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Inequality Estimations of Subclass of Univalent Functions Involving Raducanu-Orhan Operator

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Abstract. The vast number of new papers that have been written about the univalent function in recent years shows how fascinating it is. Univalent functions are injective analytic functions, which means that they do not take the same value at various places within their domain. Univalent functions are important in complex analysis and have many uses in other areas of mathematics, physics, and engineering. These days, there is a high need for operators of normalized analytic functions, particularly differential and integral operator. Operators are widely used in numerous mathematical and scientific domains. Differential equations can be solved using these operators, which are also used to describe a wide range of physical phenomena. Many researchers have reviewed and discussed a substantial amount of material for the operators. In this work, the new subclass of univalent functions is defined by the Raducanu-Orhan differential operator. In addition, the coefficient inequalities, extreme points, integral means of inequality, and Fekte-Szego inequality for the subclass have been obtained.

Key words and Phrases: univalent functions, differential operator, subordination, coefficient inequality, Fekete-Szego inequality.

1. INTRODUCTION

A univalent analytic function, sometimes called a univalent function of a complex variable, is an analytic function that is also univalent, meaning it is a one-to-one. In many branches of mathematics, such as differential equations and

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complex analysis, uniform functions play a significant role because of their unique characteristics and uses. For example, univalent functions can be used to express conformal mappings, which maintain angles locally; they are particularly relevant in complex analysis. In complex analysis, the study of the class of univalent functions is important because it has links to many other areas, including the theory of special functions, geometric function theory, and conformal mapping. To gain a better understanding of these functions' behavior and applicability in other mathematical domains, researchers frequently examine aspects of these functions, such as growth requirements, distortion theorems, and coefficient bounds. In recent years, researchers have been interested in defining a new subclass of univalent analytic functions associated with differential operators [1, 2, 3, 4, 5] because it has many applications in mathematics, physics, and engineering. The core challenges in the theory of univalent functions are comprehending the limit correspondence in conformal mapping, figuring out univalent requirements, and addressing numerous functional extreme theory problems. More precisely, defining limits on a range of values for various functions in a given class.

The first substantial results in the theory of univalent functions were obtained using the area principle. With the aid of the outer area theorem (1916), Bieberbach [6] obtained precise upper and lower bounds for $|f(z)|$ and $|f'(z)|$ for $f \in \mathcal{S}$, provided $|a_2| \leq 2$ and conjectured that $|a_n| \leq n$. He also found the exact value of the Koebe [7] constant. For a long time, mathematicians have been challenged by this conjecture. Louis De Branges [8] found a solution to the conjecture $|a_n| \leq n, (n = 2, 3, ...)$ in 1984. Following Loewner [9]'s 1923 proof of $|a_3| \leq 3$, Fekete-Szego [10] astounded mathematicians with the troublesome inequality $|a_3 - \eta a_2^2| \leq 1 + 2e^{\left(\frac{-2\eta}{1-\eta}\right)}$, $0 \leq \eta \leq 1$. In recent years, the study of geometric properties of holomorphic functions has gained significant attention due to its applications in geometric function theory and univalent function classes. One key inequality in this field is the Fekete-Szegö inequality, which provides sharp bounds for coefficients in certain classes of analytic functions. This paper focuses on extending these inequalities to more generalized domains using the Raducanu-Orhan operator, which has shown promise in preserving univalent functions across complex order starlike and convex classes. Moreover, we explore the implications of these findings, a special class of conformal maps with unique geometric properties.

A univalent function that is analytic in a domain and is also referred to as a one-to-one or injective function in complex analysis is said to be conformal. Locally, angles are preserved by conformal mappings. Formally speaking, if a function $f(z)$ defined in a domain $D \subset C$ maintains the angles between the curves that pass through z, then $z \in D$ is conformal at that point. A function is conformal in a domain D if it is both univalent (injective) and analytic in D. In line with conformal maps in complex analysis, this indicates that the mapping maintains the local structure of the domain locally by not distorting the angles between curves. Conformal mapping is a crucial method in complex analysis with a wide range of real-world applications.

If the function is harmonic, that is, it satisfies $\nabla^2 f = 0$ according to Laplace, then the conformal mapping transformation of such functions is likewise harmonic. Therefore, conformal mapping can be used to solve equations related to any field that can be represented by a potential function. Many problems arising from fluid mechanics, electrostatics, heat conduction, and many other physical situations can be formulated mathematically using Laplace's equation $\phi_{xx} + \phi_{yy} = 0$ in a certain region D of the z-plane. For example, it can be applied to scattering and diffraction problems, brain surface mapping problems, and electrostatic potential problems in the shaded region of the z-plane [11]. It can also be used in stealth technology. Although the concept of conformal mapping is not directly used in stealth technology, the development of effective stealth technologies greatly benefits from an understanding of shape optimization, material science, and electromagnetic wave behaviour [12]. In addition, the univalent function helps to analyze the frequency analysis problem [13].

In recent years, new subclasses have been defined by using the linear differential operators. The differential operator was first introduced by Ruscheweyh [14] in 1975, which cleared the path. Salagean [15] followed in 1983 with an additional variation of differential and integral operators. Many scholars have examined and debated a wide range of properties related to these two operators. Al-Oboudi [16] generalized the Salagean operator in 2004. In 2010, Raducanu and Orhan [17] generalized the Al-Oboudi differential operator. In this study, we define two new subclasses, which are defined by the Raducanu-Orhan differential operator. Also, we have discussed some properties of these subclasses.

Let $\mathcal{A}$ be the class of univalent functions consisting of the form

$$f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu, \ z \in \mathbb{U} := \{z \in \mathbb{C} \ : |z| < 1\}, \tag{1}$$

which is analytic in the unit disk $\mathbb{U}$.

For $f(z) \in \mathcal{A}$, the Raducanu-Orhan [17] differential operator is defined as $\mathcal{R}^n_{\tau,\eta} f(z)$, then

$$\begin{aligned}
\mathcal{R}^0_{\tau,\eta} &= f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu, \\
\mathcal{R}^1_{\tau,\eta} &= (1 - \tau + \eta) f(z) + (\tau - \eta) z f'(z) + (\tau\eta) z^2 f''(z) \\
&= z + \sum_{\nu=2}^{\infty} \left[1 + (\nu - 1)(\tau - \eta + \nu\tau\eta)\right] a_\nu z^\nu \\
\mathcal{R}^2_{\tau,\eta} &= \mathcal{R}_{\tau,\eta} \left(\mathcal{R}_{\tau,\eta}\right).
\end{aligned}$$

Similarly,

$$\mathcal{R}^n_{\tau,\eta} = \mathcal{R}_{\tau,\eta} \left(\mathcal{R}^{n-1}_{\tau,\eta}\right) = z + \sum_{\nu=2}^{\infty} \left[1 + (\nu - 1)(\tau - \eta + \nu\tau\eta)\right]^n a_\nu z^\nu.$$

Hence,

$$\mathcal{R}^n_{\tau,\eta}f(z) = \mathcal{R}_{\tau,\eta}\left(\mathcal{R}^{n-1}_{\tau,\eta}\right) = z + \sum_{\nu=2}^{\infty}\left[1+(\nu-1)(\tau-\eta+\nu\tau\eta)\right]^n a_\nu z^\nu, \qquad (2)$$

where $n \in \mathbb{N}_0 = \mathbb{N}\cup 0, \mathbb{N} = \{1,2,...,\}, \quad \eta,\ \tau \geq 0, z \in \mathbb{U}$.

Remark 1.1.

- $\mathcal{R}^n_{\tau,0} = \mathcal{D}^n$ *yields the of Al-Oboudi derivative operator* [16].
- $\mathcal{R}^n_{1,0} = \mathcal{D}^n$ *gives Salagean derivative operator* [15].

2. MAIN RESULTS

2.1. The subclasses $\mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$ and $\mathcal{S}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$.

Definition 2.1. *Let $\mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of a function f which satisfies the inequality*

$$Re\left(1+\frac{1}{b}\left(\frac{R^m_{\delta,\mu}f(z)}{R^n_{\delta,\mu}f(z)}-1\right)\right) > \alpha, \qquad (3)$$

for $b \in \mathbb{C}-\{0\}, m \in \mathbb{N}, n \in \mathbb{N}_0, \delta,\mu \geq 0, 0 \leq \alpha < 1$, and for all $z \in \mathbb{U}$.

Several well-known subclasses of functions are special cases of $\mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$ for suitable choices of the parameters, and are listed below.

- $\mathcal{S}^{m,n}_{b,0,\delta,0}(\alpha)$= $S_{b,m,n,\delta}(\alpha)$ studied by Thiruchran and Stalin [18].
- $\mathcal{S}^{m,n}_{1,0,1,0}(\alpha)$= $K_{m,n}(\alpha)$ studied by Sumer Eker and Owa [19].
- $\mathcal{S}^{m,n}_{1,0,\delta,0}(\alpha)$= $S_{m,n,\delta}(\alpha)$ studied by Sumer Eker and Ozlem Guney [20].
- $\mathcal{S}^{n+1,n}_{1,0,\delta,0}(\alpha)$= $S_n(\alpha)$ studied by Kadioglu [21].
- $\mathcal{S}^{1,0}_{1,0,1,0}(\alpha)$= $S^*(\alpha)$ and $S_{1,2,1,1}(\alpha) = K(\alpha)$ studied by H. Silverman [22].

Theorem 2.2. *Let $f \in \mathcal{A}$ satisfies*

$$\sum_{j=2}^{\infty}\phi(\alpha)\left|a_j\right| \leq 2(1-\alpha)b, \qquad (4)$$

for some $b \in \mathbb{C}-\{0\}, m \in \mathbb{N}, n \in \mathbb{N}_0, \delta,\mu \geq 0, 0 \leq \alpha < 1$, then $f(z) \in \mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$, where

$$\begin{aligned}\phi(\alpha) &= \left|(1+(j-1)(\delta-\mu+j\delta\mu))^m - (1+\alpha b)(1+(j-1)(\delta-\mu+j\delta\mu))^n\right| \\ &\quad + (1+(j-1)(\delta-\mu+j\delta\mu))^m + ((2-\alpha)b-1)(1+(j-1)(\delta-\mu+j\delta\mu))^n.\end{aligned} \qquad (5)$$

Proof. Suppose that

$$\sum_{j=2}^{\infty} \phi(\alpha)\,|a_j| \leq 2(1-\alpha)b,$$

it is true for some $b \in \mathbb{C} - \{0\}, m \in \mathbb{N}, n \in \mathbb{N}_0, \delta, \mu \geq 0, 0 \leq \alpha < 1$, then it is sufficient to prove that

$$\left|\frac{F(z)-1}{F(z)+1}\right| < 1.$$

For $f \in \mathcal{A}$, we define the function $F(z)$ by

$$F(z) = 1 + \frac{1}{b}\left(\frac{R^m_{\delta,\mu} f(z)}{R^n_{\delta,\mu} f(z)} - 1\right) - \alpha, \tag{6}$$

which gives,

$$F(z) - 1 = 1 + \frac{1}{b}\left(\frac{R^m_{\delta,\mu} f(z)}{R^n_{\delta,\mu} f(z)} - 1\right) - \alpha - 1 \tag{7}$$

and

$$F(z) + 1 = 1 + \frac{1}{b}\left(\frac{R^m_{\delta,\mu} f(z)}{R^n_{\delta,\mu} f(z)} - 1\right) - \alpha + 1. \tag{8}$$

From (7) and (8) we get,

$$\left|\frac{F(z)-1}{F(z)+1}\right| = \left|\frac{\frac{R^m_{\delta,\mu} f(z) - (1+\alpha b) R^n_{\delta,\mu} f(z)}{b R^n_{\delta,\mu} f(z)}}{\frac{R^m_{\delta,\mu} f(z) - [1+(\alpha-2)] R^n_{\delta,\mu} f(z)}{b R^n_{\delta,\mu} f(z)}}\right|,$$

$$\left|\frac{F(z)-1}{F(z)+1}\right| = \left|\frac{R^m_{\delta,\mu} f(z) - (1+\alpha b) R^n_{\delta,\mu} f(z)}{R^m_{\delta,\mu} f(z) - [1+(\alpha-2)] R^n_{\delta,\mu} f(z)}\right|$$

$$= \left|\frac{\begin{array}{c} z + \sum_{j=2}^{\infty}[1+(j-1)(\delta-\mu+j\delta\mu)]^m a_j z^j \\ -(1+\alpha b)[z + \sum_{j=2}^{\infty}[1+(j-1)(\delta-\mu+j\delta\mu)]^n a_j z^j] \end{array}}{\begin{array}{c} z + \sum_{j=2}^{\infty}[1+(j-1)(\delta-\mu+j\delta\mu)]^m a_j z^j \\ -(1+(\alpha-2)b)[z + \sum_{j=2}^{\infty}[1+(j-1)(\delta-\mu+j\delta\mu)]^n a_j z^j] \end{array}}\right|.$$

We know that

$$\left|\frac{F(z)-1}{F(z)+1}\right| < 1,$$

$$\left|\frac{\begin{array}{l} -\alpha bz+\sum\limits_{j=2}^{\infty}[1+(j-1)(\delta-\mu+j\delta\mu)]^m \\ \quad -(1+\alpha b)\sum\limits_{j=2}^{\infty}[1+(j-1)(\delta-\mu+j\delta\mu)]^n a_j z z^{j-1} \end{array}}{\begin{array}{l} (2-\alpha)bz+\sum\limits_{j=2}^{\infty}[1+(j-1)(\delta-\mu+j\delta\mu)]^m \\ \quad +((2-\alpha)b-1)[1+(j-1)(\delta-\mu+j\delta\mu)]^n a_j z z^{j-1} \end{array}}\right|<1,$$

and

$$\begin{aligned} &\sum_{j=2}^{\infty}[1+(j-1)(\delta-\mu+j\delta\mu)]^m |a_j| \\ &-(1+\alpha b)\sum_{j=2}^{\infty}[1+(j-1)(\delta-\mu+j\delta\mu)]^n |a_j| \\ &+\sum_{j=2}^{\infty}[1+(j-1)(\delta-\mu+j\delta\mu)]^m |a_j| \\ &+((2-\alpha)b-1)[1+(j-1)(\delta-\mu+j\delta\mu)]^n |a_j| \leq 2(1-\alpha)b. \end{aligned}$$

Therefore,

$$\sum_{j=2}^{\infty}\phi(\alpha)|a_j| \leq 2(1-\alpha)b.$$

□

Let $=0$ and $\mu=0$, then the class $\mathcal{S}_{b,\delta,\mu}^{m,n}(\alpha)$ analogues the class $\mathcal{S}_{b,m,n,\delta}(\alpha)$ was studied by Thirucheran and Stalin [18].

Corollary 2.3. *Let $f\in\mathcal{A}$ satisfies*

$$\sum_{j=2}^{\infty}\phi(m,n,\alpha,\beta,\delta,b,j)|a_j| \leq 2(1-\alpha)b, \tag{9}$$

for some $b\in\mathbb{C}-\{0\}, m\in\mathbb{N}, n\in\mathbb{N}_0, \delta,\mu\geq 0, 0\leq\alpha<1$, then $f(z)\in\mathcal{S}_{b,m,n,\delta,}(\alpha)$, where

$$\begin{aligned} \phi(m,n,\alpha,\beta,\delta,b,j) &= |(1+(j-1)\delta)^m-(1+\alpha b)(1+(j-1)\delta)^n| \\ &\quad +(1+(j-1)\delta)^m+((2-\alpha)b-1)(1+(j-1)\delta)^n. \end{aligned} \tag{10}$$

Let $b=1$, then the class $\mathcal{S}_{b,\delta,\mu}^{m,n}(\alpha)$ reduces to

$$Re\left(\frac{D_\delta^m f(z)}{D_\delta^n f(z)}\right)>\alpha,$$

which analogue the class $\mathcal{S}_{m,n,\delta}(\alpha)$ introduced by Sevtap Sumer Eker and Ozlem Guney[20].

Corollary 2.4. *Let $f \in \mathcal{A}$ satisfies the inequality*

$$\sum_{j=2}^{\infty} \phi(\alpha, m, n, \delta, j)\,|a_j| \leq 2(1-\alpha), \tag{11}$$

for some $0 \leq \alpha < 1, m \in \mathbb{N}, n \in \mathbb{N}_0, \delta, \geq 0,$ *then* $f(z) \in \mathcal{S}_{m,n,\delta}(\alpha)$,
where

$$\begin{aligned}\phi(\alpha, m, n, \delta, j) &= |(1+(j-1)\delta)^m - (1+\alpha)(1+(j-1)\delta)^n| \\ &+ (1+(j-1)\delta)^m + (1-\alpha)(1+(j-1)\delta)^n.\end{aligned} \tag{12}$$

If we put $b = 1$ and $\delta = 1$, then the class $\mathcal{S}_{b,\delta,\mu}^{m,n}(\alpha)$ reduces to

$$Re\left(\frac{D^m f(z)}{D^n f(z)}\right) > \alpha$$

which analogues the class $\mathcal{S}_{m,n}(\alpha)$ introduced by Sevtap Sumer Eker and Owa [19].

Corollary 2.5. *Let $f \in \mathcal{A}$ satisfies*

$$\sum_{j=2}^{\infty} \phi(\alpha, m, n, j)\,|a_j| \leq 2(1-\alpha), \tag{13}$$

for some $\alpha(0 \leq \alpha < 1), m \in \mathbb{N}, n \in \mathbb{N}_0,$ *then* $f(z) \in \mathcal{S}_{m,n}(\alpha),$ *where*

$$\phi(\alpha, m, n, j) = |(j)^m - (1+\alpha)(j)^n| + (j)^m + (1-\alpha)(j)^n. \tag{14}$$

If $b = 1, m = n+1,$ and $\delta = 1$, then the class $\mathcal{S}_{b,\delta,\mu}^{m,n}(\alpha)$, which reduces to the form

$$Re\left(\frac{D^{n+1} f(z)}{D^n f(z)}\right) > \alpha$$

and analogues the class $\mathcal{S}_n(\alpha)$ introduced by Kadioglu [21].

Corollary 2.6. *Let $f \in \mathcal{A}$ satisfies the coefficient inequality*

$$\sum_{j=2}^{\infty} (j^{n+1} - \alpha j^n)\,|a_j| \leq 1-\alpha,$$

for some $(0 \leq \alpha < 1)$, *then* $f \in \mathcal{S}_n(\alpha)$.

Theorem 2.7. *If* $f(z) \in \mathcal{S}_{b,\delta,\mu}^{m,n}(\alpha)$ *for* $k \geq 2$, *then*

$$|a_k| \leq \frac{\beta}{|v_k|}\{1 + \beta \sum_{j=2}^{k-1} \frac{(1+(j-1)(\delta-\mu+j\delta\mu))^n}{|v_j|}$$

$$+\beta^2 \sum_{j_2 > j_1}^{k-1} \sum_{j_1=2}^{k-2} \frac{(1+(j_1-1)(\delta-\mu+j_1 j\delta\mu+j_1))(1+(j_2-1)(\delta-\mu+j_2 j\delta\mu+j_2))^n}{|v_{j_1} v_{j_2}|}$$

$$+ ... + \beta^{k-2} \prod_{j=2}^{k-1} \frac{(1+(j-1)(\delta-\mu+j\delta\mu))^n}{|v_j|}\}, \tag{15}$$

where $\beta = 2(1-\alpha)b$ *and* $v_k = (1+(j-1)(\delta-\mu+j\delta\mu))^m - (1+(j-i)(\delta-\mu+j\delta\mu))^n$.

If replace $=0$ and $\mu=0$ in (15), then the class $\mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$, reduces to the class $\mathcal{S}_{b,\delta,\mu}(\alpha)$, which is studied by Thirucheran and Stalin [18].

Corollary 2.8. *If $f(z)\in\mathcal{S}_{b,m,n,\delta}(\alpha)$ for $k\geq 2$, then*

$$|a_k|\leq\frac{\beta}{|v_k|}\{1+\beta\sum_{j=2}^{k-1}\frac{(1+(j-1)\delta)^n}{|v_j|}+\beta^2\sum_{j_2>j_1}^{k-1}\sum_{j_1=2}^{k-2}\frac{(1+(j_1-1)\delta)(1+(j_2-1)\delta)^n}{|v_{j_1}v_{j_2}|}$$

$$+...+\beta^{k-2}\prod_{j=2}^{k-1}\frac{(1+(j-1)\delta)^n}{|v_j|}\},\tag{16}$$

where $\beta=2(1-\alpha)b$ and $v_k=(1+(j-1)\delta)^m-(1+(j-i)\delta)^n$.

If $b=1$, we get the results of subclass $\mathcal{S}_{m,n,\delta}(\alpha)$ introduced by Sevtap Sumer Eker and Ozlem Guney[20].

Corollary 2.9. *If $f(z)\in\mathcal{S}_{m,n,\delta}(\alpha)$ for $k\geq 2$, then*

$$|a_k|\leq\frac{\beta}{|v_k|}\{1+\beta\sum_{j=2}^{k-1}\frac{(1+(j-1)\delta)^n}{|v_j|}+\beta^2\sum_{j_2>j_1}^{k-1}\sum_{j_1=2}^{k-2}\frac{(1+(j_1-1)\delta)(1+(j_2-1)\delta)^n}{|v_{j_1}v_{j_2}|}$$

$$+...+\beta^{k-2}\prod_{j=2}^{k-1}\frac{(1+(j-1)\delta)^n}{|v_j|}$$

where $\beta=2(1-\alpha)$ and $v_k=(1+(j-1)\delta)^m-(1+(j-i)\delta)^n$.

If $b=1$ and $\delta=1$, we get the result of the class $\mathcal{S}_{m,n}(\alpha)$ introduced by Sevtap Sumer Eker and Owa [19].

Corollary 2.10. *If $f(z)\in\mathcal{S}_{m,n}(\alpha)$ for $k\geq 2$, then*

$$|a_k|\leq\frac{\beta}{|v_k|}\{1+\beta\sum_{j=2}^{k-1}\frac{(j)^n}{|v_j|}+\beta^2\sum_{j_2>j_1}^{k-1}\sum_{j_1=2}^{k-2}\frac{(j_1j_2)^n}{|v_{j_1}v_{j_2}|}+...+\beta^{k-2}\prod_{j=2}^{k-1}\frac{(j)^n}{|v_j|},$$

where $\beta=2(1-\alpha)$ and $v_k=(j)^m-(j)^n$.

Now we define the subclass $\tilde{\mathcal{S}}^{m,n}_{b,\delta,\mu}(\alpha)\subset\mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$, which consists of the function

$$f(z)=\sum_{j=2}^{\infty}a_jz^j,(a_j\geq 0).\tag{17}$$

Now determine the extreme points for the subclass $\tilde{\mathcal{S}}^{m,n}_{b,\delta,\mu}(\alpha)$.

Theorem 2.11. *Let*

$$f_1(z)=z,f_j(z)=z+\sum_{j=2}^{\infty}\eta_j\frac{2(1-\alpha)b}{\phi(\alpha)}z^j,(j=2,3,4,...)\tag{18}$$

then $f \in \tilde{\mathcal{S}}^{m,n}_{b,\delta,\mu}$ *if and only if it can be expressed in the form* $f(z) = \sum_{j=1}^{\infty} \eta_j f_j(z)$, *where* $\eta_j > 0$ *and*

$$\sum_{j=1}^{\infty} \eta_j = 1. \tag{19}$$

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{j=1}^{\infty} \eta_j f_j(z) \qquad (20) \\ &= z + \sum_{j=2}^{\infty} \eta_j \frac{2(1-\alpha)b}{\phi(\alpha)} z^j \\ &= z + \sum_{j=2}^{\infty} \eta_j \frac{2(1-\alpha)b}{\phi(\alpha)} \phi(\alpha) \\ &= 2(1-\alpha)b \sum_{j=2}^{\infty} \eta_j \\ &= 2(1-\alpha)b(1-\eta_1) \\ &< 2(1-\alpha)b, \qquad (21) \end{aligned}$$

which shows that f satisfies the condition (20). Hence, $f \in \tilde{\mathcal{S}}^{m,n}_{b,\delta,\mu}$. Conversely, suppose that $f \in \tilde{\mathcal{S}}^{m,n}_{b,\delta,\mu}$. Since,

$$a_j \leq \frac{2(1-\alpha)b}{\phi(\alpha)}, (j = 2, 3, ...). \tag{22}$$

Let

$$\eta_j \leq \frac{\phi(\alpha)}{2(1-\alpha)b} a_j$$

and

$$\eta_1 = 1 - \sum_{j=2}^{\infty} \eta_j,$$

then we obtain

$$f(z) = \sum_{j=1}^{\infty} \eta_j f_j(z).$$

□

Let $= 0$ and $\mu = 0$, then the class $\mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$ reduces and analogues the class $S_{b,m,n,\delta}(\alpha)$ studied by Thirucheran and Stalin [18].

Corollary 2.12. *Let*

$$f_1(z) = z, f_j(z) = z + \sum_{j=2}^{\infty} \eta_j \frac{2(1-\alpha)b}{\phi(m,n,\alpha,\beta,\delta,b,j)} z^j, (j = 2, 3, 4, ...) \tag{23}$$

then $f \in \tilde{\mathcal{S}}_{m,n}(\alpha)$ *if and only if it can be expressed in the form* $f(z) = \sum_{j=1}^{\infty} \eta_j f_j(z)$, *where* $\eta_j > 0$ *and*

$$\sum_{j=1}^{\infty} \eta_j = 1. \tag{24}$$

Let $b = 1$, we get the result of the class $\mathcal{S}_{m,n,\delta}(\alpha)$ introduced by Sevtap Sumer Eker and Ozlem Guney [20].

Corollary 2.13. *Let* $f_1(z) = z$ *and*

$$f(z) = z + \frac{2(1-\alpha)}{\phi(\alpha, m, n, \delta, j)} z^j,$$

then $f \in \tilde{\mathcal{S}}_{m,n,\delta}$ *if and only if it can be expressed* $\sum_{j=1}^{\infty} \eta_j f_j(z)$, *where* $\eta_j > 0, \sum_{j=1}^{\infty} \eta_j = 1$.

Theorem 2.14. *(Littlewood* [23]*) If* f *and* g *are analytic in* $\mathbb{U}$ *with* $f(z) \prec g(z)$, *then*

$$\int_0^{2\pi} |f(z)|^{\mu} \, d\theta \leq \int_0^{2\pi} |g(z)|^{\mu} \, d\theta,$$

for $\mu > 0$ *and* $z = re^{i\theta}$, $0 < r < 1$.

Theorem 2.15. *Let* $f \in \mathcal{S}_{b,\delta,\mu}^{m,n}(\alpha)$ *and suppose that* $f(z)$ *is defined by*

$$g(z) = z + \frac{2(1-\alpha)b\varepsilon_j}{\phi(\alpha)} z^j, (j = 2, 3, ...), |\varepsilon_j| = 1.$$

If there exists an analytic function $w(z)$ *given by*

$$\{w(z)\}^{j-1} = \frac{\phi(\alpha)}{2(1-\alpha)b\varepsilon_j} \sum_{j=2}^{\infty} a_j z^{j-1}, z = re^{i\theta}, 0 < r < 1,$$

then

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\mu} d\theta, \mu > 0.$$

Proof. We must show that

$$\int_0^{2\pi} \left|1 + \sum_{j=2}^{\infty} a_j z^{j-1}\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|1 + \frac{2(1-\alpha)b\varepsilon_j}{\phi(\alpha)} z^{j-1}\right|^{\mu} d\theta.$$

By the help of Littlewood subordination theorem, it is sufficient to show that

$$1 + \sum_{j=2}^{\infty} a_j z^{j-1} \prec 1 + \frac{2(1-\alpha)b\varepsilon_j}{\phi(\alpha)} z^{j-1}.$$

Let

$$1 + \sum_{j=2}^{\infty} a_j z^{j-1} = 1 + \frac{2(1-\alpha)b\varepsilon_j}{\phi(\alpha)} (w(z))^{j-1}).$$

$$\therefore (w(z))^{j-1} = \frac{\phi(\alpha)}{2(1-\alpha)b\varepsilon_j}\sum_{j=2}^{\infty} a_j z^{j-1}.$$

Which readily yields $w(0) = 0$. Further, we prove that the analytic function $w(z)$ satisfies $|w(z)| < 1$ using Schwarz lemma. We know that

$$\left|(w(z))^{j-1}\right| = \left|\frac{\phi(\alpha)}{2(1-\alpha)b\varepsilon_j}\sum_{j=2}^{\infty} a_j z^{j-1}\right| \le |z| < 1.$$

□

Let $= 0$ and $\mu = 0$, then the class $\mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$ reduces to the class $S_{b,m,n,\delta}(\alpha)$, which is investigated by Thirucheran and Stalin [18].

Corollary 2.16. *Let $f \in \mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$ and suppose that $f(z)$ is defined by*

$$g(z) = z + \frac{2(1-\alpha)b\varepsilon_j}{\phi(m,n,\alpha,\beta,\delta,b,j)} z^j, (j = 2, 3, ...), |\varepsilon_j| = 1.$$

If there exists an analytic function $w(z)$ given by

$$\{w(z)\}^{j-1} = \frac{\phi(\alpha)}{2(1-\alpha)b\varepsilon_j}\sum_{j=2}^{\infty} a_j z^{j-1}, z = re^{i\theta}, 0 < r < 1,$$

then

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mu} d\theta \le \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\mu} d\theta, \mu > 0.$$

Definition 2.17. *Let $\mathcal{S}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$ denote the subclass of $\mathcal{A}$ consisting of function f which satisfies the inequality*

$$Re\left(1 + \frac{1}{b}\left(\frac{R^m_{\tau,\eta}f(z)}{R^n_{\tau,\eta}f(z)} - 1\right)\right) > \mu\left|\frac{R^m_{\tau,\eta}f(z)}{R^n_{\tau,\eta}f(z)} - 1\right| + \gamma. \tag{25}$$

For some $b \in \mathbb{C} - \{0\}, m \in \mathbb{N}, n \in \mathbb{N}_0, \mu, \tau, \eta \ge 0, 0 \le \gamma < 1$ and all $z \in \mathbb{U}$.

Several well known subclasses of functions are the special cases of $\mathcal{S}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$ for suitable choices of the parameters.

Remark 2.18.
- $\mathcal{S}^{m,n}_{b,0,0,\tau}(\gamma)$= $S(m,n,\gamma,\mu,\tau,b,j)$ *studied by Stalin and Thiruchran* [24].
- $\mathcal{S}^{m,n}_{1,0,0,1}(\gamma) = K_{m,n}(\gamma)$ *studied by Sumer Eker and Owa* [19].
- $\mathcal{S}^{m,n}_{1,0,0,\tau}(\gamma) = S_{m,n,\tau}(\gamma)$ *studied by Sumer Eker and Ozlem Guney* [20].
- $\mathcal{S}^{n+1,n}_{1,0,0,1}(\gamma) = S_n(\gamma)$ *studied by Kadioglu* [21].

Theorem 2.19. *Let $f \in \mathcal{A}$ satisfies*

$$\sum_{\nu=2}^{\infty} \phi(\gamma,\mu)\,|a_\nu| \le 2(1-\gamma)b. \tag{26}$$

For some $b \in \mathbb{C} - \{0\}, m \in \mathbb{N}, n \in \mathbb{N}_0, \mu, \tau, \eta \geq 0, 0 \leq \gamma < 1$ *and all* $z \in \mathbb{U}$, *then* $f \in \mathcal{S}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$*, where*

$$\begin{aligned}\phi(\gamma,\mu) = &\left|(1+(\nu-1)(\tau-\eta+\nu\tau\eta))^m - (1+\gamma b)(1+(\nu-1)(\tau-\eta+\nu\tau\eta))^n\right| \\ &+ (1+(\nu-1)(\tau-\eta+\nu\tau\eta))^m + ((2-\gamma)b-1)(1+(\nu-1)(\tau-\eta+\nu\tau\eta))^n \\ &+ 2b\mu\left|(1+(\nu-1)(\tau-\eta+\nu\tau\eta))^m - (1+(\nu-1)(\tau-\eta+\nu\tau\eta))^n\right|.\end{aligned} \tag{27}$$

Proof. Suppose that

$$\sum_{\nu=2}^{\infty} \phi(\gamma,\mu)\,|a_\nu| \leq 2(1-\gamma)b$$

is true. For some $b \in \mathbb{C} - \{0\}, m \in \mathbb{N}, n \in \mathbb{N}_0, \mu, \tau, \eta \geq 0, 0 \leq \gamma < 1$ and all $z \in \mathbb{U}$, then it is sufficient to prove that

$$\left|\frac{F(z)-1}{F(z)+1}\right| < 1.$$

For $f \in \mathcal{A}$, then define the function $F(z)$ by

$$F(z) = 1 + \frac{1}{b}\left(\frac{R^m_{\tau,\eta}f(z)}{R^n_{\tau,\eta}f(z)} - 1\right) - \mu\left|\frac{R^m_{\tau,\eta}f(z)}{R^n_{\tau,\eta}f(z)} - 1\right| - \gamma. \tag{28}$$

Then

$$\left|\frac{F(z)-1}{F(z)+1}\right| = \left|\frac{R^m_{\tau,\eta}f(z) - (1+\gamma b)R^n_{\tau,\eta}f(z) - b\mu\left|R^m_{\tau,\eta}f(z) - R^n_{\tau,\eta}f(z)\right|}{R^m_{\tau,\eta}f(z) - (1+(\gamma-2)b)R^n_{\tau,\eta}f(z) - b\mu\left|R^m_{\tau,\eta}f(z) - R^n_{\tau,\eta}f(z)\right|}\right| < 1,$$

$$\begin{aligned}&\sum_{\nu=2}^{\infty}\{\left|(1+(\nu-1)(\tau-\eta+\nu\tau\eta))^m - (1+\gamma b)(1+(\nu-1)(\tau-\eta+\nu\tau\eta))^n\right| \\ &+ ((1+(\nu-1)(\tau-\eta+\nu\tau\eta))^m + ((2-\gamma)b-1)(1+(\nu-1)(\tau-\eta+\nu\tau\eta))^n) \\ &+ 2b\mu\left|(1+(\nu-1)(\tau-\eta+\nu\tau\eta))^m - (1+(\nu-1)(\tau-\eta+\nu\tau\eta))^n\right|\}\,|a_\nu| \\ &\leq 2(1-\gamma)b.\end{aligned}$$

$$\therefore \sum_{\nu=2}^{\infty} \phi(\gamma,\mu)\,|a_\nu| \leq 2(1-\gamma)b.$$

□

Put $\mu = 0$ and $\eta = 0$ in $\mathcal{S}^{m,n}_{b,\tau,\eta}(\gamma)$, then this class reduces as $S(m,n,\gamma,\mu,\tau,b)$, which was studied by Thirucheran and Stalin [24].

Corollary 2.20. *Let* $f \in \mathcal{A}$ *satisfies*

$$\sum_{\nu=2}^{\infty} \phi(m,n,\gamma,\mu,\tau,b,\nu)\,|a_\nu| \leq 2(1-\gamma)b. \tag{29}$$

For some $(0 \leq \gamma < 1), \mu \geq 0, m \in \mathbb{N}, n \in \mathbb{N}_0, \tau(\tau \geq 0)$, *and all* $z \in \mathbb{U}$, *then* $f \in S(m, n, \gamma, \mu, \tau, b)$, *where*

$$\begin{aligned}\phi(m, n, \gamma, \mu, \tau, b, j) &= |(1 + (\nu - 1)\tau)^m - (1 + \gamma b)(1 + (\nu - 1)\tau)^n| \\ &+ (1 + (\nu - 1)\tau)^m + ((2 - \gamma)b - 1)(1 + (\nu - 1)\tau)^n \\ &+ 2b\mu \left|(1 + (\nu - 1)\tau)^m - (1 + (\nu - 1)\tau)^n\right| .\end{aligned} \tag{30}$$

If $b = 1$ and $\mu = 0$, then the class $\mathcal{S}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$ reduces to

$$Re\left(\frac{R^m_{\tau,\eta} f(z)}{R^n_{\tau,\eta} f(z)}\right) > \gamma$$

which analogs to the class $\mathcal{S}_{m,n,\tau}(\gamma)$ introduced by Sevtap Sumer Eker and Ozlem Guney[20].

Corollary 2.21. *Let* $f \in \mathcal{A}$ *satisfies the inequality*

$$\sum_{\nu=2}^{\infty} \phi(\gamma, m, n, \tau, \nu) |a_\nu| \leq 2(1 - \gamma), \tag{31}$$

for some $b \in \mathbb{C} - \{0\}, m \in \mathbb{N}, n \in \mathbb{N}_0, \tau, \eta \geq 0, 0 \leq \gamma < 1$, *then* $f(z) \in \mathcal{S}_{m,n,\tau}(\gamma)$, *where*

$$\begin{aligned}\phi(\gamma, m, n, \tau, \nu) &= |(1 + (\nu - 1)\tau)^m - (1 + \gamma)(1 + (\nu - 1)\tau)^n| \\ &\quad + (1 + (\nu - 1)\tau)^m + (1 - \gamma)(1 + (\nu - 1)\tau)^n.\end{aligned} \tag{32}$$

If $b = 1, \mu = 0$, and $\tau = 1$, then the class $\mathcal{S}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$ given the class $\mathcal{S}_{m,n}(\gamma)$, which is discussed by Sevtap Sumer Eker and Owa [19]

Corollary 2.22. *Let* $f \in \mathcal{A}$ *satisfies*

$$\sum_{\nu=2}^{\infty} \phi(\gamma, m, n, \nu) |a_\nu| \leq 2(1 - \gamma), \tag{33}$$

for some $\gamma(0 \leq \gamma < 1), m \in \mathbb{N}, n \in \mathbb{N}_0$, *then* $f(z) \in \mathcal{S}_{m,n}(\gamma)$, *where*

$$\phi(\gamma, m, n, \nu) = |(j)^m - (1 + \gamma)(j)^n| + (j)^m + (1 - \gamma)(j)^n. \tag{34}$$

We define the subclass $\tilde{\mathcal{S}}^{m,n}_{b,\mu,\tau,\eta}(\gamma) \subset \mathcal{S}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$, which consists of the function

$$f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu, (a_\nu \geq 0). \tag{35}$$

Now we determine the extreme points of the subclass $\tilde{\mathcal{S}}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$.

Theorem 2.23. *Let* $f_1(z) = z$ *and*

$$f_\nu(z) = z + \sum_{\nu=2}^{\infty} \vartheta_\nu \frac{2(1 - \gamma)b}{\phi(\gamma, \mu)} z^\nu, (\nu = 2, 3, 4, ...), \tag{36}$$

then $f \in \tilde{\mathcal{S}}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{\nu=1}^{\infty} \vartheta_\nu f_\nu(z), \vartheta_\nu > 0$$

and

$$\sum_{\nu=1}^{\infty} \vartheta_\nu = 1. \tag{37}$$

Let $= 0$ and $\eta = 0$, then the class $\mathcal{S}^{m,n}_{b,\tau,\eta}(\gamma)$ reduces and analogs the class, which is examined by Thirucheran and Stalin [24].

Corollary 2.24. *Let* $f_1(z) = z$ *and*

$$f_\nu(z) = z + \sum_{\nu=2}^{\infty} \vartheta_\nu \frac{2(1-\gamma)b}{\phi(m,n,\gamma,\mu,\tau,b,\nu)} z^\nu, (\nu = 2, 3,4, ...), \tag{38}$$

then $f \in \tilde{\mathcal{S}}(m,n,\gamma,\mu,\tau,b)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{\nu=1}^{\infty} \vartheta_\nu f_\nu(z), \vartheta_\nu > 0$$

and

$$\sum_{\nu=1}^{\infty} \vartheta_\nu = 1. \tag{39}$$

If $b = 1$ and $\mu = 0$, we get the result of the class $\mathcal{S}_{m,n,\tau}(\gamma)$ introduced by Sevtap Sumer Eker and Ozlem Guney [20].

Corollary 2.25. *Let* $f_1(z) = z$ *and*

$$f(z) = z + \frac{2(1-\gamma)}{\phi(\gamma,m,n,\tau,\nu)} z^\nu,$$

then $f \in \tilde{\mathcal{S}}_{m,n,\tau}$ *if and only if it can be expressed*

$$\sum_{\nu=1}^{\infty} \vartheta_\nu f_\nu(z), \vartheta_\nu > 0.$$

$$\therefore \sum_{\nu=1}^{\infty} \vartheta_\nu = 1.$$

Theorem 2.26. *Let* $f \in \mathcal{S}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$ *and suppose that* $f(z)$ *is defined by*

$$g(z) = z + \frac{2(1-\gamma)b\varepsilon_\nu}{\phi(\gamma,\mu)} z^\nu, (\nu = 2, 3,...), |\varepsilon_\nu| = 1.$$

If there exists an analytic function $w(z)$ *given by*

$$(w(z))^{\nu-1} = \frac{\phi(\gamma,\mu)}{2(1-\gamma)b\varepsilon_\nu} \sum_{\nu=2}^{\infty} a_\nu z^{\nu-1}, z = re^{i\theta}, 0 < r < 1,$$

then

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta \le \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta, \eta > 0.$$

Let $= 0$ and $\eta = 0$, then the class $\mathcal{S}^{m,n}_{b,\tau,\eta}(\gamma)$ reduces and analogs to the the class studied by Thirucheran and Stalin [24].

Corollary 2.27. *Let $f \in \mathcal{S}(m, n, \gamma, \mu, \tau, b)$ and suppose that $f(z)$ is defined by*

$$f(z) = z + \frac{2(1-\gamma)b\varepsilon_\nu}{\phi(m, n, \gamma, \mu, \tau, b, \nu)} z^\nu, (\nu = 2, 3, ...), |\varepsilon_\nu| = 1.$$

If there exists an analytic function $w(z)$ given by

$$(w(z))^{\nu-1} = \frac{\phi(m, n, \gamma, \mu, \tau, b, \nu)}{2(1-\gamma)b\varepsilon_\nu} \sum_{\nu=2}^{\infty} a_\nu z^{\nu-1}, z = re^{i\theta}, 0 < r < 1,$$

then

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta \le \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta, \eta > 0.$$

2.2. **Fekete-Szego inequality for the subclasses $\mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$ and $\mathcal{S}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$.**

In 1933, Fekete-Szego [10] obtained the maximum value of $|a_3 - \eta a_2^2|$ as a function of the real parameter η, for the function of class $\mathcal{A}$. Since then, the various authors were investigated and obtained the Fekete-Szego inequalities for different subclasses of the class $\mathcal{A}$ [25] [10],[26],[27],[28],[29], [30], [31] . In this article, we introduced two new subclasses of univalent functions which are defined by using Raducanu-Ohran differential operator in the open unit disc. For these subclasses, we obtain the Fekete-Szego inequality $|a_3 - \eta a_2^2|$.

If replacing special values for the subclass, we obtained Several well known subclasses.

Remark 2.28.

- $\mathcal{S}^{1,0}_{1,0,1,0}(\alpha)$, $\mathcal{S}^{1,0}_{1,0,0,1,0}(\alpha) = \mathcal{S}^*(\alpha)$ *studied by W.Ma and D.Minda* [32].
- $\mathcal{S}^{1,0}_{b,0,1,0}(\alpha)$, $\mathcal{S}^{1,0}_{b,0,0,1,0}(\alpha) = \mathcal{S}^*_b(\alpha)$ *studied by V. Ravichandran et.al.* [27].
- $\mathcal{S}^{2,0}_{b,0,\delta,0}(\alpha)$, $\mathcal{S}^{2,0}_{b,0,0,\delta,0}(\alpha) = \mathcal{M}_{a,b}(\phi)$ *studied by K.Suchitra et.al.* [29].
- $\mathcal{S}^{2,1}_{b,0,\delta,0}(\alpha)$, $\mathcal{S}^{2,1}_{b,0,0,\delta,0}(\alpha) = \mathcal{M}_\alpha(\phi)$ *studied by T.N.Shanmugam and S.Sivasubramanian* [28].

Remark 2.29.

- $\mathcal{S}^{1,0}_{1,0,0,1}(\gamma) = \mathcal{S}^*(\gamma)$ *studied by W.Ma and D.Minda* [32].
- $\mathcal{S}^{1,0}_{b,0,0,1}(\gamma) = \mathcal{S}^*_b(\gamma)$ *studied by V. Ravichandran et.al.* [27].
- $\mathcal{S}^{2,0}_{b,0,0,\tau}(\gamma) = \mathcal{M}_{a,b}(\phi)$ *studied by K.Suchitra et.al.* [29].
- $\mathcal{S}^{2,1}_{b,0,0,\tau}(\gamma) = \mathcal{M}_\gamma(\phi)$ *studied by T.N.Shanmugam and S.Sivasubramanian* [28].

Lemma 2.30. *If $P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ is a function with positive real part in $\mathbb{U}$ and η is a complex number, then*

$$|c_2 - \eta c_1^2| \le 2Max\{1, |2\eta - 1|\}.$$

The result is sharp for the function is given by $P(z)=\frac{1+z^2}{1-z^2}$ *and* $P(z)=\frac{1+z}{1-z}$.

Lemma 2.31. *If* $P(z)=1+c_1z+c_2z^2+c_3z^3+...$ *is a function with positive real part in* $\mathbb{U}$, *then*

$$|c_2-vc_1^2|\leq\begin{cases}-4v+2 & v\leq 0\\ 2 & 0\leq v\leq 1\\ 4v+2 & v\geq 1\end{cases}$$

when $v<0$ *or* $v>1$, *the equality holds if and only if* $p_1(z)=\frac{1+z}{1-z}$ *or one of its rotations.*
If $0<v<1$, *then the equality holds if and only if* $p_1(z)=\frac{1+z^2}{1-z^2}$ *or one of its rotations.*
If $v=0$, *the equality holds if and only if* $p_1(z)=\left(\frac{1}{2}+\frac{1}{2}\lambda\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\frac{1-z}{1+z},(0\leq\lambda\leq 1)$ *or one of its rotation.*
If $v=1$, *the equality holds if and only if* $p_1(z)=\left(\frac{1}{2}+\frac{1}{2}\lambda\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\frac{1-z}{1+z},(0\leq\lambda\leq 1)$ *is the reciprocal of one of the function such that equality holds in case of* $v=0$.
Also the above upper bound is sharp and it can be improved as follows: when $0<v<1$ $|c_2-vc_1^2|+v|c_1^2|\leq 2, 0<v\leq\frac{1}{2}$ *and* $|c_2-vc_1^2|+(1-v)|c_1^2|\leq 2, \frac{1}{2}<v<1$.

Theorem 2.32. *Let* $\phi(z)=1+B_1z+B_2z^2+B_3z^3+...$, *with* $B_1\neq 0$. *If* $f\in\mathcal{S}_{b,\delta,\mu}^{m,n}(\alpha)$ *satisfies the inequality*

$$Re\left(1+\frac{1}{b}\left(\frac{R_{\delta,\mu}^m f(z)}{R_{\delta,\mu}^n f(z)}-1\right)\right)\prec\phi(z),$$

then

$$\left|a_3-\mu_2^2\right|\leq\frac{B_1|b|}{[(X_2)^m-(X_2)^n]}$$
$$\times Max\left\{1,\left|\frac{B_2}{B_1}+\left[\frac{[(X_1)^{m+n}-(X_1)^{2n}]-\mu[(X_2)^m-(X_2)^n]}{[(X_1)^m-(X_1)^n]^2}\right]bB_1\right|\right\},\qquad(40)$$

where $X_1=1+\delta+3-\mu+2j\delta\mu$ *and* $X_2=1+\delta+4-\mu+3j\delta\mu$.
Then, the result is sharp.

Proof. If $f\in\mathcal{S}_{b,\delta,\mu}^{m,n}(\alpha)$, then there is a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ in $\mathbb{U}$ such that

$$1+\frac{1}{b}\left(\frac{R_{\delta,\mu}^m f(z)}{R_{\delta,\mu}^n f(z)}-1\right)=\phi(w(z)).\qquad(41)$$

Define $P(z)$ by $P(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+c_3z^3+...$.
Since $w(z)$ is a Schwarz function, it is clear that $ReP(z)>0$ and $P(0)=1$.

$$\therefore\phi(z)=\phi\left(\frac{P(z)-1}{P(z)+1}\right)=1+\frac{B_1c_1}{2}z+\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}+\right)+\frac{B_2c_1^2}{4}\right]z^2+...$$

Using Lemma (5.2.2), we get

$$1+\frac{1}{b}\left(\frac{R^m_{\delta,\mu}f(z)}{R^n_{\delta,\mu}f(z)}-1\right)=1+\frac{B_1c_1}{2}z+\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}+\right)+\frac{B_2c_1^2}{4}\right]z^2+...$$

$$\therefore a_2=\frac{bB_1c_1}{2\left[(X_1)^m-(X_1)^n\right]}$$

and

$$a_3=\frac{bB_1c_2}{2\left[(X_2)^m-(X_2)^n\right]}$$

$$+\frac{bB_1c_1^2}{4\left[(X_2)^m-(X_2)^n\right]}\left[\frac{((X_1)^m-(X_1)^n)bB_1}{\left[(X_1)^m-(X_1)^n\right]^2}-\left(1-\frac{B_2}{B_1}\right)\right]$$

$$\therefore a_3-\mu a_2^2=\frac{bB_1}{2\left[(X_2)^m-(X_2)^n\right]}\{c_2-vc_1^2\},$$

where

$$v=\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{\mu bB_1\left[(X_2)^m-(X_2)^n\right]}{\left[(X_1)^m-(X_1)^n\right]^2}-\frac{bB_1\left[(X_1)^{m+n}-(X_1)^{2n}\right]}{\left[(X_1)^m-(X_1)^n\right]^2}\right).$$

Hence

$$\left|a_3-\mu_2^2\right|\le\frac{B_1|b|}{\left[(X_2)^m-(X_2)^n\right]}$$

$$\times max\left\{1,\left|\frac{B_2}{B_1}+\left[\frac{[(X_1)^{m+n}-(X_1)^{2n}]-\mu[(X_2)^m-(X_2)^n]}{[(X_1)^m-(X_1)^n]^2}\right]bB_1\right|\right\}.$$

Therefore, the result (40) is sharp for the function defined by

$$1+\frac{1}{b}\left(\frac{R^m_{\delta,\mu}f(z)}{R^n_{\delta,\mu}f(z)}-1\right)=\phi(z^2),1+\frac{1}{b}\left(\frac{R^m_{\delta,\mu}f(z)}{R^n_{\delta,\mu}f(z)}-1\right)=\phi(z).$$

□

Theorem 2.33. *Let $\phi(z)=1+B_1z+B_2z^2+B_3z^3+...$, with $B_1\neq 0$. If $f\in\mathcal{A}$ satisfies the inequality*

$$Re\left(1+\frac{1}{b}\left(\frac{R^m_{\delta,\mu}f(z)}{R^n_{\delta,\mu}f(z)}-1\right)\right)\prec\phi(z),$$

then

$$|a_2-\mu a_2^2|\le\begin{cases}A_1(-B_2-(1-2\mu A_2)A_3bB_1^2) & ;\mu\le\eta_1\\ A_1 & ;\eta_1\le\mu\le\eta_2\\ A_1(B_2+(1-2\mu A_2)A_3bB_1^2) & ;\mu\ge\eta_2\end{cases},$$

where

$$A_1=\frac{B_1|b|}{2\left[(X_2)^m-(X_2)^n\right]},A_2=\frac{(X_2)^m-(X_2)^n}{2[(X_2)^{m+n}-(X_2)^{2n}]},A_3=\frac{(X_1)^n}{(X_1)^n-(X_1)^m},$$

$$\eta_1 = \frac{[-(B_2+B_1)+A_3bB_1^2][(X_1)^m-(X_1)^n]^2}{[(X_2)^n-(X_1)^m]bB-1^2}$$

and

$$\eta_2 = \frac{[-(B_2-B_1)+A_3bB_1^2][(X_1)^m-(X_1)^n]^2}{[(X_2)^n-(X_1)^m]bB-1^2}.$$

Proof. Let $\mu \leq \eta_1$, then

$$|a_2-\mu a_2^2| \leq A_1(-B_2-(1-2\mu A_2)A_3bB_1^2).$$

Let $\eta_1 \leq \mu \leq \eta_2$, then

$$|a_2-\mu a_2^2| \leq A_1.$$

Let $\mu \geq \eta_2$, then

$$|a_2-\mu a_2^2| \leq A_1(B_2+(1-2\mu A_2)A_3bB_1^2).$$

To show that the bounds are sharp, for the functions $K_{\phi,l}(l \geq 2)$ well defined by

$$1+\frac{1}{b}\left(\frac{\mathcal{D}^m K_{\phi,l}}{\mathcal{D}^n K_{\phi,l}}-1\right) = \phi(z^{l-1}), K_{\phi,l}(0) = (K_{\phi,l})'(0)-1 = 0,$$

$F_\tau, G_\tau (0 \leq \tau \leq 1)$ well defined by

$$1+\frac{1}{b}\left(\frac{\mathcal{D}^m F_\tau}{\mathcal{D}^n F_\tau}-1\right) = \phi\left(\frac{z(z+(\delta-\mu+j\delta\mu))}{1+(\delta-\mu+j\delta\mu)z}\right), F_\tau(0) = (F_\tau)'(0)-1 = 0$$

and

$$1+\frac{1}{b}\left(\frac{\mathcal{D}^m G_\tau}{\mathcal{D}^n G_\tau}-1\right) = \phi\left(-\frac{z(z+(\delta-\mu+j\delta\mu))}{1+(\delta-\mu+j\delta\mu)z}\right), G_\tau(0) = (G_\tau)'(0)-1 = 0.$$

It is clear that $K_{\phi,l}(l \geq 2)$, $F_\tau, G_\tau(0 \leq \tau \leq 1)$ belongs to the class $\mathcal{S}^{m,n}_{b,\delta,\mu}(\alpha)$.
If $\mu < \eta_1$ or $\mu > \eta_2$, then equality holds if and only if $f(z)$ is $K_{\phi,2}$ or one of its rotations. If $\eta_1 < \mu < \eta_2$, then equality holds if and only if $f(z)$ is $K_{\phi,3}$ or one of its rotations. If $\mu = \eta_1$, then equality holds if and only if $f(z)$ is F_τ or one of its rotations. If $\mu = \eta_2$, then equality holds if and only if $f(z)$ is G_τ or one of its rotations. □

Theorem 2.34. *let* $\phi(z) = 1+B_1z+B_2z^2+B_3z^3+...$, *with* $B_1 \neq 0$. *If* $f \in \mathcal{S}^{m,n}_{b,\mu,\tau,\eta}(\gamma)$ *satisfies the inequality*

$$Re\left(1+\frac{1}{b}\left(\frac{R^m_{\tau,\eta}f(z)}{R^n_{\tau,\eta}f(z)}-1\right)-\mu\left|\frac{R^m_{\tau,\eta}f(z)}{R^n_{\tau,\eta}f(z)}-1\right|\right) \prec \phi(z),$$

then

$$\left|a_3-\eta_2^2\right| \leq \frac{B_1|b|}{Y_2-b\mu|Y_2|} max\left\{1, \left|\frac{B_2}{B_1}+\left[\frac{[Y_3-b\mu|Y_3|]-\eta\left[[Y_2]-b\mu|Y_2|\right]}{\left[Y_1-b\mu[Y_1]\right]^2}\right]bB_1\right|\right\}, \tag{42}$$

where

$$Y_1 = ((1+(\tau-\eta+2\tau\eta)))^m-((1+(\tau-\eta+2\tau\eta)))^n,$$
$$Y_2 = ((1+2(\tau-\eta+3\tau\eta)))^m-((1+2(\tau-\eta+3\tau\eta)))^n,$$

and

$$Y_3 = ((1 + (\tau - \eta + 2\tau\eta)))^{m+n} - ((1 + (\tau - \eta + 2\tau\eta)))^{2n}.$$

Then, the result is sharp.

3. CONCLUDING REMARKS

In this paper, we obtained several results for a new subclass of normalized analytic univalent functions and compared them with existing results in the literature. The coefficient bounds for the proposed subclass were explicitly established. Moreover, by incorporating the Răducanu–Orhan operator, we extended the Fekete–Szegö inequality to a broader class of holomorphic functions. The results provide new insights into the sharp bounds for the introduced subclass of univalent functions. Future work may explore applications of these findings in related areas such as geometric function theory and conformal mapping. Further investigations could also examine whether these inequalities remain valid for more generalized operators or domains.

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