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Bipolar Fuzzy Filters in BL-Algebras

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Abstract. This research introduces and examines the novel concept of filters in BL-algebras constructed upon bipolar fuzzy structures. Specifically, we present the formulation of bipolar fuzzy filters (BFFs) within the context of BL-algebras and conduct a comprehensive investigation of their associated properties. This work extends the traditional notion of filters in BL-algebras by incorporating the bipolarity aspect of fuzzy logic, thereby providing a more nuanced framework for analyzing logical structures. Our study not only establishes the fundamental definitions but also explores the theoretical implications and characteristics of these bipolar fuzzy filters, contributing to the broader understanding of many-valued logic systems and their algebraic representations.

Key words and Phrases: bipolar fuzzy filters, bipolar fuzzy sets, BL-algebras.

1. INTRODUCTION

The introduction of BL-algebras by Petr Hájek [1] marked a significant advancement in the algebraic investigation of many-valued logic. These structures, which leverage continuous triangular norms—a fundamental concept in fuzzy logic—provide an algebraic perspective on logical systems. Within BL-algebras, filters play a pivotal role, corresponding to sets of demonstrably valid formulas in the associated logical framework. Hájek [1] established the concepts of filters and prime filters (PEFs), ultimately demonstrating the completeness of basic logic (BL) through the utilization of PEFs.

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Subsequent investigations by Esko Turunen [2, 3, 4] further explored the properties of filters and PEFs in BL-algebras. Notably, Turunen [3] introduced and characterized Boolean filters, establishing a connection between their existence and the bipartiteness of BL-algebras. Haveshki et al. [5] extended this analysis by introducing the concept of positive implicative filters.

The foundational notion of fuzzy sets (FSs) and their associated operations were initially proposed by Lotfi A. Zadeh [6], with further developments elaborated in subsequent works [7, 8]. Zadeh's 1965 introduction of fuzzy sets represented a paradigm shift from classical set theory, paving the way for Liu and Li [9] to introduce fuzzy filters (FFs) in BL-algebras. This concept was subsequently extended to encompass fuzzy Boolean and positive implicative filters [10], playing a crucial role in elucidating the structure and behavior of BL-algebras.

The concept of bipolar fuzzy sets (BFSs), a generalization of FSs, was first introduced by Weihua Zhang [11] in 1998, followed by the development of bipolar fuzzy logic [12]. While FSs characterize elements over the unit interval [0, 1], bipolar fuzzy sets extend this characterization to the interval [-1, 1]. This extension allows for a more nuanced representation of membership, distinguishing between elements with irrelevant and contrary characteristics to a given property. For a comprehensive comparison of these concepts, readers are referred to Lee [13].

The traditional fuzzy set theory, with membership degrees ranging over [0, 1], has limitations in distinguishing between elements with irrelevant and contrary characteristics. To address this, K. M. Lee [14] introduced bipolar-valued fuzzy sets (BV-FSs). This concept has been applied to various algebraic structures, including BCK/BCI-algebras by K. J. Lee [15], semigroups by Kim et al. [16], CI-algebras by Jun et al. [17], and other mathematical structures by Akram et al. [18, 19, 20].

Recent developments have seen the application of bipolar fuzzy set theory to Γ-near rings and ordered Γ-near rings [21]. This research introduces the concepts of BFFs and bipolar fuzzy prime ideals within these structures, investigating the correspondence between BFFs and crisp filters, and examining homomorphisms of ordered Γ-near rings.

Building upon these foundational studies, our research focuses on the novel concept of BFFs within BL-algebras. We define these as extensions of traditional fuzzy filters, incorporating the bipolarity aspect to handle both positive and negative memberships. Through a comprehensive examination of the characteristics and behaviors of BFFs, we aim to enrich the theoretical understanding of fuzzy logic in algebraic contexts and uncover new insights into their practical applications.

2. Preliminaries

Definition 2.1. [1] *A **BL-algebra** is an algebra* $(\mathfrak{A}, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *satisfying the following axioms for every* $\zeta, \eta, \gamma \in \mathfrak{A}$*:*

(1) $(\mathfrak{A}, \wedge, \vee, 0, 1)$ *is a bounded lattice.*
(2) $(\mathfrak{A}, \odot, 1)$ *is a commutative monoid.*

(3) $\zeta \odot \eta \leq \gamma$ *if and only if* $\zeta \leq \eta \rightarrow \gamma$.
(4) $\zeta \wedge \eta = \zeta \odot (\zeta \rightarrow \eta)$.
(5) $(\zeta \rightarrow \eta) \vee (\eta \rightarrow \zeta) = 1$.

Lemma 2.2. [1] *For any BL-algebra* $\mathfrak{A}$ *and any* $\zeta, \eta, \gamma \in \mathfrak{A}$ *the following properties hold:*

(BL-1) $\zeta \leq \eta \iff \zeta \rightarrow \eta = 1$.
(BL-2) $0^- = 1$ *and* $1^- = 0$.
(BL-3) $1 \rightarrow \zeta = \zeta$ *and* $0 \rightarrow \zeta = \zeta \rightarrow 1 = \zeta \rightarrow \zeta = 1$.
(BL-4) $\zeta \odot \eta \leq \zeta \wedge \eta$ *and* $\eta \leq \zeta \rightarrow \eta$.
(BL-5) $\zeta \leq \eta \Longrightarrow \zeta \odot \gamma \leq \eta \odot \gamma$.
(BL-6) $\zeta \rightarrow \eta \leq (\eta \rightarrow \gamma) \rightarrow (\zeta \rightarrow \gamma)$.
(BL-7) $\zeta \rightarrow \eta \leq (\gamma \rightarrow \zeta) \rightarrow (\gamma \rightarrow \eta)$.
(BL-8) $\zeta \leq \zeta^{--}$ *and* $\zeta^- = \zeta^{---}$.
(BL-9) $\zeta \odot \zeta^- = 0$ *and* $\zeta \odot 0 = 0$.
(BL-10) $\zeta \rightarrow \eta \leq \zeta \odot \gamma \rightarrow \eta \odot \gamma$.
(BL-11) $(\zeta \odot \eta)^{--} = \zeta^{--} \odot \eta^{--}$.
(BL-12) $(\zeta \vee \eta)^- = \zeta^- \wedge \eta^-$ *and* $(\zeta \wedge \eta)^- = \zeta^- \vee \eta^-$.
(BL-13) $\zeta \odot (\eta \vee \gamma) = \zeta \odot \eta \vee \zeta \odot \gamma$.
(BL-14) $\zeta \rightarrow (\eta \rightarrow \gamma) = (\zeta \odot \eta) \rightarrow \gamma$.
(BL-15) $(\zeta \rightarrow \eta) \odot (\eta \rightarrow \gamma) \leq \zeta \rightarrow \gamma$.
(BL-16) $\zeta \vee \eta = ((\zeta \rightarrow \eta) \rightarrow \eta) \wedge ((\eta \rightarrow \zeta) \rightarrow \zeta)$.
(BL-17) $\zeta \rightarrow (\eta \rightarrow \gamma) = \eta \rightarrow (\zeta \rightarrow \gamma)$.

Definition 2.3. [1] *A* ***filter*** $\mathfrak{F}$ *in a BL- algebra is a non empty subset of* $\mathfrak{A}$ *satisfying:*

(F-1) If $\zeta, \eta \in \mathfrak{F}$*, then* $\zeta \odot \eta \in \mathfrak{F}$.
(F-2) If $\zeta \leq \eta$ *and* $\zeta \in \mathfrak{F}$*, then* $\eta \in \mathfrak{F}$.

Remark 2.4. *A filter* $\mathfrak{F}$ *in a BL-algebra* $\mathfrak{A}$ *is* ***proper*** *whenever* $\mathfrak{F} \neq \mathfrak{A}$.

Proposition 2.5. [1] *Let* $\mathfrak{F}$ *be a non-empty subset in a BL- algebra* $\mathfrak{A}$*. Then* $\mathfrak{F}$ *is a filter of* $\mathfrak{A}$ *if and only if the following conditions hold:*

(F-3) $1 \in \mathfrak{F}$.
(F-4) $\zeta, \zeta \rightarrow \eta \in \mathfrak{F}$ *implies* $\eta \in \mathfrak{F}$.

Definition 2.6. [6] *A* ***fuzzy set*** *of a set* $\mathfrak{X}$ *is a function* $\Im : \mathfrak{X} \rightarrow [0,1]$*. Let* $\Im$ *be a fuzzy set in* $\mathfrak{X}$ *and* $\alpha \in [0,1]$*, the set* $\Im_\alpha = \{x \in \mathfrak{X} / \Im(x) \geq \alpha\}$ *is called a* ***level or cut subset*** *of* $\Im$.

Definition 2.7. [9] *A fuzzy set* $\Im$ *in a BL- algebra* $\mathfrak{A}$ *is a* ***fuzzy filter*** *whenever for every* $\alpha \in [0,1]$ *the* α***-level subset*** $\Im_\alpha$ *is either empty or a filter of* $\mathfrak{A}$.

Theorem 2.8. [9] *If $\Im$ is a fuzzy set in a BL-algebra $\mathfrak{A}$, the following statements are equivalent:*

(a) *$\Im$ is a fuzzy filter in $\mathfrak{A}$.*
(b) *For any $\zeta, \eta \in \mathfrak{A}$ we have:*
(FF-1) $\Im(1) \geq \Im(\zeta)$.
(FF-2) $\Im(\eta) \geq \Im(\zeta) \wedge \Im(\zeta \to \eta)$.
(c) *For any $\zeta, \eta, \gamma \in \mathfrak{A}$ if $\zeta \to (\eta \to \gamma) = 1$ then $\Im((\gamma) \geq \Im(\zeta) \wedge \Im(\eta)$.*
(d) *For any $\zeta,\ \eta, \gamma \in \mathfrak{A}$ if $\zeta \odot \eta \leq \gamma$ then $\Im((\gamma) \geq \Im(\zeta) \wedge \Im(\eta)$.*
(e) *For any $\zeta, \eta \in \mathfrak{A}$ we have:*
(FF-3) *If $\zeta \leq \eta$, then $\Im(\zeta) \leq \Im(\eta)$.*
(FF-4) $\Im(\zeta \odot \eta) \geq \Im(\zeta) \wedge \Im(\eta)$.

Definition 2.9. [6] *Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are two BL-algebras, $\Im$ and ω are fuzzy sets in $\mathfrak{A}$ and $\mathfrak{B}$, respectively, and $f : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism.*

*The **image** of $\Im$ under f, denoted by $f(\Im)$, is a fuzzy set in $\mathfrak{B}$ defined by:*

$$\forall \eta \in \mathfrak{B}, \quad f(\Im)(\eta) = \begin{cases} \sup\limits_{\zeta \in f^{-1}(\eta)} \Im(\zeta) & \text{if } f^{-1}(\eta) \neq \varnothing, \\ 0 & \text{if } f^{-1}(\eta) = \varnothing. \end{cases}$$

*The **preimage** of ω under f, denoted by $f^{-1}(\omega)$, is a fuzzy set in $\mathfrak{A}$ defined by:*

$$\forall \zeta \in \mathfrak{A}, \quad f^{-1}(\omega)(\zeta) = \omega(f(\zeta)).$$

Definition 2.10. [13] *A* bipolar-valued fuzzy set *(BV-FS) $\Im$ in $\mathfrak{X}$ is characterized by a pair of functions $\Im = \{(x; \Im^p(x), \Im^n(x)) \mid x \in \mathfrak{X}\}$, where:*

- $\Im^p : \mathfrak{X} \to [0, 1]$ *is the* positive membership function*,*
- $\Im^n : \mathfrak{X} \to [-1, 0]$ *is the* negative membership function*.*

The positive membership function $\Im^p(x)$ of a BV-FS $\Im = \{(x; \Im^p(x), \Im^n(x)) \mid x \in \mathfrak{X}\}$ quantifies the degree to which a given element x satisfies the property described by the BV-FS. Correspondingly, the negative membership function $\Im^n(x)$ encodes the degree to which x exhibits some implicit counter-property associated with $\Im$. In the case where $\Im^p(x) \neq 0$ and $\Im^n(x) = 0$, the element x is deemed to possess only positive satisfaction with respect to the property characterized by $\Im^p(x)$. Conversely, if $\Im^p(x) = 0$ and $\Im^n(x) \neq 0$, the element x is considered to not satisfy the property of $\Im$, but rather to exhibit some non-zero degree of fulfillment of the associated counter-property. It is also possible for an element x to exhibit $\Im^p(x) \neq 0$ and $\Im^n(x) \neq 0$, indicating that the membership functions of the property and its counter-property overlap for that element over a portion of the domain of discourse. For the sake of notational concision, we shall henceforth utilize the shorthand $\Im = (\Im^p, \Im^n)$ to represent the BV-FS $\Im = \{(x; \Im^p(x), \Im^n(x)) \mid x \in \mathfrak{X}\}$, and refer to these constructs as bipolar fuzzy sets (BFSs) rather than BV-FSs.

3. Motivation and Contribution

The motivation and contributions of this study reflect the evolving landscape of fuzzy logic and algebraic structures. As BL-algebras continue to serve as a foundational framework for non-classical logic, there is a growing need to expand their applicability to more intricate logical systems. This research aims to address that gap by integrating bipolar fuzzy set theory into BL-algebras, providing a more refined approach to handling logical contradictions and complexity. By introducing BFFs and exploring their properties, this study not only expands theoretical understanding but also opens up new avenues for practical applications in various domains of fuzzy logic and decision-making.

3.1. **Motivation.**

The motivation for this research stems from several key factors:

Table 1. Key motivating factors for BFF, along with corresponding authors.

No.	Authors	Motivating Factors
1	Hájek (1998) [1]	The necessity to extend the foundational framework of BL-algebras to encompass more nuanced and comprehensive representations of logical structures, enhancing their utility in modeling non-classical logical systems.
2	Zhang (1998) [11]	The potential of bipolar fuzzy sets to offer a more sophisticated framework for modeling complex logical systems, overcoming limitations in conventional fuzzy set theory.
3	Lee (2000) [14]	The shortcomings of traditional fuzzy set theory in effectively distinguishing between irrelevant and contrary elements, highlighting the need for advanced models to handle logical contradictions.
4	Liu and Li (2005) [9]	The demonstrated success of fuzzy filters (FFs) in BL-algebras, and the potential of bipolar fuzzy concepts in other algebraic structures, encouraging further development in this area.

3.2. **Contribution.**

This study contributes to the field of BL-algebra by:

- Introducing the novel concept of BFFs in BL-algebras, bridging bipolar fuzzy theory with established algebraic structures.
- Establishing fundamental properties and theorems of BFFs, providing a theoretical foundation for future research.
- Developing a framework applicable to various domains of fuzzy logic and related disciplines, enhancing the modeling of complex decision-making processes.
- Extending the application of bipolar fuzzy set theory in algebraic structures, building upon previous work in related mathematical domains.

This research not only advances theoretical understanding but also offers potential practical applications [22] in artificial intelligence, expert systems, and decision analysis, particularly in scenarios requiring consideration of both positive and negative aspects.

4. Bipolar fuzzy filters

Definition 4.1. ***A bipolar fuzzy set*** *$\Im$ in a BL-algebra $\mathfrak{A}$ is a pair $(\Im^p, \Im^n)$ of maps: $\Im^p : \mathfrak{A} \to [0,1]$ and $\Im^n : \mathfrak{A} \to [-1,0]$.*

Definition 4.2. *A bipolar fuzzy set $\Im$ in a BL-algebra $\mathfrak{A}$ is a* ***bipolar fuzzy filter*** *whenever it satisfies:*

(BF-1) $\left.\begin{array}{l}\Im^p(1) \geq \Im^p(\zeta) \text{ and} \\ \Im^n(1) \leq \Im^n(\zeta),\end{array}\right\}$ *for any $\zeta \in \mathfrak{A}$.*

(BF-2) $\left.\begin{array}{l}\Im^p(\eta) \geq \Im^p(\zeta) \wedge \Im^p(\zeta \to \eta) \text{ and} \\ \Im^n(\eta) \leq \Im^n(\zeta) \vee \Im^n(\zeta \to \eta),\end{array}\right\}$ *for any $\zeta, \eta \in \mathfrak{A}$.*

Remark 4.3. *Observe that if $\Im$ is a bipolar fuzzy filter, then $\Im$ determines two fuzzy filters: $\Im^p$ and $-\Im^n$, and conversely, given two fuzzy filters $\Im_1$ and $\Im_2$, then $(\Im_1, -\Im_2)$ is a bipolar fuzzy filter.*

Example 4.4. [9] *Consider the BL-algebra $\mathfrak{A} = \{0, a, b, 1\}$ where $0 < a < b < 1$ with the following operations:*

Table 2. Product Operation

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Table 3. Implication Operation

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Now, let's define a BFS $\Im = (\Im^p, \Im^n)$ on this BL-algebra:

TABLE 4. $\Im^p$ and $\Im^n$

ζ	0	a	b	1
$\Im^p$	0	0.4	0.7	1
$\Im^n$	0	-0.2	-0.5	-1

Let's verify that the given function $\Im = (\Im^p, \Im^n)$ is indeed a BFF on the given BL-algebra.

(1) *First, we verify that $\Im$ is a BFS:*
- $\Im^p(\zeta) \in [0, 1]$ *for all* $\zeta \in \mathfrak{A}$.
- $\Im^n(\zeta) \in [-1, 0]$ *for all* $\zeta \in \mathfrak{A}$.

These conditions are satisfied for all elements in $\mathfrak{A}$.

(2) *Now, we check the BFF conditions:*

BF-1. $\Im^p(1) \geq \Im^p(\zeta)$ *and* $\Im^n(1) \leq \Im^n(\zeta)$ *for all* $\zeta \in \mathfrak{A}$.
We observe that $\Im^p(1) = 1$, *which is the maximum value of* $\Im^p$, *and* $\Im^n(1) = -1$, *which is the minimum value of* $\Im^n$. *Thus, this condition is satisfied.*

BF-2. $\Im^p(\eta) \geq \Im^p(\zeta) \wedge \Im^p(\zeta \to \eta) = \min\{\Im^p(\zeta), \Im^p(\zeta \to \eta)\}$ *and* $\Im^n(\eta) \leq \Im^p(\zeta) \vee \Im^p(\zeta \to \eta) = \max\{\Im^n(\zeta), \Im^n(\zeta \to \eta)\}$ *for all* $\zeta, \eta \in \mathfrak{A}$.
To verify this, we need to check all possible combinations of ζ *and* η. *Let's examine all cases:*

TABLE 5. Verification of BF-2 condition for $\Im^p$

ζ	η	$\zeta \to \eta$	$\Im^p(\zeta) \wedge \Im^p(\zeta \to \eta) = \min\{\Im^p(\zeta), \Im^p(\zeta \to \eta)\}$	$\Im^p(\eta)$	*Condition*
0	0	1	$\min\{0, 1\} = 0$	0	$0 \geq 0$
0	a	1	$\min\{0, 1\} = 0$	0.4	$0.4 \geq 0$
0	b	1	$\min\{0, 1\} = 0$	0.7	$0.7 \geq 0$
0	1	1	$\min\{0, 1\} = 0$	1	$1 \geq 0$
a	0	0	$\min\{0.4, 0\} = 0$	0	$0 \geq 0$
a	a	1	$\min\{0.4, 1\} = 0.4$	0.4	$0.4 \geq 0.4$
a	b	1	$\min\{0.4, 1\} = 0.4$	0.7	$0.7 \geq 0.4$
a	1	1	$\min\{0.4, 1\} = 0.4$	1	$1 \geq 0.4$
b	0	0	$\min\{0.7, 0\} = 0$	0	$0 \geq 0$
b	a	a	$\min\{0.7, 0.4\} = 0.4$	0.4	$0.4 \geq 0.4$
b	b	1	$\min\{0.7, 1\} = 0.7$	0.7	$0.7 \geq 0.7$
b	1	1	$\min\{0.7, 1\} = 0.7$	1	$1 \geq 0.7$
1	0	0	$\min\{1, 0\} = 0$	0	$0 \geq 0$
1	a	a	$\min\{1, 0.4\} = 0.4$	0.4	$0.4 \geq 0.4$
1	b	b	$\min\{1, 0.7\} = 0.7$	0.7	$0.7 \geq 0.7$
1	1	1	$\min\{1, 1\} = 1$	1	$1 \geq 1$

As we can see from Tables 5 and 6, all combinations of ζ *and* η *satisfy the conditions for both* $\Im^p$ *and* $\Im^n$. *Therefore, we can conclude that* $\Im$ *is indeed a BFF on this BL-algebra.*

Figure 1 of Example 4.4 illustrates the behavior of a BFF across the BL-algebra elements {0, a, b, 1}, where $a = 0.33$ *and* $b = 0.67$. *The x-axis represents these BL-algebra elements, the y-axis distinguishes between positive (upper surface)*

TABLE 6. Verification of BF-2 condition for $\Im^n$

ζ	η	$\zeta \to \eta$	$\Im^n(\zeta) \vee \Im^n(\zeta \to \eta) = \max\{\Im^n(\zeta), \Im^n(\zeta \to \eta)\}$	$\Im^n(\eta)$	*Condition*
0	*0*	*1*	$\max\{0, -1\} = 0$	*0*	$0 \leq 0$
0	*a*	*1*	$\max\{0, -1\} = 0$	*-0.2*	$-0.2 \leq 0$
0	*b*	*1*	$\max\{0, -1\} = 0$	*-0.5*	$-0.5 \leq 0$
0	*1*	*1*	$\max\{0, -1\} = 0$	*-1*	$-1 \leq 0$
a	*0*	*0*	$\max\{-0.2, 0\} = 0$	*0*	$0 \leq 0$
a	*a*	*1*	$\max\{-0.2, -1\} = -0.2$	*-0.2*	$-0.2 \leq -0.2$
a	*b*	*1*	$\max\{-0.2, -1\} = -0.2$	*-0.5*	$-0.5 \leq -0.2$
a	*1*	*1*	$\max\{-0.2, -1\} = -0.2$	*-1*	$-1 \leq -0.2$
b	*0*	*0*	$\max\{-0.5, 0\} = 0$	*0*	$0 \leq 0$
b	*a*	*a*	$\max\{-0.5, -0.2\} = -0.2$	*-0.2*	$-0.2 \leq -0.2$
b	*b*	*1*	$\max\{-0.5, -1\} = -0.5$	*-0.5*	$-0.5 \leq -0.5$
b	*1*	*1*	$\max\{-0.5, -1\} = -0.5$	*-1*	$-1 \leq -0.5$
1	*0*	*0*	$\max\{-1, 0\} = 0$	*0*	$0 \leq 0$
1	*a*	*a*	$\max\{-1, -0.2\} = -0.2$	*-0.2*	$-0.2 \leq -0.2$
1	*b*	*b*	$\max\{-1, -0.5\} = -0.5$	*-0.5*	$-0.5 \leq -0.5$
1	*1*	*1*	$\max\{-1, -1\} = -1$	*-1*	$-1 \leq -1$

and negative (lower surface) membership types, and the z-axis shows the membership values ranging from -1 to 1.

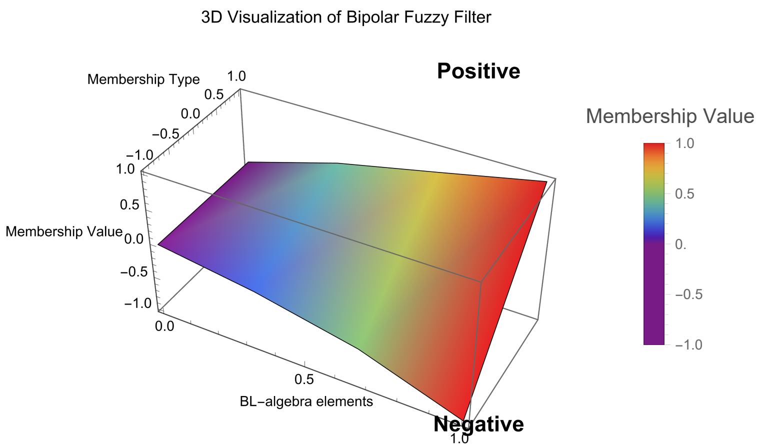

FIGURE 1. 3D Visualization of BFF

The visualization in Figure 1 reveals several important characteristics of the BFF:

- ***Monotonicity:*** *The positive membership function increases monotonically from 0 to 1, while the negative membership function decreases monotonically from 0 to -1 (BF-1).*
- ***Non-linear Progression:*** *The change in membership values is not linear across the BL-algebra, allowing for nuanced representation of fuzzy concepts.*

- ***Smoothness of Surfaces:*** *The smooth, continuous nature of both surfaces suggests that the functions satisfy condition BF-2, as the membership values change gradually and consistently across the BL-algebra.*

(1) ***Upper Surface (Positive Membership Function*** $\Im^p$***):***
   - *Starts at 0 when* $x = 0$.
   - *Increases monotonically as x increases.*
   - *Reaches 1 when* $x = 1$.
   - *This behavior satisfies condition BF-1.*

(2) ***Lower Surface (Negative Membership Function*** $\Im^n$***):***
   - *Starts at 0 when* $x = 0$.
   - *Decreases monotonically as x increases.*
   - *Reaches -1 when* $x = 1$.
   - *This behavior satisfies condition BF-1.*

(3) ***Specific Points:***
   - *At* $x = 0 : \Im^p(0) = 0$, $\Im^n(0) = 0$.
   - *At* $x = a(0.33) : \Im^p(a) = 0.4$, $\Im^n(a) = -0.2$.
   - *At* $x = b(0.67) : \Im^p(b) = 0.7$, $\Im^n(b) = -0.5$.
   - *At* $x = 1 : \Im^p(1) = 1$, $\Im^n(1) = -1$ *(maximum certainty).*

These specific points illustrate how the membership values change across the BL-algebra, satisfying the monotonicity requirements of BF-1. The figure thus provides a visual representation of how a BFF satisfies its defining conditions across the BL-algebra, offering insights into its behavior and properties.

Example 4.5. *Consider the BL-algebra* $\mathfrak{A} = \{0, a, b, c, 1\}$ *with the partial order* $0 < a < b < 1$ *and* $0 < a < c < 1$, *with the following operations:*

TABLE 7. Product ($\odot$) Operation

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

The Hasse diagram visually represents the partial order of the elements in this BL-algebra.

TABLE 8. Implication ($\rightarrow$) Operation

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

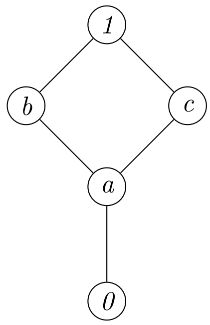

FIGURE 2. Hasse Diagram for BL-algebra {0, a, b, c, 1}

Now, let's define the BFS $\Im = (\Im^p, \Im^n)$ *on this BL-algebra as follows:*

TABLE 9. $\Im^p$ and $\Im^n$

ζ	0	a	b	c	1
$\Im^p$	0.2	0.5	0.8	0.7	1
$\Im^n$	-0.1	-0.4	-0.6	-0.8	-1

It is easily verified that $\Im$ *is a BFF.*

Example 4.6. *Let* $\mathfrak{A}$ *be the BL-algebra in Example 4.4. Now, define the BFS* $\Im = (\Im^p, \Im^n)$ *as follows:*

TABLE 10. BFS $(\Im^p, \Im^n)$

ζ	0	a	b	1
$\Im^p$	0	0.4	0.3	1
$\Im^n$	0	-0.2	-0.5	-1

We check the condition (BF-2):

For $\zeta = a$ and $\eta = b$:

- $a \to b = 1$.
- $\Im^p(a) \wedge \Im^p(a \to b) = \min\{\Im^p(a), \Im^p(a \to b)\} = \min\{0.4, 1\} = 0.4$.
- *However,* $\Im^p(b) = 0.3$*, which is less than 0.4.*

Since $\Im^p(b) < \min\{\Im^p(a), \Im^p(a \to b)\}$*, the function* $\Im$ *does not satisfy (BF-2), and thus it is not a bipolar fuzzy filter.*

Theorem 4.7. *Let $\Im$ be a bipolar fuzzy set in $\mathfrak{A}$, the following statements are equivalent:*

(a) $\Im$ is a bipolar fuzzy filter in $\mathfrak{A}$.
(b) For any $\zeta, \eta \in \mathfrak{A}$ we have

$$\zeta \to (\eta \to \gamma) = 1 \Rightarrow \begin{cases} \Im^p(\gamma) \geq \Im^p(\zeta) \wedge \Im^p(\eta) \text{ and} \\ \Im^n(\gamma) \leq \Im^n(\zeta) \vee \Im^n(\eta). \end{cases}$$

Proof. (a) $\Rightarrow$ (b). We have $\Im^p(\gamma) \geq \Im^p(\eta) \wedge \Im^p(\eta \to \gamma)$ and $\Im^p(\eta \to \gamma) \geq \Im^p(\zeta) \wedge \Im^p(\zeta \to (\eta \to \gamma)$). Therefore

$$\Im^p(\gamma) \geq \Im^p(\eta) \wedge \Im^p(\eta \to \gamma) \geq \Im^p(\eta) \wedge \Im^p(\zeta) \wedge \Im^p(\zeta \to (\eta \to \gamma)) = \Im^p(\eta) \wedge \Im^p(\zeta),$$

whenever $\zeta \to (\eta \to \gamma) = 1$. Similarly we have $\Im^n(\gamma) \leq \Im^n(\zeta) \vee \Im^n(\eta)$.
(b) $\Rightarrow$ (a). Since $\zeta \to (\zeta \to 1) = 1$, then $\Im^p(1) \geq \Im^p(\zeta) \wedge \Im^p(\zeta) = \Im^p(\zeta)$. On the other hand we have: $(\zeta \to \eta) \to (\zeta \to \eta) = 1$, hence $\Im^r(\eta) \geq \Im^p(\zeta) \wedge \Im^p(\zeta \to \eta)$. Similarly we have $\Im^n(1) \leq \Im^n(\zeta)$ and $\Im^n(\eta) \leq \Im^n(\zeta) \vee \Im^n(\zeta \to \eta)$ for every $\zeta, \eta \in \mathfrak{A}$. ☐

Throughout this text $\mathfrak{A}$ will be a BL-algebra, and since for every bipolar fuzzy subset $\Im$ the properties of $\Im^p$ and the properties of $\Im^n$ are dual, therefore, if it is not necessary, we avoid proofs in the case of $\Im^n$.

Since $\zeta \to (\eta \to \gamma) = \zeta \odot \eta \to \gamma$, then we have:

Corollary 4.8. *Let $\Im$ be a bipolar fuzzy set in $\mathfrak{A}$, the following statements are equivalent:*

(a) $\Im$ is a bipolar fuzzy filter in $\mathfrak{A}$.
(b) For any $\zeta, \eta \in \mathfrak{A}$ we have

$$\zeta \odot \eta \leq \gamma \Rightarrow \begin{cases} \Im^p(\gamma) \geq \Im^p(\zeta) \wedge \Im^p(\eta) \text{ and} \\ \Im^n(\gamma) \leq \Im^n(\zeta) \vee \Im^n(\eta). \end{cases}$$

Theorem 4.9. *Let $\Im$ be a bipolar fuzzy set in $\mathfrak{A}$, the following statements are equivalent:*

(a) $\Im$ is a bipolar fuzzy filter in $\mathfrak{A}$.

(b) *(BF-3) If* $\zeta \leq \eta$ *then* $\begin{cases} \Im^p(\zeta) \leq \Im^p(\eta) \text{ and} \\ \Im^n(\zeta) \geq \Im^n(\eta). \end{cases}$

(BF-4) $\left.\begin{cases} \Im^p(\zeta \odot \eta) \geq \Im^p(\zeta) \wedge \Im^p(\eta) \text{ and} \\ \Im^n(\zeta \odot \eta) \leq \Im^n(\zeta) \vee \Im^n(\eta), \end{cases}\right\}$ *for every* $\zeta, \eta \in \mathfrak{A}$.

Proof. (a) ⇒ (b). If $\zeta \leq \eta$, then $\zeta \odot \zeta \leq \zeta \leq \eta$, hence $\Im^p(\eta) \geq \Im^p(\zeta) \wedge \Im^p(\zeta) = \Im^p(\zeta)$. On the other hand, $\zeta \odot \eta \leq \zeta \odot \eta$, hence $\Im^p(\zeta \odot \eta) \geq \Im^p(\zeta) \wedge \Im^p(\eta)$.
(b) ⇒ (a). If $\zeta \odot \eta \to \gamma = 1 (\Leftrightarrow \zeta \odot \eta \leq \gamma)$, then $\Im^p(\gamma) \geq \Im^p(\zeta \odot \eta) \geq \Im^p(\zeta) \wedge \Im^p(\eta)$, and apply Corollary 4.8. □

Proposition 4.10. *If* $\Im$ *is a bipolar fuzzy filter in* $\mathfrak{A}$*, then the following statements hold:*

(1) *If* $\zeta \to \eta = 1$*, then* $\begin{cases} \Im^p(\zeta) \leq \Im^p(\eta) \text{ and} \\ \Im^n(\zeta) \geq \Im^n(\eta). \end{cases}$

(2) $\begin{cases} \Im^p(\zeta \odot \eta) = \Im^p(\zeta) \wedge \Im^p(\eta) \text{ and} \\ \Im^n(\zeta \odot \eta) = \Im^n(\zeta) \vee \Im^n(\eta). \end{cases}$

(3) $\begin{cases} \Im^p(0) = \Im^p(\zeta) \wedge \Im^p(\zeta^-) \text{ and} \\ \Im^n(0) = \Im^n(\zeta) \vee \Im^n(\zeta^-). \end{cases}$

Proof. (1) We have $\zeta \to (\zeta \to \eta) = 1$, hence $\Im^p(\eta) \geq \Im^p(\zeta) \wedge \Im^p(\zeta) = \Im^p(\zeta)$.

(2) Since $\zeta \odot \eta \leq \zeta, \eta$, then $\Im^p(\zeta \odot \eta) \leq \Im^p(\zeta) \wedge \Im^p(\eta)$. On the other hand, $\Im^p(\zeta \odot \eta) \geq \Im^p(\zeta) \wedge \Im^p(\eta)$, by Theorem 4.9.

(3) Since $\zeta \odot \zeta^- = 0$, hence $\Im^p(0) = \Im^p(\zeta) \wedge \Im^p(\zeta^-)$.

□

Theorem 4.11. *The intersection of a family of bipolar fuzzy filters in a BL-algebra* $\mathfrak{A}$ *is a bipolar fuzzy filter.*

Proof. Let $\{(\Im_i^p, \Im_i^n) \mid i \in I\}$ be a family of bipolar fuzzy filters, define $\Im$ as follows:

$$\Im^p(\zeta) = \wedge_i \Im_i^p(\zeta), \text{ and } \Im^n(\zeta) = \vee_i \Im_i^n(\zeta),$$

for any $\zeta \in \mathfrak{A}$. In this case, for any $\zeta, \eta \in \mathfrak{A}$ we have:

$$\Im^p(\zeta) = \wedge_i \Im_i^p(\zeta) \leq \wedge_i \Im_i^p(1) = \Im^p(1).$$

$$\begin{aligned} \Im^p(\eta) = \wedge_i \Im_i^p(\eta) \geq \wedge_i \left(\Im_i^p(\zeta) \wedge \Im_i^p(\zeta \to \eta)\right) &\geq (\wedge_i \Im_i^p(\zeta)) \wedge (\wedge_i \Im_i^p(\zeta \to \eta)) \\ &= \Im^p(\zeta) \wedge \Im^p(\zeta \to \eta). \end{aligned}$$

□

Corollary 4.12. *For any bipolar fuzzy subset* $\Im$ *in a BL-algebra* $\mathfrak{A}$ *there exists a smallest bipolar fuzzy filter containing* $\Im$.

Definition 4.13. *Let $\Im$ be a bipolar fuzzy subset in a BL-algebra $\mathfrak{A}$, the (α, β)-**level subset** of $\Im$ is*

$$\Im_{(\alpha,\beta)} = \{\zeta \in \mathfrak{A} \mid \Im^p(\zeta) \geq \alpha \text{ and } \Im^n(\zeta) \leq \beta\}$$

for $\alpha \in [0, 1]$ and $\beta \in [-1, 0]$.

Theorem 4.14. *Let $\Im$ be a bipolar fuzzy subset in $\mathfrak{A}$, the following statements are equivalent:*

(a) *$\Im$ is a bipolar fuzzy filter.*
(b) *For any α, β the (α, β)-level is either empty or a filter in $\mathfrak{A}$.*

Proof. (a) $\Rightarrow$ (b). Let (α, β) such that $\Im_{(a,\beta)} \neq \varnothing$. For any $\zeta, \eta \in \mathfrak{A}$ such that $\zeta \leq \eta$ and $\zeta \in \Im_{(\alpha,\beta)}$ we have $\Im^p(\eta) \geq \Im^p(\zeta) \geq \alpha$, and $\Im^n(\eta) \leq \Im^n(\zeta) \leq \beta$, hence $\eta \in \Im_{(\alpha,\beta)}$. On the other hand, for any $\zeta, \eta \in \Im_{(\alpha,\beta)}$, since $\Im^p(\zeta), \Im^p(\eta) \geq \alpha$, then $\Im^p(\zeta \odot \eta) \geq \Im^p(\zeta) \wedge \Im^p(\eta) \geq \alpha$, and similarly $\Im^n(\zeta \odot \eta) \leq \beta$. Therefore, $\zeta \odot \eta \in \Im_{(\alpha,\beta)}$, and $\Im_{(\alpha,\beta)}$ is a filter in $\mathfrak{A}$.
(b) $\Rightarrow$ (a). We'll use Theorem 4.9. Let $\zeta \leq \eta$, and $\alpha = \Im^p(\zeta), \beta = \Im^n(\zeta)$, then $\zeta \in \Im_{(\alpha,\beta)}$; hence $\eta \in \Im_{(\alpha,\beta)}$, so $\Im^p(\eta) \geq \alpha = \Im^p(\zeta)$ and $\Im^n(\eta) \leq \beta = \Im^n(\zeta)$. On the other hand, for any $\zeta, \eta \in \mathfrak{A}$ consider $\Im^p(\zeta) = \alpha_1, \Im^p(\eta) = \alpha_2, \Im^n(\zeta) = \beta_1$ and $\Im^n(\eta) = \beta_2$, and define $\alpha = \alpha_1 \wedge \alpha_2$ and $\beta = \beta_1 \vee \beta_2$. Therefore, $\zeta, \eta \in \Im_{(\alpha,\beta)}$, so $\zeta \odot \eta \in \Im_{(\alpha,\beta)}$. This means $\Im^p(\zeta \odot \eta) \geq \alpha = \Im^p(\zeta) \wedge \Im^p(\eta)$, and $\Im^n(\zeta \odot \eta) \leq \beta = \Im^n(\zeta) \vee \Im^n(\eta)$. □

Definition 4.15. *Let $T \subseteq \mathfrak{A}$ be a subset, we have a bipolar fuzzy subset $\Im$ associated to T and defined as*

$$\Im^p(\zeta) = \begin{cases} 1, & \text{if } \zeta \in T, \\ 0, & \text{if } \zeta \notin T \end{cases} \qquad \Im^n(\zeta) = \begin{cases} -1, & \text{if } \zeta \in T \\ 0, & \text{if } \zeta \notin T \end{cases}$$

We can characterize when $\Im$ is a bipolar fuzzy filter.

Theorem 4.16. *Let $T \subseteq \mathfrak{A}$ be a non-empty subset; the following statements are equivalent:*

(a) *$\Im$ is a bipolar fuzzy filter.*
(b) *$\Im^p$ is a fuzzy filter.*
(c) *T is a filter in $\mathfrak{A}$.*

Proof. (a) $\Rightarrow$ (b). It is clear that $\Im^p$ is a fuzzy filter, indeed, if $\zeta \leq \eta$ then $\Im^p(\zeta) \leq \Im^p(\eta)$, and $\Im^p(\zeta \odot \eta) \geq \Im^p(\zeta) \wedge \Im^p(\eta)$, for any $\zeta, \eta \in \mathfrak{A}$.
(b) $\Rightarrow$ (c). Let $\zeta, \eta \in \mathfrak{A}$. If $\zeta, \eta \in T$, since $\Im^p(\zeta \odot \eta) \geq \Im^p(\zeta) \wedge \Im^p(\eta) = 1$, then $\zeta \odot \eta \in T$. On the other hand, if $\zeta \leq \eta$ and $\zeta \in T$, since $\Im^p(\eta) \geq \Im^p(\zeta) = 1$, then $\eta \in T$. Therefore, $T \subseteq \mathfrak{A}$ is a filter.
(c) $\Rightarrow$ (a). We'll use Theorem 4.9. If $\zeta \leq \eta$ and $\zeta \in T$, then $\eta \in T$, and $1 = \Im^p(\zeta) = \Im^p(\eta)$; similarly $-1 = \Im^n(\zeta) = \Im^n(\eta)$. If $\zeta \notin T$ then $\Im^p(\zeta) = 0 \leq \Im^p(\eta)$ and $\Im^n(\zeta) = 0 \geq \Im^n(\eta)$. On the other hand, we consider $\zeta \odot \eta$; since $\zeta \odot \eta \in T$,

then $\Im^p(\zeta \odot \eta) = 1 \geq \Im^p(\zeta) \wedge \Im^p(\eta)$; similarly for $\Im^n$. If $\zeta \odot \eta \notin T$, then either $\zeta \notin T$ or $\eta \notin T$, hence $\Im^p(\zeta) \wedge \Im^p(\eta) = 0 \leq \Im^p(\zeta \odot \eta)$; similarly for $\Im^n$. □

Definition 4.17. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a BL-algebra map, for any bipolar fuzzy subset $\Im$ in $\mathfrak{B}$ define a bipolar fuzzy subset $f^{-1}(\Im)$ in $\mathfrak{A}$ as follows (the **preimage** of $\Im$) :*

$$\left(f^{-1}(\Im)\right)^p(\zeta) = \Im^p(f(\zeta)), \text{ and } \left(f^{-1}(\Im)\right)^n(\zeta) = \Im^n(f(\zeta)), \text{ for any } \zeta \in \mathfrak{A}.$$

Theorem 4.18. *In the above situation, for any bipolar fuzzy filter $\Im$ in $\mathfrak{B}$ we have that $f^{-1}(\Im)$ is a bipolar fuzzy filter in $\mathfrak{A}$.*

Proof. For any $\zeta \in \mathfrak{A}$, we have:

$$\left(f^{-1}(\Im)\right)^p(\zeta) = \Im^p(f(\zeta)) \leq \Im^p(1) = \Im^p(f(1)) = \left(f^{-1}(\Im)\right)^p(1).$$

On the other hand, for any $\zeta, \eta \in \mathfrak{A}$ we have:

$$\begin{aligned}(f^{-1}(\Im))^p(\eta) = \Im^p(f(\eta)) &\geq \Im^P(f(\zeta)) \wedge \Im^p(f(\zeta) \to f(\eta)) \\ &= \Im^p(f(\zeta)) \wedge \Im^p(f(\zeta \to \eta)) \\ &= \left(f^{-1}(\Im)\right)^p(\zeta) \wedge \left(f^{-1}(\Im)\right)^p(\zeta \to \eta).\end{aligned}$$

□

Definition 4.19. *In the above situation, if $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a surjetive BL-algebra map and $\Im$ is a bipolar fuzzy subset in $\mathfrak{A}$, then we can define a bipolar fuzzy subset $f(\Im)$ in $\mathfrak{B}$ as follows (the **image** of $\Im$) :*

$$\begin{cases} f(\Im)^p(\upsilon) = \wedge\{\Im^p(\zeta) \mid f(\zeta) = \upsilon\}, \text{ and} \\ f(\Im)^n(\upsilon) = \vee\{\Im^n(\zeta) \mid f(\zeta) = \upsilon\}. \end{cases}$$

Remark 4.20. *In this case we have if $\Im$ is a bipolar fuzzy filter of $\mathfrak{A}$ we can not assure that $f(\Im)$ is a bipolar fuzzy filter in $\mathfrak{B}$; a sufficient condition to obtain a bipolar fuzzy filter is that f is an isomorphism.*

5. Comparative Analysis and Discussion

The concept of BFFs in BL-algebras marks a substantial advancement in the study of fuzzy algebraic structures. This section provides a comparative analysis with existing approaches, discussing the implications, advantages, and potential applications of this novel framework.

5.1. Comparative Analysis.

The following table presents a comparative overview of Classical Filters, Fuzzy Filters, and Bipolar Fuzzy Filters in BL-algebras:

TABLE 11. Comparative Analysis of Filters in BL-algebras

Aspect	**Classical Filters** [1, 2, 3, 4, 5]	**Fuzzy Filters** [9, 10]	**Bipolar Fuzzy Filters (Proposed)**
Membership Theory	Crisp set theory (binary)	Gradual membership	Bipolar membership (positive and negative)
Strengths	Well-established, mathematically rigorous	Models uncertainty and vagueness	Comprehensive uncertainty handling, aligns with cognitive processes
Limitations	Cannot handle uncertainty or gradual membership	Unipolar representation; lacks negative information	Higher computational complexity, requires new interpretation methodologies

5.2. **Discussion.**

The introduction of BFFs in BL-algebras offers several significant advantages and opens up new research directions:

(1) **Enhanced Representation of Uncertainty:** By incorporating both positive and negative membership functions, BFFs provide a more nuanced representation of uncertainty in BL-algebras. This is particularly valuable in scenarios where the absence of a property is not equivalent to its negation.
(2) **Cognitive Alignment:** The bipolar approach aligns more closely with human cognitive processes, which often involve simultaneous consideration of positive and negative aspects. This makes BFFs potentially more intuitive and applicable in decision-making systems.
(3) **Unified Framework:** BFFs offer a generalized framework that can subsume both classical and fuzzy filters as special cases. This unification provides a more comprehensive approach to filter theory in BL-algebras.
(4) **Novel Algebraic Properties:** The introduction of bipolarity leads to new algebraic properties and theorems, enriching the theoretical landscape of BL-algebras. This opens up possibilities for new insights and applications in abstract algebra.
(5) **Interdisciplinary Connections:** BFFs provide a natural bridge to other theories in fuzzy mathematics and beyond, potentially fostering interdisciplinary research.

The framework of BFFs in BL-algebras offers significant potential across diverse disciplines. It provides a robust foundation for addressing complex decision-making, logical reasoning, and systems with bipolar characteristics. Applications span areas such as multi-criteria decision-making, artificial intelligence, control systems, medical diagnosis, and social network analysis [6, 7].

BFFs offer a rigorous mathematical model to handle duality, uncertainty, and complex trade-offs, enabling the development of advanced algorithms for nuanced inputs and outcomes. This enhances decision-making, system optimization, and the accurate modeling of complex phenomena, with profound implications for fields like artificial intelligence and control theory [22]. In conclusion, BFFs in BL-algebras represent a significant theoretical advancement, offering a more sophisticated and flexible approach to filter theory. This concept not only enhances our understanding of BL-algebras but also provides a powerful tool for modeling complex, real-world scenarios characterized by bipolarity and uncertainty. As research in this area progresses, we anticipate seeing further theoretical developments and practical applications that leverage the unique capabilities of BFFs.

6. Concluding Remarks

This research presented a significant advancement in the fields of algebraic logic and fuzzy set theory through the introduction and rigorous examination of BFFs within BL-algebras. Our work established fundamental definitions and a key theorem, extending the theoretical foundations laid by Hájek's seminal work on BL-algebras [1] and Zhang's pioneering development of BFS [11].

The formulation of BFFs represented a substantial step forward in synthesizing bipolar fuzzy structures with BL-algebras, addressing limitations in traditional fuzzy set theory identified by Lee [14]. This novel construct offered a more sophisticated framework for analyzing and representing logical structures, particularly in contexts where both positive and negative aspects required consideration.

Our findings not only enhanced the theoretical underpinnings of fuzzy logic and algebraic structures but also presented potential practical applications. BFFs improved our capacity to model complex decision-making processes and logical systems, potentially driving advancements in artificial intelligence, expert systems, and related fields.

The integration of bipolar fuzzy concepts with BL-algebras through BFFs bridged multiple areas of study, providing new analytical tools for both theoretical exploration and practical problem-solving. This research contributed to the ongoing discourse in fuzzy logic and algebraic theory, offering a more nuanced approach to representing and analyzing complex logical structures.

In conclusion, this study advanced our theoretical understanding of fuzzy logic systems and established a robust foundation for more sophisticated modeling of logical structures. The introduction of BFFs in BL-algebras enriched the existing body of knowledge and opened new avenues for research and application in fuzzy logic domains.

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