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A New Slope Averaging Technique in Heun's Method: Combining Harmonic and Root Mean Square Means

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Abstract. This paper introduces a modified version of Heun's method for solving initial value problems (IVPs) in ordinary differential equations (ODEs). The proposed method replaces the standard arithmetic mean used for the average slope in Heun's method with a novel combination of harmonic mean and root mean square (RMS) for the slopes of the tangent lines. This alteration is designed to enhance the accuracy and performance of the method. Through theoretical analysis, we establish that the modified method retains stability and consistency properties, crucial for reliable numerical solutions. Extensive numerical experiments demonstrate that the method performs well across a range of step sizes, showing notable improvements in accuracy for both small and large step sizes when compared to the standard Heun's method. These results suggest that the new approach offers a robust alternative for solving IVPs, particularly in scenarios where adaptive step sizing or high precision is required.

Key words and phrases: initial value problems, Heun's method, Euler's method, harmonic mean, root mean square.

1. INTRODUCTION

Numerical solutions to initial value problems (IVPs),

$$\begin{aligned} \frac{dy}{dt} &= \xi(t, y), \\ y(0) &= y_0, \end{aligned} \tag{1}$$

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play a critical role in scientific and engineering applications, as exact analytical solutions are often unattainable [1]. These problems typically involve solving differential equations that model various real-world systems, ranging from fluid dynamics to population growth models. Over the years, a variety of classical methods have been developed to provide approximate solutions to IVPs, including the Euler method, Heun's method, and the widely-used Runge-Kutta family of methods. Each of these approaches aims to generate numerical solutions by approximating the integral of the function over small time intervals, relying on different strategies for calculating the slope of the function.

The Euler method, one of the simplest techniques, estimates the solution using the slope at the initial point. While it provides a straightforward approximation, its first-order accuracy often results in significant errors for stiff or complex problems [2]. Heun's method, sometimes referred to as the improved Euler method, offers a refinement by averaging the slope at the beginning and the end of the interval, thereby increasing the order of accuracy of two. Runge-Kutta methods, on the other hand, provide a general framework that further improves the accuracy and stability by computing intermediate slopes within the interval. These classical methods form the foundation for many numerical solvers in use today.

Despite their widespread use, these traditional methods such as Euler's method, Heun's method, and Runge-Kutta method, are not without their limitations, and over time, mathematicians have proposed modifications to enhance their accuracy, stability, and efficiency. For instance, adaptive step-size control, higher-order methods, combining transcendental functions on finite difference, and enhanced stability conditions have been incorporated into existing techniques to handle more complex or stiff differential equations as can be studied in [3, 4, 5, 6, 7, 8, 9, 1]. These improvements have proven particularly useful for problems where the classical methods exhibit poor convergence or suffer from excessive computational cost.

In this paper, we propose a novel modification to Heun's method by altering how the average slope of the tangents is computed. Instead of using the arithmetic mean, we replace it with the mean of harmonic mean and the root mean. The harmonic mean, which emphasizes smaller values [10], and the root mean square, which provides a balanced approach between the arithmetic and geometric means [11], both offer potential improvements in accuracy and stability for specific classes of problems.

The motivation for choosing this combination arises from the nature of stiff initial value problems in differential equations, where maintaining numerical stability while controlling local truncation error is crucial. As known, the harmonic mean emphasize the smaller of two values, which suggests it may reduce the effect of large tangent slopes and hence enhance numerical stability of stiff problems. Similarly, the root mean square gives more weight to larger values of slope magnitude, which suggests a potential for better approximation in regions of rapid variation. Motivated by these complementary properties, we propose combining them and develop a new method for solving initial value problems, in particular stiff problem as the model.

The rest of the paper is structured as follows. In Section 2, we provide the derivation of our proposed method, the stability and consistency analysis. Section 3 we present a series of numerical experiments comparing the performance of the modified methods against traditional approaches. Finally, Section 4 offers a conclusion and discusses potential future directions for research, including further refinements and possible applications in more complex differential equation models.

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2. MAIN RESULTS

2.1. **Modified Heun's Method.**

There are several methods to solve IVPs, such as Euler's method (EuM) [12], which is defined by

$$y_{n+1}^{E} = y_n + h\xi(t_n, y_n) \tag{2}$$

for $n = 0, 1, 2, \ldots$. Heun's method (HeM) [12] is called a predictor-corrector method where it uses Euler's method as the predictor and the corrector is the average of the slope. This method is defined as

$$y_{n+1} = y_n + \frac{h}{2}\left(\xi(t_n, y_n) + \xi(t_{n+1}, y_{n+1}^{E})\right) \tag{3}$$

where y_{n+1}^{E} is given by (2). In [13], modification to Heun's method is done by changing the average slope with harmonic mean. This method is referred to as HM and is given by the following equation:

$$y_{n+1} = y_n + h\left(\frac{J_1^2 + J_2^2}{J_1 + J_2}\right) \tag{4}$$

where $J_1 = \xi(t_n, y_n)$ and $J_2 = \xi(t_{n+1}, y_{n+1}^{E})$. Another modification to Heun's method with the same manner is given in [6] where it incorporates arithmetic mean and second-order harmonic mean as the the average of the slope of the tangent of (1). This method is given by

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{J_1 + J_2}{2} + \frac{J_1^2 + J_2^2}{J_1 + J_2}\right) \tag{5}$$

and is called AHM.

The proposed method is developed by assuming that the average of the slope of the tangent in (3) is a combination of the harmonic mean and the root mean square.

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{2J_1 J_2}{J_1 + J_2} + \frac{\sqrt{2{J_1}^2 + 2{J_2}^2}}{2}\right). \tag{6}$$

This combination is motivated by the complementary properties of the two means. The harmonic mean gives greater weight to smaller slopes, which helps enhance numerical stability, especially for stiff problems, while the root mean square emphasizes larger slopes, improving accuracy in rapidly changing regions. By combining the two means, the proposed method aims to balance stability and accuracy, making it suitable for stiff and oscillatory initial value problems. This method will be identified as HRM for the rest of this study.

2.2. **Stability Analysis.**

Examining the stability of solution given by a numerical method is for a general problem given by (1) is too complicated. Instead, we apply a model problem given in [12], that is defined as follows:

$$\frac{dy}{dt} = \lambda y, \qquad y(t_0) = 1, \tag{7}$$

and assume that $\lambda < 0$ or λ has negative real part if $\lambda \in \mathbb{C}$, $\mathbb{C}$ is the set of complex numbers. The stability of the iterative method is achieved if the method is applied to (7), the numerical solution satisfies

$$y_h(t_n) \longrightarrow 0, \quad \text{when} \quad t_n \longrightarrow \infty,$$

for any stepsize $h > 0$ and $n = 0, 1, 2, \ldots$.
Suppose

$$\frac{dy}{dt} = \xi(t_n, y_n) = \lambda y_n. \tag{8}$$

By applying (6) to (8), we have

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}\left(\frac{2\xi(t_n, y_n)\xi(t_{n+1}, y_{n+1}^E)}{\xi(t_n, y_n) + \xi(t_{n+1}, y_{n+1}^E)} + \frac{\sqrt{2\xi(t_n, y_n)^2 + 2\xi(t_{n+1}, y_{n+1}^E)^2}}{2}\right) \\ &= y_n + \frac{h}{2}\left(\frac{2\lambda y_n \lambda y_{n+1}^E}{\lambda y_n + \lambda y_{n+1}^E} + \frac{\sqrt{2(\lambda y_n)^2 + 2(\lambda y_{n+1}^E)^2}}{2}\right) \\ &= y_n + \frac{h}{2}\left(\frac{2\lambda y_n \lambda(y_n + h\lambda y_n)}{\lambda y_n + \lambda(y_n + h\lambda y_n))} + \frac{\sqrt{2(\lambda y_n)^2 + 2(\lambda(y_n + h\lambda y_n)}}{2}\right). \end{aligned} \tag{9}$$

Expanding and simplifying (9), we obtain

$$y_{n+1} = \left(\frac{4(\lambda h)^2 + 8\lambda h + 8 + \sqrt{2}(\lambda h)^2\sqrt{(\lambda h)^2 + 2\lambda h + 2} + 2\sqrt{2}\lambda h\sqrt{(\lambda h)^2 + 2\lambda h + 2}}{4\lambda h + 8}\right) y_n. \tag{10}$$

By running the iteration for $n \longrightarrow \infty$, assuming $z = \lambda h$, and the fact that $y(0) = 1$, we have

$$y_{n+1} = \left(\frac{4z^2 + 8z + 8 + \left(\sqrt{2}z^2 + 2\sqrt{2}z\right)\sqrt{z^2 + 2z + 2}}{4z + 8} \right)^n .$$

Suppose $G(z) = \dfrac{4z^2 + 8z + 8 + \left(\sqrt{2}z^2 + 2\sqrt{2}z\right)\sqrt{z^2 + 2z + 2}}{4z + 8}$, then the stability of HRM method is achieved when

$$|G(z)| \leq 1. \tag{11}$$

The stability region from (11) is represented by the shaded-less region in Figure 1 below.

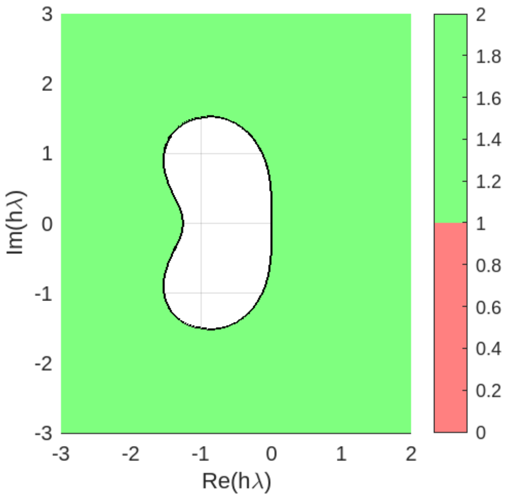

FIGURE 1. Stability region of HRM method

2.3. **Consistency Analysis.**

According to [14], consistency of a numerical algorithm for solving IVPs is satisfied if it converges to the differential equation for all step-size $h \longrightarrow 0$. Specifically, suppose IVP in (1) is approached with a numerical method described as

$$y_{n+1} = y_n + h\phi(t_n, y_n, h), \tag{12}$$

then consistency of the method is attained if

$$\lim_{h \longrightarrow 0} \phi(t_n, y_n, h) = \xi(t_n, y_n).$$

Hence, in our case, from equating (6) with (12), we obtain

$$\phi(t_n, y_n, h) = \frac{1}{2}\left(\frac{2\xi(t_n, y_n)\xi(t_{n+1}, y^E_{n+1})}{\xi(t_n, y_n) + \xi(t_{n+1}, y^E_{n+1})} + \frac{\sqrt{2\xi(t_n, y_n)^2 + 2\xi(t_{n+1}, y^E_{n+1})^2}}{2}\right). \tag{13}$$

By means of Taylor expansion, assuming that $\xi(t_n, y_n) = \xi$, and applying (2) we have

$$\xi(t_{n+1}, y^E_{n+1}) = \xi + \xi_t h + \xi_y \xi h + \frac{1}{2}\xi_{yy}\xi^2 h^2 + \frac{1}{2}\xi_{tt}h^2 + \xi_{ty}h^2\xi + O(h^3), \tag{14}$$

where $\xi_t = \dfrac{\partial \xi}{\partial t}, \xi_y = \dfrac{\partial^2 \xi}{\partial y}, \xi_{yy} = \dfrac{\partial^2 \xi}{\partial y \partial y}$, and $\xi_{tt} = \dfrac{\partial^2 \xi}{\partial t \partial t}$.
Substituting (14) into (13) and simplifying, we have

$$\begin{aligned}\phi(t_n, y_n, h) = \frac{1}{2}\Bigg(&\xi + \left(\frac{\xi_x}{2} + \frac{\xi_y \xi}{2}\right)h + \left(-\frac{\xi_y^2\xi}{4} - \frac{\xi_t^2}{4\xi} + \frac{\xi_{tt}}{4} + \frac{\xi_{yy}\xi^2}{4} - \frac{\xi_y\xi_t}{2} + \frac{\xi_{ty}\xi}{2}\right)h^2 \\ &+ \frac{\sqrt{16\xi^2 + 16\xi_y\xi^2 h + 16\xi\xi_t h + 8\xi_{yy}\xi^3 h^2 + 8\xi_y^2\xi^2 h^2 + 16\xi_{ty}\xi^2 h^2 + 16\xi_t\xi_y\xi\, h^2 + 8\xi\xi_{tt}h^2 + 8\xi_t^2 h^2}}{4}\Bigg) \\ &+ O(h^3). \end{aligned} \tag{15}$$

Taking the limit of (15) resulting in

$$\lim_{h \longrightarrow 0} \phi(t_n, y_n, h) = \xi,$$

which shows that our proposed method is consistent.

3. NUMERICAL RESULTS

In this section, we test our proposed method to solve three IVPs with different step-size h. In order to see the performance of the method, we compare its results with several numerical methods, i.e. EuM, HeM, HM, and AHM.

Example 3.1. *Consider the IVP*

$$\frac{dy}{dt} = \frac{1}{y(t)} \tag{16}$$
$$y(0) = 1.$$

The exact solution is $y(t) = \sqrt{2t+1}$, $0 \le t \le 1$. *The solution is given in the following tables.*

Table 1: Error comparison of EuM, HeM, HM, AHM, and HRM for IVP (16) with $h = 0.1$

t	EuM	HeM	HM	AHM	HRM
0.100	$5.346522e-03$	$1.534781e-04$	$1.534781e-04$	$2.903478e-03$	$1.457435e-03$
0.200	$7.961484e-03$	$3.035877e-04$	$7.297711e-03$	$4.342679e-03$	$1.974052e-03$
0.300	$9.858246e-03$	$4.478703e-04$	$2.249504e-02$	$5.573349e-03$	$2.385002e-03$
0.400	$1.192778e-02$	$5.816589e-04$	$4.699446e-02$	$6.806972e-03$	$2.813163e-03$
0.500	$1.436895e-02$	$6.970676e-04$	$8.294958e-02$	$8.138060e-03$	$3.319746e-03$
0.600	$1.734100e-02$	$7.815183e-04$	$1.331632e-01$	$9.634144e-03$	$3.958304e-03$
0.700	$2.102439e-02$	$8.155925e-04$	$2.012491e-01$	$1.135848e-02$	$4.790266e-03$
0.800	$2.564585e-02$	$7.698856e-04$	$2.918645e-01$	$1.337946e-02$	$5.894602e-03$
0.900	$3.150123e-02$	$6.004090e-04$	$4.110245e-01$	$1.577656e-02$	$7.377777e-03$
1.000	$3.898340e-02$	$2.418500e-04$	$5.665221e-01$	$1.864557e-02$	$9.386511e-03$

Table 2. Error comparison of EuM, HeM, HM, AHM, and HRM for IVP (16) with $h = 0.01$

t	EuM	HeM	HM	AHM	HRM
0.010	$4.950600e-05$	$1.000000e-09$	$4.975123e-03$	$1.240000e-07$	$6.100000e-08$
0.020	$4.800600e-05$	$2.000000e-09$	$9.877555e-03$	$2.410000e-07$	$1.180000e-07$
.030	$4.657900e-05$	$3.000000e-09$	$1.470973e-02$	$3.520000e-07$	$1.710000e-07$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.360	$2.144700e-05$	$1.200000e-08$	$1.456228e-01$	$2.036000e-06$	$9.960000e-07$
0.370	$2.104700e-05$	$1.200000e-08$	$1.489526e-01$	$2.056000e-06$	$1.006000e-06$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.600	$1.432500e-05$	$1.000000e-08$	$2.187878e-01$	$2.344000e-06$	$1.147000e-06$
0.610	$1.411100e-05$	$9.000000e-09$	$2.215725e-01$	$2.350000e-06$	$1.151000e-06$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.990	$8.588000e-06$	$7.000000e-09$	$3.161976e-01$	$2.450000e-06$	$1.201000e-06$
1.000	$8.489000e-06$	$6.000000e-09$	$3.184437e-01$	$2.449000e-06$	$1.202000e-06$

From Table 1, it can be seen that HM shows the highest error across all values of t, indicating that it performs the worst for this step-size. EuM and AHM progressively reduce error, with HeM consistently showing better results than HM and AHM. On the other hand, HRM demonstrates its superiority, with significantly lower error compared to the other methods. At a smaller step-size as can be viewed in Table 2, the errors of all methods decrease significantly where HRM continues to provide consistent accuracy, followed by AHM, EuM and finally HM. As step-size decreases, the errors of all methods decreases while HRM and HeM consistently perform well.

Example 3.2. *Given IVP*

$$\frac{dy}{dt} = \cos(t)^2 \tag{17}$$
$$y(0) = 0.$$

The exact solution is $y(t) = \frac{t}{2} + \frac{\sin(2 \cdot t)}{4}$, $0 \le t \le 10$. *The following tables present the numerical solutions for* $h = 0.1$ *and* $h = 0.01$.

Table 3 presents the results in which HRM again shows a consistent accuracy across the entire time range. HM performs poorly with the largest errors among all methods while EuM, HeM, and AHM performs moderately, with AHM closer to HRM in accuracy. At smaller the step-size, as can be seen in Table 5, the errors across all methods reduce and HRM maintains the consistency despite not as accurate as EuM and HeM.

TABLE 3. Error comparison of EuM, HeM, HM, AHM, and HRM for IVP (17) with $h = 0.1$

t	EuM	HeM	HM	AHM	HRM
0.100	$3.326673e-04$	$1.656682e-04$	$4.991525e-02$	$1.644204e-04$	$1.662922e-04$
0.200	$1.149784e-03$	$3.247318e-04$	$9.882725e-02$	$3.123280e-04$	$3.309341e-04$
0.300	$1.916083e-03$	$4.708494e-04$	$1.457728e-01$	$4.278717e-04$	$4.923411e-04$
⋮	⋮	⋮	⋮	⋮	⋮
3.500	$1.322168e-03$	$5.478540e-04$	$9.502217e-01$	$6.627667e-03$	$4.232565e-03$
3.600	$7.077680e-04$	$6.618310e-04$	$9.922851e-01$	$6.592451e-03$	$4.385941e-03$
3.700	$2.284330e-04$	$7.494230e-04$	$1.030341e+00$	$6.623149e-03$	$4.532724e-03$
⋮	⋮	⋮	⋮	⋮	⋮
9.900	$7.923435e-03$	$6.785120e-04$	$2.555563e+00$	$2.080711e-02$	$1.171755e-02$
10.000	$7.469339e-03$	$7.612940e-04$	$2.592888e+00$	$2.084982e-02$	$1.186313e-02$

Table 4: Error comparison of EuM, HeM, HM, AHM, and HRM for IVP (17) with $h = 0.01$

t	EuM	HeM	HM	AHM	HRM
0.010	$3.333280e-07$	$1.666570e-07$	$4.999917e-03$	$1.666420e-07$	$1.666620e-07$
0.020	$1.166490e-06$	$3.332500e-07$	$9.998833e-03$	$3.331300e-07$	$3.333100e-07$
0.030	$1.999150e-06$	$4.997000e-07$	$1.499575e-02$	$4.992700e-07$	$4.999200e-07$
⋮	⋮	⋮	⋮	⋮	⋮
0.210	$1.646610e-05$	$3.398000e-06$	$1.034716e-01$	$3.245000e-06$	$3.474400e-06$
0.220	$1.721730e-05$	$3.549500e-06$	$1.082440e-01$	$3.373700e-06$	$3.637300e-06$
⋮	⋮	⋮	⋮	⋮	⋮
1.630	$2.505830e-05$	$9.843000e-07$	$3.926916e-01$	$4.250750e-05$	$1.989780e-05$
1.640	$2.613520e-05$	$1.149500e-06$	$3.927117e-01$	$4.316770e-05$	$1.998150e-05$
⋮	⋮	⋮	⋮	⋮	⋮

t	EuM	HeM	HM	AHM	HRM
3.600	$6.997000e-06$	$6.620000e-06$	$9.991324e-01$	$7.274200e-05$	$4.642200e-05$
3.610	$6.537000e-06$	$6.720000e-06$	$1.003133e+00$	$7.274200e-05$	$4.657200e-05$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
4.440	$2.977400e-05$	$4.322000e-06$	$1.174677e+00$	$9.948100e-05$	$5.634300e-05$
4.450	$3.071700e-05$	$4.178000e-06$	$1.175025e+00$	$1.000900e-04$	$5.643200e-05$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
9.640	$1.001590e-04$	$3.483000e-06$	$2.461928e+00$	$2.299690e-04$	$1.205780e-04$
9.650	$9.941200e-05$	$3.634000e-06$	$2.466689e+00$	$2.298420e-04$	$1.207410e-04$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
9.990	$8.088600e-05$	$7.544000e-06$	$2.610343e+00$	$2.285630e-04$	$1.259710e-04$
10.000	$8.060000e-05$	$7.614000e-06$	$2.613886e+00$	$2.286390e-04$	$1.261130e-04$

TABLE 5. Error comparison of EuM, HeM, HM, AHM, and HRM for IVP (17) with $h = 0.01$

t	EuM	HeM	HM	AHM	HRM
0.010	$3.333280e-07$	$1.666570e-07$	$4.999917e-03$	$1.666420e-07$	$1.666620e-07$
0.020	$1.166490e-06$	$3.332500e-07$	$9.998833e-03$	$3.331300e-07$	$3.333100e-07$
0.030	$1.999150e-06$	$4.997000e-07$	$1.499575e-02$	$4.992700e-07$	$4.999200e-07$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.210	$1.646610e-05$	$3.398000e-06$	$1.034716e-01$	$3.245000e-06$	$3.474400e-06$
0.220	$1.721730e-05$	$3.549500e-06$	$1.082440e-01$	$3.373700e-06$	$3.637300e-06$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1.630	$2.505830e-05$	$9.843000e-07$	$3.926916e-01$	$4.250750e-05$	$1.989780e-05$
1.640	$2.613520e-05$	$1.149500e-06$	$3.927117e-01$	$4.316770e-05$	$1.998150e-05$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3.600	$6.997000e-06$	$6.620000e-06$	$9.991324e-01$	$7.274200e-05$	$4.642200e-05$
3.610	$6.537000e-06$	$6.720000e-06$	$1.003133e+00$	$7.274200e-05$	$4.657200e-05$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
4.440	$2.977400e-05$	$4.322000e-06$	$1.174677e+00$	$9.948100e-05$	$5.634300e-05$
4.450	$3.071700e-05$	$4.178000e-06$	$1.175025e+00$	$1.000900e-04$	$5.643200e-05$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
9.640	$1.001590e-04$	$3.483000e-06$	$2.461928e+00$	$2.299690e-04$	$1.205780e-04$
9.650	$9.941200e-05$	$3.634000e-06$	$2.466689e+00$	$2.298420e-04$	$1.207410e-04$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
9.990	$8.088600e-05$	$7.544000e-06$	$2.610343e+00$	$2.285630e-04$	$1.259710e-04$
10.000	$8.060000e-05$	$7.614000e-06$	$2.613886e+00$	$2.286390e-04$	$1.261130e-04$

Example 3.3. *The exact solution of IVP*

$$\frac{dy}{dx} = x^2 + yx \tag{18}$$

$$y(0) = 1$$

is $y(x) = -x + \frac{e^{\frac{x^2}{2}} \cdot \sqrt{\pi} \cdot \sqrt{2} \cdot \mathrm{erf}\left(\frac{\sqrt{2} \cdot x}{2}\right)}{2} + e^{\frac{x^2}{2}}$*. The tables below presents the errors given by the mentioned methods for* $0 \leq t \leq 1$*.*

Table 6. Error comparison of EuM, HeM, HM, AHM, and HRM for IVP (18) with $h = 0.1$

t	EuM	HeM	HM	AHM	HRM
0.100	$5.346522e-03$	$1.534781e-04$	$1.534781e-04$	$2.903478e-03$	$1.457435e-03$
0.200	$7.961484e-03$	$3.035877e-04$	$7.297711e-03$	$4.342679e-03$	$1.974052e-03$
0.300	$9.858246e-03$	$4.478703e-04$	$2.249504e-02$	$5.573349e-03$	$2.385002e-03$
0.400	$1.192778e-02$	$5.816589e-04$	$4.699446e-02$	$6.806972e-03$	$2.813163e-03$
0.500	$1.436895e-02$	$6.970676e-04$	$8.294958e-02$	$8.138060e-03$	$3.319746e-03$
0.600	$1.734100e-02$	$7.815183e-04$	$1.331632e-01$	$9.634144e-03$	$3.958304e-03$
0.700	$2.102439e-02$	$8.155925e-04$	$2.012491e-01$	$1.135848e-02$	$4.790266e-03$
0.800	$2.564585e-02$	$7.698856e-04$	$2.918645e-01$	$1.337946e-02$	$5.894602e-03$
0.900	$3.150123e-02$	$6.004090e-04$	$4.110245e-01$	$1.577656e-02$	$7.377777e-03$
1.000	$3.898340e-02$	$2.418500e-04$	$5.665221e-01$	$1.864557e-02$	$9.386511e-03$

Table 7. Error comparison of EuM, HeM, HM, AHM, and HRM for IVP (18) with $h = 0.01$

t	EuM	HeM	HM	AHM	HRM
0.010	$5.033433e-05$	$1.656700e-07$	$1.656700e-07$	$2.541567e-05$	$1.462533e-05$
0.020	$6.597387e-05$	$3.311300e-07$	$6.722887e-05$	$3.428413e-05$	$1.892887e-05$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.060	$8.032384e-05$	$9.921600e-07$	$8.875078e-04$	$5.019116e-05$	$2.594084e-05$
0.070	$8.260844e-05$	$1.157560e-06$	$1.232278e-03$	$5.271456e-05$	$2.695844e-05$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.180	$1.034335e-04$	$2.968500e-06$	$9.107826e-03$	$7.195850e-05$	$3.389950e-05$
0.190	$1.052670e-04$	$3.131000e-06$	$1.022915e-02$	$7.339500e-05$	$3.437800e-05$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.510	$1.810713e-04$	$7.949700e-06$	$9.541977e-02$	$1.202127e-04$	$5.084630e-05$
0.520	$1.842819e-04$	$8.071100e-06$	$1.000552e-01$	$1.219251e-04$	$5.153390e-05$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.980	$4.513070e-04$	$5.488000e-06$	$5.541857e-01$	$2.418560e-04$	$1.164820e-04$
0.990	$4.611220e-04$	$5.067000e-06$	$5.716989e-01$	$2.457660e-04$	$1.191060e-04$
1.000	$4.711920e-04$	$4.618000e-06$	$5.896761e-01$	$2.497540e-04$	$1.218120e-04$

Similar to previous examples, HRM still provides the smallest errors, after HeM. At this step-size, HM shows the largest error values, followed by EuM and AHM, as seen in Table 6. Reducing the step-size for this example, results in improved accuracy of all methods, but the ranking remains consistent which is apparent in Table 7.

Overall, from the results of the numerical simulations, HRM shows the best performance in terms of accuracy. It consistently has the smallest error for both large end small step-size. HRM can also handle larger step-size without sacrificing much accuracy, unlike HM, which performs poorly with large step-size. In addition, HRM works well for different types of differential equations as demonstrated across all examples. This shows the versatility of HRM and marks consistency of the method for solving IVPs that require precise solutions.

4. CONCLUSIONS

This paper introduces a novel numerical technique for solving initial value problems (IVPs) by modifying Heun's method. The modification involves using the harmonic and root mean square to calculate the average of the slopes of the tangents in Heun's method. Stability and consistency analyses demonstrate the reliability of the approach, while numerical simulations emphasize its accuracy in producing approximate solutions with improved precision. The proposed method shows competitive potential compared to existing methods, offering a viable alternative for solving IVPs. Future advancements could involve applying other averaging techniques to different numerical methods, expanding their use for solving real-world problems.

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