertex in the graph. In other words, a center vertex in G is one that is connected by an edge to all other vertices. The degree of v, $deg(v)$, represents total number of edges that incident to v. If $n = |V(G)|$ is the total number of vertices in G, and w is a center vertex, then $deg(w) = n - 1$. A graph G that contains all of its center vertices is called a centered graph, denoted G_c. A centered graph can be a basic graph or a graph produced from an operation. Some centered graphs for which we will determine their nonlocal-adjacency metric dimensions are: graphs K_n and S_n (including basic graphs), and graphs $K_1 + P_n$, $K_1 + C_n$, and $K_m + \overline{K_n}$ (including the graphs resulting from operations). From Figure 3, we can see that the only center vertex of graph S_n is vertex v_n. All vertices of graph K_n are center vertices, and a center vertex of graphs $K_1 + P_n$, $K_1 + C_n$, and $K_m + \overline{K_n}$ is the vertex of K_1, respectively.

2.4. Adjacency Metric Dimension.

The concept of adjacency distance is introduced by Jannesari and Omoomi as described in Definition 2.7.

Definition 2.7. [14] *Given graph G, and let $T = \{t_1, t_2, \ldots, t_i\} \subseteq V(G)$ be an ordered set. For every vertex $v \in V(G)$, the adjacency metric representation of v with respect to T is $r_A(v|T) = (d_A(v, t_1), d_A(v, t_2), \ldots, d_A(v, t_i))$ with*

$$d_A(v, t_i) = \begin{cases} 0 & , \text{for} \quad v = t_i \\ 1 & , \text{for} \quad v \sim t_i \\ 2 & , \text{for} \quad v \nsim t_i \end{cases}$$

where $v \sim t_i$ means v is adjacent to t_i, and $v \nsim t_i$ means v is not adjacent to t_i, $i \in \{1, 2, ..., k\}$.

The adjacency metric dimension of graph G is defined as follows.

Definition 2.8. [19] *For an ordered subset $T = \{t_1, t_2, \ldots, t_k\} \subseteq V(G)$ and $p \in V(G)$, the adjacency metric representation of p with respect to T is k-vector $r_A(p|T) = (d_A(p, t_1), d_A(p, t_2), \ldots, d_A(p, t_k))$. The set T is an adjacency metric resolving set of G, if $\forall p, q \in V(G)$ with $p \neq q$ holds $r_A(p|T) \neq r_A(q|T)$. The adjacency resolving set with the minimum number of vertices is called the adjacency basis of G. The adjacency metric dimension of G, written as $dim_A(G)$, is defined as the number of elements in its adjacency basis.*

Several researchers, as referenced in [19], have investigated and determined the adjacency metric dimensions for numerous types of connected graphs. For any graph G, two distinct vertices p and q can either be adjacent or non-adjacent. Based on Definition 2.8, a resolving set T in the context of nonlocal-adjacency metric only resolves between vertex pairs that are not adjacent. The minimum number of elements in such a resolving set T is referred to the nonlocal-adjacency metric dimension, represented by $dim_{Anl}(G)$. This study combines the adjacency metric resolving set concept and the nonlocal property in a graph. Thus, we can construct a concept that is produced from this development, as we can see in Definition 3.1.

3. MAIN RESULTS

A formal definition of nonlocal-adjacency metric dimension for graph G is provided below.

Definition 3.1. *Let G be a connected graph and suppose $T = \{t_1, t_2, \ldots, t_\kappa\}$ is an ordered subset of its vertex set. For any vertex $u \in G$, the adjacency metric representation of u with respect to T is given by the κ-vector $r_A(u \mid T) = (d_A(u, t_1), d_A(u, t_2), \ldots, d_A(u, t_\kappa))$, where $d_A(u, t_i)$ denotes the adjacency distance from vertex u to vertex t_i, for $i \in \{1, 2, \ldots, \kappa\}$. The set T is called a nonlocal-adjacency metric resolving set for G if, for every pair of distinct non-adjacent vertices u and w in G, their representations are distinct, or $r_A(u \mid T) \neq r_A(w \mid T)$. The* *minimum cardinality among all such resolving sets is referred as the nonlocal-adjacency basis of G. The number of vertices in this basis is called the nonlocal-adjacency metric dimension of G, denoted by $\dim_{Anl}(G)$.*

3.1. **Nonlocal-Adjacency Metric Dimension of Basic Graph.**

In this section, we determine the nonlocal-adjacency metric dimension of basic graph G_b, specifically non center vertex, that is path graph P_n and cycle graph C_n. Theorem 3.2 describes nonlocal-adjacency metric dimension of P_n, $n \geq 5$.

Theorem 3.2. *Nonlocal-adjacency metric dimension of P_n, $n \geq 5$ is*

$$dim_{Anl}(P_n) = \begin{cases} 2 & \text{, for } n \in \{5, 6\} \\ \lfloor \frac{n+1}{2} \rfloor & \text{, for } n \geq 7. \end{cases}$$

Proof. Let P_n be labeled as shown in the Figure 1. The vertices set of P_n is $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, $deg(v_1) = 1 = deg(v_n)$ and $deg(v_i) = 2$ for $i \in \{2, 3, \ldots, n-1\}$
The proof is analyzed under two separate conditions:
Case 1. For $n \in \{5, 6\}$. We choose $W_5 = \{v_1, v_3\} = W_6$. The vertices in $V(P_n)$ have adjacency metric representations with respect to W_5.

$$r_A(v_1|W_5) = (0, 2) = r_A(v_1|W_6)$$
$$r_A(v_2|W_5) = (1, 1) = r_A(v_2|W_6)$$
$$r_A(v_3|W_5) = (2, 0) = r_A(v_3|W_6)$$
$$r_A(v_4|W_5) = (2, 1) = r_A(v_4|W_6)$$
$$r_A(v_5|W_5) = (2, 2) = r_A(v_5|W_6).$$

$r_A(v_5|W_6) = (2, 2) = r_A(v_6|W_6)$, but v_5 is adjacent to v_6 in P_6. The cardinality of $W_n, |W_n| = \lfloor \frac{n-1}{2} \rfloor = 2$, for $n \in \{5, 6\}$ is minimum, because if we take a set $W'_5 = W'_6$ with cardinality one, then $r_A(v_3|W'_5) = r_A(v_4|W'_5) = r_A(v_5|W'_5) = (2)$, but $v_3 \nsim v_5$. In P_6, $r_A(v_i|W'_6) = (2)$ for $3 \leq i \leq 6$, $v_3 \nsim v_5$, $v_3 \nsim v_6$, and $v_4 \nsim v_6$. So $W'_5 = W'_6$ is not a nonlocal-adjacency metric resolving set of P_5 and P_6. So, $|W_5| = |W_6| = 2$ is minimum, such that $dim_{Anl}(P_n) = 2$ for $n \in \{5, 6\}$.
Case 2. For $n \geq 7$. An ordered set $T = \{v_4, v_6, v_8, \ldots v_{2(\lfloor \frac{n+1}{3} \rfloor + 1)}\}$ is chosen with cardinality T, $|T| = \lfloor \frac{n+1}{3} \rfloor$. The adjacency metric representation vertex $v_i \in V(P_n)$ with respect to T is

$$r_A(v_i \mid T) = \underbrace{\Big(d_A(v_i, v_4), d_A(v_i, v_6), d_A(v_i, v_8), \ldots d_A(v_i, v_{2(\lfloor \frac{n+1}{3} \rfloor + 1)})\Big)}_{\lfloor \frac{n+1}{3} \rfloor}, \text{ with}$$

$$d_A(v_i, v_j) = \begin{cases} 0 & \text{, if } i = j \\ 1 & \text{, if } |i-j| = 1 \\ 2 & \text{, if } |i-j| \geq 2 \end{cases}$$

for $i \in \{1,2,\ldots,n\}$ and $j \in \{4,6,8,\ldots,2\left(\lfloor\frac{n+1}{3}\rfloor+1\right)\}$. We have $r_A(v_1|T) = \underbrace{(2,2,,\ldots 2)}_{\lfloor\frac{n+1}{3}\rfloor} = r_A(v_2|T)$ but $v_1 \sim v_2$ and $r_A(v_3|T) = \underbrace{(1,2,,\ldots 2)}_{\lfloor\frac{n+1}{3}\rfloor}$. Hence, all non-adjacent vertices in P_n, $n \geq 7$, have different adjacency metric representations. Therefore, T is a nonlocal-adjacency metric resolving set of P_n. The cardinality of T, $|T| = \lfloor\frac{n+1}{3}\rfloor$ is minimum. If we take any ordered subset $T' \subseteq T \subseteq V(P_n)$ with $|T'| < |T|$, and we choose $T' = T \setminus \{v_i\}$, $i \in \{4,6,\ldots,2(\lfloor\frac{n+1}{3}\rfloor+1)\}$, then there exists a vertex $v_j \in V(P_n)$ such that $r_A(v_i \mid T') = r_A(v_j \mid T') = \underbrace{(2,2,\ldots,2)}_{\lfloor\frac{n+1}{3}\rfloor-1}$ with $1 \leq i,j \leq n$, $i \neq j$, and $v_i \not\sim v_j$. Therefore, T is a minimum nonlocal-adjacency metric resolving set of P_n. So, the nonlocal-adjacency metric dimension of P_n is $dim_{A_{nl}}(P_n) = |T| = \lfloor\frac{n+1}{3}\rfloor$. □

Nonlocal-adjacency metric dimension of C_n, $n \geq 5$, is given in Theorem 3.3.

Theorem 3.3. *Nonlocal-adjacency metric dimension of C_n, $n \geq 5$ is*

$$dim_{Anl}(C_n) = \begin{cases} \lfloor\frac{n+1}{3}\rfloor & \text{, for odd } n \geq 5 \\ \lfloor\frac{n-1}{2}\rfloor & \text{, for even } n \geq 6. \end{cases}$$

Proof. Suppose the graph C_n is labeled as shown in Figure 2, so that the vertex set of C_n is $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. An ordered subset $T \subseteq V(C_n)$ is chosen, that is $T = \{v_1, v_3, v_5, \ldots\}$. It will be shown that T is a nonlocal-adjacency metric resolving set of C_n. The proof is divided into two cases, that is for odd $n \geq 5$ and for even $n \geq 6$.

Case 1 For odd $n \geq 5$.

Choose the set $T = \underbrace{\{v_1, v_3, v_5, \ldots\}}_{\lfloor\frac{n+1}{3}\rfloor} = \{v_{2k-1} \mid k \in \{1,2,3,\ldots,\lfloor\frac{n+1}{3}\rfloor\}\}$. The adjacency metric representation of $v_i \in V(C_n)$ with respect to T, is $r_A(v_i \mid T) = (d_A(v_i,v_1), d_A(v_i,v_3), d_A(v_i,v_5), \ldots, d_A(v_i,v_{2k-1}))$, with

$$d_A(v_i,v_j) = \begin{cases} 0 & \text{, if } i = j \\ 1 & \text{, if } |i-j| = 1 \\ 2 & \text{, if } |i-j| \geq 2 \end{cases}$$

for $i \in \{1,2,\ldots,n\}$ and $j = 2k-1$, $k \in \{1,2,\ldots,\lfloor\frac{n+1}{3}\rfloor\}$. All vertices have distinct adjacency metric representations, except for v_{n-1} and v_{n-2}, which have the same representation $\underbrace{\{2,2,\ldots,2\}}_{\lfloor\frac{n+1}{3}\rfloor}$, but v_{n-1} is adjacent to v_{n-2}. Hence, the adjacency metric representations of non-adjacent vertices are all different. Therefore, T is a nonlocal-adjacency metric resolving set for C_n when $n \geq 5$ and n is odd. The cardinality of $T = |T| = \lfloor\frac{n+1}{3}\rfloor$ is minimum because if an ordered subset $T' \subseteq T \subseteq$

$V(C_n)$ is taken with $|T'| < |T|$, say $T' = T\backslash\{v_i\}$ for some $i \in \{1, 3, \ldots, 2\left\lfloor\frac{n+1}{3}\right\rfloor - 1\}$, then there exists a vertex $v_j \in V(C_n)$ such that

$$r_A(v_i \mid T') = r_A(v_j \mid T') = \underbrace{\{2, 2, \ldots, 2\}}_{\left\lfloor\frac{n+1}{3}\right\rfloor - 1}$$

with $1 \leq i, j \leq n$, $i \neq j$, and $v_i \nsim v_j$.

Thus, any ordered subset $T' \subseteq T \subseteq V(C_n)$ with $|T'| < |T|$ is not a nonlocal-adjacency metric resolving set of C_n. Consequently, the nonlocal-adjacency metric dimension of C_n is $dim_{Anl}(C_n) = |T| = \left\lfloor\frac{n+1}{3}\right\rfloor$.

Case 2 For $n \geq 6$, n even.

Choose the ordered set $T = \{v_{2k-1} \mid k \in \{1, 2, \ldots, \left\lfloor\frac{n-1}{2}\right\rfloor\}\} \subseteq V(C_n)$. The adjacency metric representation of the vertices $v_i \in V(C_n)$ with respect to T is

$$r_A(v_i \mid T) = \Big(d_A(v_i, v_1), d_A(v_i, v_3), \ldots, d_A(v_i, v_{2\left\lfloor\frac{n-1}{2}\right\rfloor - 1})\Big) \text{, with}$$

$$d_A(v_i, v_j) = \begin{cases} 0 & \text{, if } i = j \\ 1 & \text{, if } |i - j| = 1 \\ 2 & \text{, if } |i - j| \geq 2 \end{cases}$$

for $i \in \{1, 2, \ldots, n\}$ and $j = 2k - 1$, $k \in \{1, 2, \ldots, \left\lfloor\frac{n-1}{2}\right\rfloor\}$.

Hence, every vertex has a distinct nonlocal-adjacency metric representation. So, T is a nonlocal-adjacency metric resolving set of C_n. The cardinality $|T| = \left\lfloor\frac{n-1}{2}\right\rfloor$ is minimum because if any ordered subset $T' \subseteq T \subseteq V(C_n)$ is taken with $|T'| < |T|$, say $T' = T \setminus \{v_i\}$, for some $i \in \{1, 3, \ldots, 2\left\lfloor\frac{n-1}{2}\right\rfloor - 1\}$, then there exists a vertex $v_j \in V(C_n)$ such that $r_A(v_i \mid T') = r_A(v_j \mid T') = \underbrace{\{2, 2, \ldots, 2\}}_{\left\lfloor\frac{n-1}{2}\right\rfloor - 1}$, with $1 \leq i, j \leq n$, $i \neq j$, and $v_i \nsim v_j$.

Hence, any ordered subset $T' \subseteq T \subseteq V(C_n)$ with $|T'| < |T|$ cannot serve as a nonlocal-adjacency metric resolving set for C_n. Consequently, the nonlocal-adjacency metric dimension of C_n is given by $dim_{\mathrm{Anl}}(C_n) = |T| = \left\lfloor\frac{n-1}{2}\right\rfloor$, where n is even and $n \geq 6$. □

3.2. **Nonlocal-Adjacency Metric Dimension of Centered Graphs G_c.**

Non local-adjacency metric dimension of centered graphs that are basic graphs are presented in Theorem 3.4 and Theorem 3.5. The following theorems, up to Theorem 3.8, address the nonlocal-adjacency metric dimension of centered graph that are the result of an operation. Characterization of nonlocal-adjacency metric dimension of a graph is presented in Theorem 3.4.

Theorem 3.4. *Nonlocal-adjacency metric dimension of G_c, $dim_{Anl}(G_c) = 0$ if and only if $G_c = K_n$.*

Proof. (⇐) It is known that $G_c = K_n$. From Definition 2.4, we know that all vertices in $G_c = K_n$ are adjacent to each other. So, there is no vertex in G_c that is not adjacent to another. Hence, there is no vertex that can be chosen as a candidate for the element of the nonlocal-adjacency metric resolving set of G_c. Therefore, the nonlocal-adjacency metric resolving set of G_c is the empty set. Thus, the minimum cardinality of the nonlocal-adjacency metric resolving set is zero or $dim_{Anl}(G_c) = 0$.

(⇒) It is known that the nonlocal-adjacency metric dimension of G_c is zero. This means the cardinality of the nonlocal-adjacency metric resolving set of G_c is zero. Therefore, the nonlocal-adjacency metric resolving set of G_c is the empty set. This means there is no vertex in G_c that is not adjacent to another. Hence, all vertices in G_c are adjacent to each other. The only connected graph that has this property is K_n. □

So, zero is lower bound of the nonlocal-adjacency metric dimension of graphs. This lower bound is sharp because $dim_{Anl}(G) = 0$ is achieved by $G = K_n$. Thus, we can write $0 \leq dim_{Anl}(G)$.

Theorem 3.5 presents the nonlocal-adjacency metric dimension of star graph S_n, $n \geq 4$.

Theorem 3.5. *Nonlocal-adjacency metric dimension of* S_n, $n \geq 4$ *is* $dim_{Anl}(S_n) = n - 2$.

Proof. Suppose the graph S_n is labeled as shown in Figure 3. The vertex set of S_n is $V(S_n) = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$, with $\deg(v_i) = 1$ for $i \in \{1, 2, 3, \ldots, n-1\}$, and $\deg(v_n) = n - 1$. Every vertex v_i with $i \in \{1, 2, 3, \ldots, n-1\}$ is not adjacent to one another. Let us choose the set $T = \{v_1, v_2, v_3, \ldots, v_{n-2}\} \subseteq V(S_n)$. We will show that T is a nonlocal-adjacency metric resolving set for S_n. The adjacency metric representations of the vertices in S_n with respect to the set T are:

$$r_A(v_n \mid T) = \underbrace{(1, 1, 1, \ldots, 1)}_{(n-2)}, \quad r_A(v_{n-1} \mid T) = \underbrace{(2, 2, 2, \ldots, 2)}_{(n-2)},$$

$$r_A(v_i \mid T) = \left(d_A(v_i, v_1), d_A(v_i, v_2), \ldots, d_A(v_i, v_{n-2})\right),$$

with

$$d_A(v_i, v_j) = \begin{cases} 0 & , \text{if } i = j \\ 2 & , \text{if } i \neq j \end{cases}$$

and $v_i \nsim v_j$ for all $i, j \in \{1, 2, 3, \ldots, n-1\}, i \neq j$, while $v_n \sim v_i$ for all $i \in \{1, 2, 3, \ldots, n-1\}$.

Hence, all pairs of non-adjacent vertices in S_n have distinct representations. Therefore, T is a nonlocal-adjacency metric resolving set for S_n. The cardinality of T, $|T| = n - 2$ is minimum because if we take any ordered subset $T' \subset T \subseteq V(S_n)$ with $|T'| < |T| = n-2$, and we choose $T' = T \setminus \{v_i\}$, $i \in \{1, 2, ..., n-1\}$ then there is exist a vertex $v_j \in V(S_n)$ such that $r_A(v_i \mid T') = r_A(v_j \mid T') = \underbrace{(2, 2, 2, \ldots, 2)}_{n-3}$, with $1 \leq i, j \leq n-1, i \neq j, v_i \nsim v_j$. Therefore, T is a minimum nonlocal-adjacency metric resolving set for S_n. Hence, the nonlocal-adjacency metric dimension of S_n is $dim_{A_{nl}}(S_n) = n - 2$. □

The next centered graph to be discussed is the fan graph. Let G_c be a fan graph $K_1 + P_n$ whose vertices are labeled as shown in figure 6 .The vertex set of $K_1 + P_n$ is $V(K_1 + P_n) = \{v_1, v_2, v_3, \ldots, v_n, v_{n+1}\}$, so the order of the graph $K_1 + P_n$ is $|V(K_1 + P_n)| = n + 1 = m$. The nonlocal-adjacency metric dimension of the graph $K_1 + P_n$ is given in the following theorem.

Theorem 3.6. *Nonlocal-adjacency metric dimension of fan graph $F_m = K_1 + P_n$, for $n \geq 4$ is*

$$dim_{Anl}(F_m) = \begin{cases} 1 & , \text{for } m = 5 \\ 2 & , \text{for } m = 6 \\ \left\lfloor \dfrac{n-3}{2} \right\rfloor & , \text{for } m \geq 7 \end{cases}$$

with $m = n + 1$.

Proof. Let the vertices of graph $K_1 + P_n$ labeled as $v_0, v_1, \ldots, v_n$, where v_0 is the vertex of degree n, v_1 and v_n are the vertices of degree two, and $v_2, v_3, \ldots, v_{n-1}$ are the vertices of degree three. Hence, the vertex set of $K_1 + P_n$ is $V(K_1 + P_n) = \{v_0, v_1, v_2, \ldots, v_n\}$. The proof is given in three cases.
Case 1. For $m = 5$.
Then $n = m - 1 = 4$, so $V(K_1 + P_4) = \{v_0, v_1, v_2, v_3, v_4\}$.
Choose the set $T = \{v_1\}$. The vertices in $K_1 + P_4$ have the following adjacency metric representations with respect to the set T.

$$r_A(v_1 \mid T) = (0), \quad r_A(v_2 \mid T) = r_A(v_0 \mid T) = (1), \quad r_A(v_3 \mid T) = r_A(v_4 \mid T) = (2)$$

However, $v_2 \sim v_0$ and $v_3 \sim v_4$. Therefore, T is a nonlocal-adjacency metric resolving set of $K_1 + P_4$, and the cardinality T, $|T| = 1$ is minimum. Thus, the nonlocal-adjacency metric dimension of $K_1 + P_4$ is $dim_{Anl}(K_1 + P_4) = 1$.
Case 2. For $m = 6$.
Then $n = 6 - 1 = 5$, so $V(K_1 + P_5) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$.
Choose the set $T = \{v_1, v_3\}$. The vertices in $K_1 + P_5$ have the following adjacency metric representations with respect to the set T

$$r_A(v_1 \mid T) = (0, 2), \quad r_A(v_2 \mid T) = r_A(v_0 \mid T) = (1, 1),$$
$$r_A(v_3 \mid T) = (2, 0), \quad r_A(v_4 \mid T) = (2, 1), \quad r_A(v_5 \mid T) = (2, 2).$$

Nevertheless, v_2 is adjacent to v_0. Therefore, T constitutes a nonlocal-adjacency metric resolving set for the graph $K_1 + P_5$, and its cardinality $|T| = 2$ is minimum. To justify minimumity, assume there exists a set T' with $|T'| = 1$, for instance $T' = T \setminus \{v_i\}$, where $i \in \{1, 3\}$. In such a case, at least one pair of distinct vertices in the graph would share an identical nonlocal-adjacency metric representation with respect to T'.

If $T' = T \setminus \{v_1\} = \{v_3\}$, then $r_A(v_3 \mid T') = r_A(v_5 \mid T') = (2)$, and $v_3 \nsim v_5$
If $T' = T \setminus \{v_3\} = \{v_1\}$, then $r_A(v_1 \mid T') = r_A(v_5 \mid T') = (2)$, and $v_1 \nsim v_5$.

Thus, T' is not a nonlocal-adjacency metric resolving set. Therefore, $dim_{Anl}(K_1 + P_5) = 2$.

Case 3. For $m \geq 7$.
Choose the ordered set $T = \{v_4, v_6, v_8, \ldots, v_{2\lfloor \frac{m-1}{2} \rfloor}\}$. The vertices $v_i \in V(K_1 + P_n)$, have the adjacency metric representations with respect to T.

$$r_A(v_i \mid T) = \left(d_A(v_i, v_4), d_A(v_i, v_6), d_A(v_i, v_8), \ldots, d_A\left(v_i, v_{2\lfloor \frac{m-1}{2} \rfloor}\right)\right)$$

with the distance defined by:

$$d_A(v_i, v_j) = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } |i - j| = 1 \\ 2, & \text{if } |i - j| \geq 2. \end{cases}$$

Hence, we obtain the adjacency metric representation $r_A(v_1 \mid T) = r_A(v_2 \mid T) = \underbrace{(2, 2, \ldots, 2)}_{\lfloor \frac{m-3}{2} \rfloor}$, with $v_1 \sim v_2$. While all other vertices in $V(K_1 + P_n) \setminus \{v_1, v_2\}$ have distinct representations. Thus, T is a nonlocal-adjacency metric resolving set, and its cardinality $|T| = \left\lfloor \frac{m-3}{2} \right\rfloor$ is minimum, because if we choose any subset $T' \subseteq T$ with $|T'| < |T|$ will result the adjacency metric representasion $r_A(v_1 \mid T') = r_A(v_2 \mid T') = \underbrace{(2, 2, \ldots, 2)}_{\lfloor \frac{m-1}{2} \rfloor - 1}$, with $v_1 \nsim v_2$. Thus, T' is not a nonlocal-adjacency metric resolving set. Therefore, we conclude $dim_{Anl}(K_1 + P_n) = \left\lfloor \frac{m-3}{2} \right\rfloor$. □

Let G_c be a wheel graph $K_1 + C_n$ with vertices labeled as shown in the following figure.

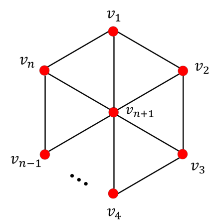

FIGURE 7. $K_1 + C_n$

The vertex set of $K_1 + C_n$ is $V(K_1 + C_n) = \{v_1, v_2, \ldots, v_{n+1}\}$. The nonlocal-adjacency metric dimension of $K_1 + C_n$ is given in the following theorem.

Theorem 3.7. *Nonlocal-adjacency metric dimension of $K_1 + C_n$, for $n \geq 5$, is*

$$dim_{Anl}(K_1 + C_n) = \begin{cases} \left\lfloor \frac{n+1}{3} \right\rfloor & \text{, for odd } n \geq 5 \\ \left\lfloor \frac{n-1}{2} \right\rfloor & \text{, for even } n \geq 6. \end{cases}$$

Proof. Proof is divided into two cases.
Case 1. For odd $n \geq 5$. Choose the ordered set $T = \{v_1, v_3, v_5, \ldots, v_{2\lfloor \frac{n+1}{3} \rfloor - 1}\} \subseteq V(K_1 + C_n)$. For every vertex v_i in $K_1 + C_n$, the adjacency metric representation with respect to T is

$$r_A(v_i|T) = \Big(d_A(v_i, v_1), d_A(v_i, v_3), d_A(v_i, v_5), \ldots, d_A(v_i, v_{2\lfloor \frac{n+1}{3} \rfloor - 1})\Big)$$

with

$$d_A(v_i, v_j) = \begin{cases} 0 & \text{, if } i = j \\ 1 & \text{, if } |i - j| = 1 \\ 2 & \text{, if } |i - j| \geq 2 \end{cases}$$

for $i \in \{1, 2, \ldots, n+1\}$ and $j \in \{1, 3, 5, \ldots, 2\left\lfloor \frac{n+1}{3} \right\rfloor - 1\}$. Specifically for $n = 5$, $r_A(v_1|T) = r_A(v_{n+1}|T) = (1, 1)$ but v_1 and v_{n+1} are adjacent. Similarly for $n = 7$, $r_A(v_1|T) = r_A(v_{n+1}|T) = (1, 1)$ and $r_A(v_{n-1}|T) = r_A(v_{n-2}|T) = (2, 2)$, but the vertices with the same adjacency metric representation are still adjacent. For $n \geq 9$, $r_A(v_{n-1}|T) = r_A(v_{n-2}|T) = \underbrace{(2, 2, \ldots, 2)}_{\lfloor \frac{n+1}{3} \rfloor}$, but $v_{n-1} \sim v_{n-2}$. It follows that the adjacency metric representation of non-adjacent vertices in $K_1 + C_n$ is all distinct. Consequently, T is a nonlocal-adjacency metric resolving set of $K_1 + C_n$. The cardinality of T, $|T| = \left\lfloor \frac{n+1}{3} \right\rfloor$, is minimum, since if any ordered set $T' \subseteq T \subseteq V(K_1 + C_n)$ with $|T'| < |T|$ is taken, say $T' = T \setminus \{v_j\}$, $j \in \{1, 3, \ldots, \left\lfloor \frac{n+1}{3} \right\rfloor - 1\}$, then there exist vertices v_i, $i \in \{1, 2, ..., n\}$ such that $r_A(v_i|T') = r_A(v_j|T') = \underbrace{(2, 2, \ldots, 2)}_{\lfloor \frac{n+1}{3} \rfloor - 1}$, where $1 \leq i, j \leq n$, and $i \neq j$, that is T' not being a nonlocal-adjacency metric resolving set of $K_1 + C_n$. Therefore, T is the nonlocal-adjacency metric basis of $K_1 + C_n$, and $dim_{Anl}(K_1 + C_n) = |T| = \left\lfloor \frac{n+1}{3} \right\rfloor$, for odd $n \geq 5$.
Case 2. For even $n \geq 6$. Choose the ordered set $T = \{v_1, v_2, v_4, v_6, \ldots, v_{2\lfloor \frac{n-3}{2} \rfloor}\}$. The adjacency metric representation of the vertices in $K_1 + C_n$ is

$$r_A(v_i|T) = \Big(d_A(v_i, v_1), d_A(v_i, v_2), d_A(v_i, v_4), \ldots, d_A(v_i, v_{2\lfloor \frac{n-3}{2} \rfloor})\Big)$$

with

$$d_A(v_i, v_j) = \begin{cases} 0 & \text{, if } i = j \\ 1 & \text{, if } |i - j| = 1 \\ 2 & \text{, if } |i - j| \geq 2 \end{cases}$$

for $i \in \{1, 2, \ldots, n{+}1\}$ and $j \in \{1, 2, 4, \ldots, 2\left\lfloor \frac{n-3}{2} \right\rfloor\}$. For every $n \geq 6$, $r_A(v_{n-1}|T) = r_A(v_{n-2}|T) = \underbrace{(2, 2, \ldots, 2)}_{\lfloor \frac{n-3}{2} \rfloor}$, but $v_{n-1} \sim v_{n-2}$. Therefore, the adjacency metric representation of non-adjacent vertices is distinct. Thus, T is a nonlocal-adjacency metric resolving set of $K_1 + C_n$. The cardinality of T, $|T| = \left\lfloor \frac{n-3}{2} \right\rfloor$, is minimum, since if any ordered set $T' \subseteq T \subseteq V(K_1 + C_n)$ with $|T'| < |T|$ is taken, say $T' = T - \{v_j\}$, where $j \in \{1, 2, 4, 6, \ldots, 2\left\lfloor \frac{n-3}{2} \right\rfloor\}$, then there exist vertices $v_i \in V(K_1 + C_n)$ such that $r_A(v_i|T') = r_A(v_j|T') = \underbrace{(2, 2, \ldots, 2)}_{\lfloor \frac{n-3}{2} \rfloor - 1}$, with $1 \leq i, j \leq n$, $i \neq j$, and $v_i \nsim v_j$. Therefore, T is the nonlocal-adjacency metric basis of $K_1 + C_n$, and $dim_{Anl}(K_1 + C_n) = |T| = \left\lfloor \frac{n-3}{2} \right\rfloor$, for even $n \geq 6$. □

Given a complete graph K_m with the vertex set $V(K_m) = \{v_1, v_2, \ldots, v_m\}$ and a graph $\overline{K_n}$ with the vertex set $V(\overline{K_n}) = \{x_1, x_2, \ldots, x_n\}$. The graph $K_m + \overline{K_n}$ is obtained by taking the graph K_m and the graph $\overline{K_n}$. Then, every vertex in K_m is connected to all the vertices in $\overline{K_n}$. The graph $K_m + \overline{K_n}$ has vertex set $V(K_m + \overline{K_n}) = V(K_m) \cup V(\overline{K_n}) = \{v_1, v_2, \ldots, v_m, x_1, x_2, \ldots, x_n\}$ and edge set $E(K_m + \overline{K_n}) = E(K_m) \cup \{v_i x_j \mid i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}\}$. This can be expanded as $E(K_m + \overline{K_n}) = \{v_1v_2, v_1v_3, \ldots, v_1v_m\} \cup \{v_2v_3, v_2v_4, \ldots, v_2v_m\} \cup \cdots \cup \{v_{m-1}v_m\} \cup \{v_1x_1, v_2x_1, \ldots, v_mx_1\} \cup \{v_1x_2, v_2x_2, \ldots, v_mx_2\} \cup \cdots \cup \{v_1x_n, v_2x_n, \ldots, v_mx_n\}$.

As an illustration, the graph $K_m + \overline{K_n}$ can be seen in Figure 8.

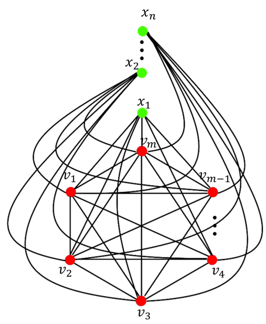

FIGURE 8. Graph $K_m + \overline{K_n}$

In Figure 8], the red vertex represents the vertex of K_m and the green vertex represents the vertex of $\overline{K_n}$. The Nonlocal-adjacency metric dimension of $K_m + \overline{K_n}$, with $m \geq 3$ and $n \geq 1$, is presented below.

Theorem 3.8. *Let G_c be a complete graph K_m with $m \geq 3$ and let H be an empty graph $\overline{K_n}$, $n \geq 1$. Nonlocal-adjacency metric dimension of $K_m + \overline{K_n}$ is $dim_{Anl}(K_m + \overline{K_n}) = n - 1$.*

Proof. Let $T = \{x_j \mid j \in \{1,2,\dots,n-1\}\} \subseteq V(\overline{K_n})$. Adjacency metric representations of the vertices of $K_m + \overline{K_n}$ with respect to T are $r_A(v_i \mid T) = (d_A(v_i,x_1), d_A(v_i,x_2),\dots,d_A(v_i,x_{n-1}))$, $i \in \{1,2,\dots,m\}$ with $d_A(v_i,x_j) = 1$, $j \in \{1,2,...,n\}$ and $r_A(x_h \mid T) = (d_A(x_h,x_1), d_A(x_h,x_2),\dots,d_A(x_h,x_{n-1}))$, $h \in \{1,2,\dots,n\}$ with

$$d_A(x_h,x_j) = \begin{cases} 0 & \text{, if } j = h \\ 2 & \text{, if } j \neq h. \end{cases}$$

Since all of vertices that $r_A(v_i \mid T)$, $i \in \{1,2,\dots,m\}$ are all the same, which is equal to $\underbrace{(1,1,\dots,1)}_{(n-1)}$ where v_i are all adjacent to each other, then the adjacency metric representation of every two adjacent vertices is not different. However, for every $h,j \in \{1,2,\dots,n\}$ with $h \neq j$, $x_h \nsim x_j$ and $r_A(x_h \mid T) \neq r_A(x_j \mid T)$. Hence, although T serves as a nonlocal-adjacency metric resolving set for the graph $K_m + \overline{K_n}$, this does not establish it as a lower bound. Therefore, the nonlocal-adjacency metric dimension satisfies $\dim_{Anl}(K_m + \overline{K_n}) \leq n-1$.

Now, we show that $dim_{Anl}(K_m + \overline{K_n}) \geq n-1$. Let $T' = \{x_j \mid j \in \{1,2,\dots,n-1\}\}$ be a nonlocal-adjacency metric resolving set with $|T'| = n-1$. Assume that an ordered set T' is another minimum nonlocal-adjacency metric resolving set, or $|T'| < |T| = n-1$. If we select an ordered set $T' \subseteq T - \{x_h, x_j\}$, $1 \leq h,j \leq n$, $h \neq j$, so there is exist two vertices $x_h, x_j \in V(K_m + \overline{K_n})$ such that $r_A(x_h \mid T') = r_A(x_j \mid T') = \underbrace{(2,2,\dots,2)}_{<(n-1)}$. It should be noted that T is not a nonlocal-adjacency metric resolving set, which is contrary to the assumption. So, the lower bound is $dim_{Anl}(K_m + \overline{K_n}) \geq n-1$. We conclude that $dim_{Anl}(K_m + \overline{K_n}) = n-1$. □

3.3. **Nonlocal-Adjacency Metric Dimension of Degree Corona Graphs.**

This subsection explained definition of degree corona graph and the nonlocal-adjacency metric dimension of the graph resulting from degree corona operation. Degree corona operation is an extension of the corona operation discovered by [4]. The following is the definition of a degree corona graph.

Definition 3.9. *The degree corona graph of two graphs G and H, denoted by $G \odot_{\deg} H$, is a graph obtained by taking a graph G and $\sum_{i=1}^{|V(G)|} \deg(v_i)$ copies of the graph H, denoted by H_{ij} (the ij^{th} copy of H). For each i^{th} vertex $v_i \in V(G)$, connect v_i to every vertex in each H_{ij}, where $i \in \{1,2,\dots,|V(G)|\}$ and $j \in \{1,2,\dots,\deg(v_i)\}$.*

As an illustration, the degree corona graph can be seen in Figure 9

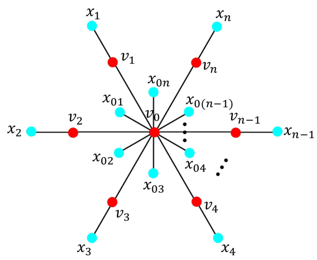

FIGURE 9. Graph $S_n \odot_{\deg} K_1$

In Figure 9, the red vertex represents the vertex of the graph S_n and the blue vertex represents the vertex of the graph K_1.

Observation 3.10. *Given a connected graph G, and H is a graph with at least two vertices. In any subgraph H_i of $G \odot_{\deg} H$, the vertices u and v are considered similar adjacency distance with respect to H_i.*

Based on the above observation, we can formulate the lemma below.

Lemma 3.11. *Let $G = (V(G), E(G))$ be a connected graph with order $n \geq 2$, and let H be a graph of order at least two such that H is not isomorphic to the complete graph K_n. For each i, let $H_i = (V_i(H_i), E_i(H_i))$ represent the subgraph in $G \odot_{\deg} H$ associated with the i^{th} copy of H.*

(1) For any pair of vertices $\alpha, \beta \in V_i(H_i)$, it holds that $d_{A_{G\odot_{\deg}H}}(\alpha, p) = d_{A_{G\odot_{\deg}H}}(\beta, p)$ for every vertex $p \in V(G \odot_{\deg} H) \setminus V_i(H_i)$.
(2) If T is a nonlocal-adjacency metric resolving set for the graph $G \odot_{\deg} H$, then for each $i \in \{1, 2, \ldots, n\}$, we have $V_i(H_i) \cap T \neq \emptyset$.
(3) If T is a nonlocal-adjacency metric resolving set with minimum cardinality for $G \odot_{\deg} H$, then we have $V(G) \cap T = \emptyset$.
(4) If H be a graph, and let T be a nonlocal-adjacency metric resolving set for the graph $G \odot_{\deg} H$. Then, for each $i \in \{1, 2, \ldots, n\}$, the subset $V_i(H_i) \cap T$ serves as a nonlocal-adjacency metric resolving set for the subgraph H_i.

Proof. (1) Suppose $q = \beta_i \in V(G)$. The conclusion follows immediately from the fact that $d_{A_{G\odot_{\deg}H}}(\alpha, p) = d_{A_{G\odot_{\deg}H}}(\alpha, q) + d_{A_{G\odot_{\deg}H}}(q, p) = d_{A_{G\odot_{\deg}H}}(\beta, q) + d_{A_{G\odot_{\deg}H}}(q, p) = d_{A_{G\odot_{\deg}H}}(\beta, p)$.
(2) We suppose $V_i(H_i) \cap T = \emptyset$ for some $i \in \{1, 2, \ldots, n\}$. Let $p, q \in V_i(H_i)$. By (1), we have $d_{A_{G\odot_{\deg}H}}(p, \alpha) = d_{A_{G\odot_{\deg}H}}(q, \alpha)$ for every vertex $\alpha \in T$, which is a contradiction.
(3) In the following, we show that $T' = T \setminus V(G)$ is a nonlocal-adjacency metric resolving set for $G \odot_{\deg} H$. Let p, $q \in V(\odot_{\deg} H)$ with $p \neq q$. We examine the proof by considering the following cases.

Case 1. $p, q \in V_i(H_i)$. By (1), we conclude that there exists $\beta \in V_i(H_i) \cap T'$ such that $d_{A_{G\odot_{\deg}H}}(p,\beta) \neq d_{A_{G\odot_{\deg}H}}(q,\beta)$.

Case 2. $p \in V_i$, and $q \in V_j$, with $i \neq j$. Let $\beta \in V_i(H_i) \cap T'$. Then we have $d_{A_{G\odot_{\deg}H}}(p,\beta) = 2 = d_{A_{G\odot_{\deg}H}}(q,\beta)$.

Case 3. $p, q \in V(G)$. Let $p = \beta_i$ and $\beta \in V_i \cap T'$. Then we have $d_{A_{G\odot_{\deg}H}}(p,\beta) = 1 < (1 + d_{A_{G\odot_{\deg}H}}(q,p)) = d_{A_{G\odot_{\deg}H}}(q,\beta) = 2$.

Case 4. $q \in V_i(H_i)$, and $q \in V(G)$. If $q \sim q$, then $q = \beta_i$. Let $\beta_j \in V$, with $j \neq i$, and let $\beta \in V_j \cap T'$. Then we have $d_{A_{G\odot_{\deg}H}}(p,\beta) = 1 + d_{A_{G\odot_{\deg}H}}(q,\beta) = 2 = d_{A_{G\odot_{\deg}H}}(q,\beta)$. For $q \not\sim q = \beta_i$, we take $\beta \in V_i \cap T'$ and obtain $d_{A_{G\odot_{\deg}H}}(p,\beta) = d_{A_{G\odot_{\deg}H}}(p,q) + d_{A_{G\odot_{\deg}H}}(q,\beta) > d_{A_{G\odot_{\deg}H}}(q,\beta)$. Therefore, T' is a nonlocal-adjacency metric resolving set for $G \odot_{\deg} H$.

(4) Let $T_i = T \cap V_i(H_i)$. For any $q \in T_i$, the conclusion is immediate. Now suppose $p, q \in V_i \setminus T_i$. Since T is a nonlocal-adjacency metric resolving set for $G \odot_{\deg} H$, it follows that $r(p \mid T) \neq r(q \mid T)$. According to (1), for every vertex α in $G \odot_{\deg} H$ that does not belong to $V_i(H_i)$, we have $d_{A_{G\odot_{\deg}H}}(p,\alpha) = d_{A_{G\odot_{\deg}H}}(q,\alpha)$. Therefore, there must exist some $\beta \in T_i$ such that $d_{A_{G\odot_{\deg}H}}(p,\beta) \neq d_{A_{G\odot_{\deg}H}}(q,\beta)$. This implies that either $\beta \sim p$ and $\beta \not\sim q$, or the other way around. In the first case, we obtain $d_{A_{G\odot_{\deg}H}}(p,\beta) = d_{H_i}(p,\beta) = 1$, while $d_{A_{G\odot_{\deg}H}}(q,\beta) > 1$. The second case, where $\beta \sim q$ and $\beta \not\sim p$, yields a similar conclusion. Hence, T_i forms a nonlocal-adjacency metric resolving set for H_i. □

An observation regarding the element of basis of the degree corona graph was also obtained as follows.

Observation 3.12. *Given a connected graph G, and H be an empty graph containing one or more vertices. In any subgraph H_i (the i^{th}-copy of H) of the graph $G \odot_{\deg} H$, the vertex $q \in V_i(H_i)$ is the element of basis of the graph $G \odot_{\deg} H$.*

In addition to the definition of the degree corona graph, this research also presents a theorem on the nonlocal-adjacency metric dimension of the degree corona graph of centered graphs $G_c \in \{S_n, K_1 + P_n, K_1 + C_n, K_m + \overline{K_n}\}$ and the trivial graph K_1.

Theorem 3.13. *Let G_c be a centered graph, $G_c \in \{S_n, K_1+P_n, K_1+C_n, K_m+\overline{K_n}\}$ with $n \geq 4$ and H is the trivial graph K_1. The nonlocal-adjacency metric dimension of the degree corona graph $G_c \odot_{\deg} K_1$,*

$$dim_{Anl}(G_c \odot_{\deg} K_1) = \left\lceil \sum_{i=1}^{|V(G_c)|} \deg(v_i) \right\rceil - 1.$$

Proof. Based on Observation 3.12, let T be the nonlocal-adjacency metric resolving set of $G_c \odot_{\deg} K_1$, that is $T = \{x_1, x_2, \ldots, x_{\left[\sum_{i=1}^{|V(G_c)|} \deg(v_i)\right]-1}\}$ but this is insufficient to establish it as the lower bound. So, the upper bound is $dim_{Anl}(G_c \odot_{\deg} K_1) \leq \left[\sum_{i=1}^{|V(G_c)|} \deg(v_i)\right] - 1$. If the basis selection, aside from the vertex of H_i, then it is not a basis.

Now, we show that $dim_{Anl}(G_c \odot_{\deg} K_1) \geq \left[\sum_{i=1}^{|V(G_c)|} \deg(v_i)\right] - 1$. Let $T = \{x_1, x_2, \ldots, x_{\left(\left[\sum_{i=1}^{|V(G_c)|} \deg(v_i)\right]-1\right)}\}$ be a nonlocal-adjacency metric resolving set with $|T'| = \left[\sum_{i=1}^{|V(G_c)|} \deg(v_i)\right] - 1$. Assume that an ordered set T' is another minimum nonlocal-adjacency metric resolving set, or $|T'| < |T| = \left[\sum_{i=1}^{|V(G_c)|} \deg(v_i)\right] - 1$. If we select an ordered set $T' \subseteq T \backslash \{x_i, x_j\}, \quad 1 \leq i, j \leq \left[\sum_{i=1}^{|V(G_c)|} \deg(v_i)\right], \quad i \neq j$, so that there exist two vertices $x_i, x_j \in V(G_c \odot_{\deg} K_1)$ such that $r_A(x_i \mid T') = r_A(x_j \mid T') = \underbrace{(2, 2, \ldots, 2)}_{<\left(\left[\sum_{i=1}^{|V(G_c)|} \deg(v_i)\right]-1\right)}$. It should be noted that T' is not a nonlocal-adjacency metric resolving set, which is contrary to the assumption. So, the lower bound is $dim_{Anl}(G_c \odot_{\deg} K_1) \geq \left[\sum_{i=1}^{|V(G_c)|} \deg(v_i)\right] - 1$. We conclude that $dim_{Anl}(G_c \odot_{\deg} K_1) = \left[\sum_{i=1}^{|V(G_c)|} \deg(v_i)\right] - 1$. □

3.4. Upper Bound of $dim_{Anl}(G)$.

Based on the above research results, the upper bound of the nonlocal-adjacency metric dimension is obtained as follows

$$dim_{Anl}(G) \leq n - r, \quad \text{achieved by} \quad G = S_n, \quad r = \text{radius}(G).$$

3.5. Example of Application.

In the modern era, where online shopping platforms are rapidly growing, the logistics distribution system has become a key focus for optimization by shipping and expedition companies. In a distribution network, when two major cities are directly connected through a main transportation route, the shipping schedule can be carried out routinely and efficiently. However, for cities that are not directly connected (non-adjacent), shipment planning must be done carefully; the logistics must pass through other cities, and the shipping schedules need to be adjusted to minimize travel time and costs. In determining the metric dimension of nonlocal-adjacency, the vertices in the graph represent the cities. Cities that have a direct transportation route are represented as adjacent vertices in the graph, while cities without a direct transportation route are represented as non-adjacent vertices. These non-adjacent vertices need to be resolved, as they require a transit warehouse for collecting logistics, which are then scheduled for shipment according to the delivery schedule. By modeling the distribution network as a graph and applying the concepts of adjacency and metric dimension, we can identify key points such as warehouse locations and transit warehouses that can improve overall distribution efficiency.

4. CONCLUDING REMARKS

This study presents the determination and analysis of the nonlocal-adjacency metric dimension for basic graphs P_n and C_n, centered graphs including K_n, S_n, $K_1 + P_n$, $K_1 + C_n$, $K_m + \overline{K_n}$, and for the degree corona product of a centered graph $G_c \in \{S_n, K_1 + P_n, K_1 + C_n, K_m + \overline{K_n}\}$ with the trivial graph K_1. From the discussion, it can be concluded that: Adding a vertex as a central vertex of a connected graph G one by one can increase the order of the graph, while the size of the nonlocal-adjacency metric dimension of graphs obtained remains the same as the original graph, thus $dim_{Anl}(K_1 + (K_1 + \cdots + (K_1 + (K_1 + G)) \cdots)) = dim_{Anl}(G)$. Also, we obtained the upper bound, characteristics, and an example of the application of nonlocal-adjacency metric dimension concept of graphs.

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