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Jacobsthal Trigonometric Functions

Apisit Pakapongpun[1] and Natdanai Chailangka[2*]

[1]Department of Mathematics, Burapha University, Thailand, apisit.buu@gmail.com

[2]Branch of Sciences and Mathematics, Rajamangala University of Technology Tawan-ok, Thailand, natdanai_ch@rmutto.ac.th

Abstract. Jacobsthal numbers satisfy a second order homogeneous recurrence relation $J_n = J_{n-1} + 2J_{n-2}$ where J_n denotes the n^{th} Jacobsthal number. In this paper, the Jacobsthal sine, cosine, tangent and cotangent are defined, and some identities of Jacobsthal trigonometric functions are provided.

Key words and Phrases: Jacobsthal numbers, Jacobsthal trigonometric functions.

1. INTRODUCTION

The well-known generalized Fibonacci sequence [1] is defined by recurrence relation $F_n = pF_{n-1} + qF_{n-2}$, $n \geq 2$ with initial condition $F_0 = a$ and $F_1 = b$ where p, q are positive integers and a, b are non-negative integers.

The Fibonacci sequence [2] is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$. The Jacobsthal sequence [3] is defined by the recurrence relation $J_n = J_{n-1} + 2J_{n-2}$, $n \geq 2$ with $J_0 = 0$ and $J_1 = 1$. The Binet's formula is given by $J_n = \frac{2^n - (-1)^n}{3}$ where 2 and -1 are the roots of the characteristic equation $x^2 - x - 2 = 0$.

In 2001, Smith R.M. [4] studied the Fibonometric function by the initial value problem $y'' - y' - y = 0$ with $y(0) = 0$ and $y'(0) = 1$ which is analogous to the definition of Fibonacci numbers $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ where $F_0 = 0$ and $F_1 = 1$. He defined the Fibonacci sine, cosine, tangent and cotangent, and established some theorems and elementary identities for Fibonometry.

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*Corresponding author

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In 2020, Srimuk, V. and Pakapongpun, A. [5] studied Identities of k-Fibonometric functions which are obtained from a second order linear differential equation $y'' - ky' - y = 0$ with $y(0) = 0$ and $y'(0) = 1$. k-Fibonometric differential equation is analogous to the formula for k-Fibonacci numbers $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$, $k \geq 1$ where $F_{k,0} = 0$ and $F_{k,1} = 1$.

Recently, the investigation of the Jacobsthal function has continued to attract interest. Abd-Elhameed, W. M., Alqubori, O. M., and Amin, A. K. (2025) [6] introduced a class of Jacobsthal-type polynomials involving one parameter. They presented new formulas, including expressions for derivatives, moments, and linear relations. Simultaneously, Yesilyurta, I., and Degirmenb, N. (2025) [7] proposed a new version of the Jacobsthal and Jacobsthal–Lucas sequences, along with their characteristics, formulas, and several identities—especially Cassini's identity, d'Ocagne's identity, Binet's formula, the Gelin–Cesàro identity, Honsberger's identity, and Melham's identity.

In this paper, we have studied the Jacobsthal trigonometric functions by the initial value problem $y'' - y' - 2y = 0$ with $y(0) = 0$ and $y'(0) = 1$ which is analogous to the definition of Jacobsthal numbers $J_n = J_{n-1} + 2J_{n-2}$ for $n \geq 2$ where $J_0 = 1$ and $J_1 = 1$. The Jacobsthal sine, cosine, tangent and cotangent are defined, and some elementary identities for Jacobsthal trigonometric functions are provided.

2. MAIN RESULTS

The solution of Jacobsthal trigonometric differential equation is $y = \dfrac{e^{2x} - e^{-x}}{3}$ where 2 and -1 are the solutions of the equation $r^2 - r - 2 = 0$. By the well-known formula $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, we define the Jacobsthal sine function as follows:

Definition 2.1. *The Jacobsthal sine function is denoted by sinJ:*

$$\sin J(x) = \frac{e^{2x} - e^{-x}}{3}.$$

It is interesting to examine the relationship between the power series coefficients and the sequence of Jacobsthal numbers as the theorem.

Lemma 2.2. *If* $y = \sum_{n=0}^{\infty} c_n x^n$ *is the solution of Jacobsthal trigonometric differential equation* $y'' - y' - 2y = 0$ *with* $y(0) = 0$ *and* $y'(0) = 1$*, then* $(n+2)(n+1)c_{n+2} - (n+1)c_{n+1} - 2c_n = 0$ *for all* $n \geq 0$*, where* c_n *is the* n^{th} *coefficient of the solutions of the Jacobsthal trigonometric differential equation.*

Proof. Suppose the solution of the Jacobsthal trigonometric differential equation $y'' - y' - 2y = 0$ with $y(0) = 0$ and $y'(0) = 1$ is $y = \sum_{n=0}^{\infty} c_n x^n$.

Thus,

$$y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

Hence

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^{n-1} - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

so,

$$\sum_{n=0}^{\infty}\big[(n+2)(n+1))c_{n+2} - (n+1)c_{n+1} - 2c_n\big]x^n = 0.$$

Therefore, $(n+2)(n+1))c_{n+2} - (n+1)c_{n+1} - 2c_n = 0$ for all $n \geq 0$. □

Lemma 2.3. *If* $y = \sum_{n=0}^{\infty} c_n x^n$ *is the solution of Jacobsthal trigonometric differential equation* $y'' - y' - 2y = 0$ *with* $y(0) = 0$ *and* $y'(0) = 1$ *then*

$$c_n = \frac{J_n c_1 + 2J_{n-1}c_0}{n!},$$

where J_n *is the* n^{th} *Jacobsthal number.*

Proof. We use strong mathematical induction to prove this lemma. If $n = 1$ then $\frac{J_1c_1 + 2J_0c_0}{1!} = (1)c_1 + 2(0)c_0 = c_1$. Suppose that the hypothesis is true for $n = 1, 2, 3, \cdots, r, r+1$. Namely, $c_r = \frac{J_rc_1 + 2J_{r-1}c_0}{r!}$ and $c_{r+1} = \frac{J_{r+1}c_1 + 2J_rc_0}{(r+1)!}$. We now demonstrate that it is true for $n = r+2$. Now, by Lemma 2.2 and from the hypothesis, we obtain

$$\begin{aligned}
(r+2)(r+1)c_{r+2} &= (r+1)c_{r+1} + 2c_r \\
&= \frac{J_{r+1}c_1 + 2J_rc_0}{r!} + \frac{2(J_rc_1 + 2J_{r-1}c_0)}{r!} \\
&= \frac{c_1(J_{r+1} + 2J_r) + 2c_0(J_r + 2J_{r-1})}{r!}. \\
&= \frac{J_{r+2}c_1 + 2c_0J_{r+1}}{r!}
\end{aligned}$$

Hence,

$$c_{r+2} = \frac{J_{r+2}c_1 + 2J_{r+1}c_0}{(r+2)!},$$

which follows the proof of the theorem. □

Theorem 2.4. *The Jacobsthal sine function is*

$$\sin J(x) = \sum_{n=0}^{\infty} \frac{J_n x^n}{n!}$$

where J_n is the n^{th} Jacobsthal number.

Proof. Since $y = \dfrac{e^{2x} - e^{-x}}{3} = \displaystyle\sum_{k=0}^{\infty} c_n x^n$, we have

$$y = c_0 + c_1 x + \left(\frac{J_2 c_1 + 2J_1 c_0}{2!}\right)x^2 + \left(\frac{J_3 c_1 + 2J_2 c_0}{3!}\right)x^3 + \cdots + \left(\frac{J_n c_1 + 2J_{n-1} c_0}{n!}\right)x^n + \cdots .$$

Applying the initial conditions $y(0) = 0$ and $y'(0) = 1$ on the series, we get $c_0 = 0$ and $c_1 = 1$. Therefore, we obtain

$$\begin{aligned} y &= x + \frac{J_2 x^2}{2!} + \frac{J_3 x^3}{3!} + \cdots + \frac{J_n x^n}{n!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{J_n x^n}{n!}. \end{aligned}$$

Therefore,

$$\sin J(x) = \sum_{n=0}^{\infty} \frac{J_n x^n}{n!}.$$

□

Next, we are going to define the Jacobsthal cosine, tangent and cotangent. We will show that $\sin J(x)$ is absolutely convergent for all real numbers x as the following lemmas.

Lemma 2.5. *If $J_n = \dfrac{2^n - (-1)^n}{3}$ then $\displaystyle\lim_{n\to\infty} \frac{J_{n+1}}{J_n} = 2$ where J_n is the n^{th} Jacobsthal number.*

Proof. Since

$$\begin{aligned} \lim_{n\to\infty} \frac{J_{n+1}}{J_n} &= \lim_{n\to\infty} \left[\left(\frac{2^{n+1} - (-1)^{n+1}}{3}\right)\left(\frac{3}{2^n - (-1)^n}\right)\right] \\ &= \lim_{n\to\infty} \left(\frac{2^{n+1} - (-1)^{n+1}}{2^n - (-1)^n}\right) \\ &= 2. \end{aligned}$$

□

Lemma 2.6. *The series expansion for $\sin J(x)$ is absolutely convergent for all real numbers x.*

Proof. By ratio test, we obtain

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\frac{J_{n+1}x^{n+1}}{(n+1)!}}{\frac{J_nx^n}{n!}}\right| &= \lim_{n\to\infty}\left|\frac{x}{n+1}\cdot\frac{J_{n+1}}{J_n}\right| \\ &= \lim_{n\to\infty}\frac{|x|}{n+1}\cdot\lim_{n\to\infty}\frac{J_{n+1}}{J_n} \\ &= 0\cdot 2 = 0 < 1,\end{aligned}$$

implies the series $|\sin J(x)|$ is convergent. Therefore, the series expansion for $\sin J(x)$ is absolutely convergent for all real numbers x. $\square$

Next, we introduce the Jacobsthal cosine function as the derivative of Jacobsthal sine function. Since an absolutely convergent power series is infinitely differentiable within its interval of convergence, Jacobsthal sine function is differentiable. Then we have $\frac{d}{dx}\sin J(x) = \cos J(x)$.

Definition 2.7. *The Jacobsthal cosine is denoted by* $\cos J$:

$$\cos J(x) = \frac{2e^{2x}+e^{-x}}{3}.$$

Theorem 2.8. *The expansion of the Jacobsthal cosine is*

$$\cos J(x) = \sum_{n=0}^{\infty}\frac{J_{n+1}x^n}{n!}.$$

Proof. Since $\sin J(x) = \sum_{k=0}^{\infty}\frac{J_nx^n}{n!}$ is absolutely convergent for all real numbers x. We have

$$\begin{aligned}\frac{d}{dx}\sin J(x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{J_nx^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\frac{J_nx^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{nJ_nx^{n-1}}{n!} \\ &= 0 + J_1 + J_2x + \frac{J_3x^2}{2!} + \frac{J_4x^3}{3!} + \cdots + \frac{J_nx^{n-1}}{(n-1)!} + \cdots \\ &= \sum_{n=0}^{\infty}\frac{J_{n+1}x^n}{n!}.\end{aligned}$$

Therefore, $\cos J(x) = \sum_{k=0}^{\infty}\frac{J_{n+1}x^n}{n!}$. $\square$

Next, we will define Jacobsthal tangent and cotangent functions which are similar to the definition of the trigonometric tangent and cotangent functions. At first, we have to show that $\cos J(x) \neq 0$ for all real numbers x as the following lemma.

Lemma 2.9. $\cos J(x) \neq 0$ *for all real numbers* x.

Proof. Suppose that $\cos J(x) = 0$ for some x. Thus, $\cos J(x) = \dfrac{2e^{2x} + e^{-x}}{3} = 0$. Hence, $e^{3x} = -\dfrac{1}{2}$, this is a contradiction. Therefore, $\cos J(x) \neq 0$ for all real numbers x. □

The following theorems demonstrate the Jacobsthal tangent and cotangent functions in terms of a power series of e^x.

Definition 2.10. *The Jacobsthal tangent and Jacobsthal cotangent functions are denoted by* $\tan J$ *and* $\cot J$ *respectively:*

$$\tan J(x) = \frac{\sin J(x)}{\cos J(x)} = \frac{e^{2x} - e^{-x}}{2e^{2x} + e^{-x}}$$

and

$$\cot J(x) = \frac{\cos J(x)}{\sin J(x)} = \frac{2e^{2x} + e^{-x}}{e^{2x} - e^{-x}}, x \neq 0.$$

The following theorems demonstrate the Jacobsthal tangent and cotangent functions in terms of a power series of e^x.

Theorem 2.11. *The Jacobsthal tangent function is of the form*

$$\tan J(x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 3e^{-3(n+1)x}}{2^{n+2}}.$$

Proof. By the definition 2.10, we have

$$\begin{aligned}
\tan J(x) &= \frac{e^{2x} - e^{-x}}{2e^{2x} + e^{-x}} = \frac{1 - e^{-3x}}{2 + e^{-3x}} \\
&= \frac{1}{2} - \frac{3e^{-3x}}{4} + \frac{3e^{-6x}}{8} - \frac{3e^{-9x}}{16} + \frac{3e^{-12x}}{32} - \frac{3e^{-15x}}{64} + \cdots \\
&= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 3e^{-3(n+1)x}}{2^{n+2}}.
\end{aligned}$$

□

Theorem 2.12. *The Jacobsthal cotangent function is of the form*

$$\cot J(x) = 2 + 3\sum_{n=0}^{\infty} e^{-3(n+1)x}.$$

Proof. Since,

$$\begin{aligned}\cot J(x) &= \frac{2e^{2x}+e^{-x}}{e^{2x}-e^{-x}} = \frac{2+e^{-3x}}{1-e^{-3x}}\\ &= 2+3e^{-3x}+3e^{-6x}+3e^{-9x}+\cdots\\ &= 2+3\sum_{n=0}^{\infty} e^{-3(n+1)x}.\end{aligned}$$

□

We will define a definition of Jacobsthal secant and cosecant, and some elementary identities of Jacobsthal trigonometric functions are obtained.

Definition 2.13. *The Jacobsthal secant and Jacobsthal cosecant are denoted by* $\sec J$ *and* $cosecJ$ *respectively:*

$$\sec J(x) = \frac{1}{\cos J(x)} \text{ and } cosecJ(x) = \frac{1}{\sin J(x)}, x \neq 0$$

The following theorems are the expansion of the Jacobsthal tangent and cotangent.

Theorem 2.14. *The expansion of the Jacobsthal secant and Jacobsthal cosecant respectively are*

$$\sec J(x) = 3\sum_{n=0}^{\infty} \frac{(-1)^n e^{-(3n+2)x}}{2^{n+1}}$$

and

$$cosecJ(x) = 3\sum_{n=1}^{\infty} e^{-(3n+2)x}.$$

Proof. The proofs are similar to the proof of the Jacobsthal tangent and Jacobsthal cotangent. □

3. Some identities of Jacobsthal trigonometric functions

The trigonometric identities $\sin^2 x + \cos^2 x = 1$, $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$, $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ and the hyperbolic trigonometric identities $\cosh^2 x - \sinh^2 x = 1$, $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$, $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$ are quite well known. Then the following theorem looks like trigonometric formulas.

Theorem 3.1. *The Fundamental Identities for Jacobsthal trigonometric functions are*

- **(1):** $\cos J^2(x) - \cos J(x)\sin J(x) - 2\sin J^2(x) = e^x$,
- **(2):** $\cot J^2(x) - \cot J(x) - 2 = e^x cosecJ^2(x)$,
- **(3):** $1 - \tan J(x) - 2\tan J^2(x) = e^x \sec J^2(x)$,
- **(4):** $\sin J(x+y) = \sin J(x)\cos J(y) - \sin J(x)\sin J(y) + \cos J(x)\sin J(y)$,
- **(5):** $\sin J(x-y) = \sin J(x)\cos J(-y) - \sin J(x)\sin J(-y) + \cos J(x)\sin J(-y)$,
- **(6):** $\cos J(x+y) = \cos J(x)\cos J(y) + 2\sin J(x)\sin J(y)$,

(7): $\cos J(x-y) = \cos J(x)\cos J(-y) + 2\sin J(x)\sin J(-y)$,
(8): $\sin J(2x) = 2\sin J(x)\cos J(x) - \sin J^2(x)$,
(9): $\cos J(2x) = \cos J^2(x) + 2\sin J^2(x)$.

Proof. Since we have defined $\sin J(x) = \dfrac{e^{2x} - e^{-x}}{3}$ and $\cos J(x) = \dfrac{2e^{2x} + e^{-x}}{3}$, we obtain the results as follows:

(1)

$$
\begin{aligned}
&\cos J^2(x) - \cos J(x)\sin J(x) - 2\sin J^2(x)\\
&= \left(\frac{2e^{2x} + e^{-x}}{3}\right)^2 - \left(\frac{2e^{2x} + e^{-x}}{3}\right)\left(\frac{e^{2x} - e^{-x}}{3}\right) - 2\left(\frac{e^{2x} - e^{-x}}{3}\right)^2\\
&= \frac{1}{9}(4e^{4x} + 4e^{x} + e^{-2x}) - \frac{1}{9}(2e^{4x} - 2e^{x} + e^{x} - e^{-2x}) - \frac{2}{9}(4e^{4x} - 2e^{x} + e^{-2x})\\
&= \frac{1}{9}(4e^{4x} - 2e^{4x} - 2e^{4x}) + \frac{1}{9}(4e^{x} + e^{x} + 4e^{x}) + \frac{1}{9}(e^{-2x} + e^{-2x} - 2e^{-2x})\\
&= e^{x}.
\end{aligned}
$$

(2) From (1) we get,

$$
\begin{aligned}
\cot J^2(x) - \cot J(x) - 2 &= \frac{\cos J^2(x)}{\sin J^2(x)} - \frac{\cos J(x)}{\sin J(x)} - 2\\
&= \frac{\cos J^2(x) - \cos J(x)\sin J(x) - 2\sin J^2(x)}{\sin J^2(x)}\\
&= e^{x}\ \mathit{cosec}\ J^2(x).
\end{aligned}
$$

(3) From (1) we get,

$$
\begin{aligned}
1 - \tan J(x) - 2\tan J^2(x) &= \frac{\cos J^2(x) - \sin J(x)\cos J(x) - 2\sin J^2(x)}{\cos J^2(x)}\\
&= \frac{e^{x}}{\cos J^2(x)}\\
&= e^{x}\sec J^2(x).
\end{aligned}
$$

(4)

$$
\begin{aligned}
\sin J(x)\cos J(y) - \sin J(x)\sin J(y) + \cos J(x)\sin J(y)& \\
&= \left(\frac{e^{2x}-e^{-x}}{3}\right)\left(\frac{2e^{2y}-e^{-y}}{3}\right) - \left(\frac{e^{2x}-e^{-x}}{3}\right)\left(\frac{e^{2y}-e^{-y}}{3}\right) \\
&\ + \left(\frac{2e^{2x}+e^{-x}}{3}\right)\left(\frac{e^{2y}-e^{-y}}{3}\right) \\
&= \frac{1}{9}\left[\left(2e^{2(x+y)} + e^{2x-y} - 2e^{2y-x} - e^{-x-y}\right) - \left(e^{2(x+y)} - e^{2x-y}\right.\right. \\
&\qquad \left.\left. -e^{2y-x} + e^{-x-y}\right) + \left(e^{2(x+y)} - 2e^{2x-y} + e^{2y-x} - e^{-x-y}\right)\right] \\
&= \frac{e^{2(x+y)} - e^{-(x+y)}}{3} \\
&= \sin J(x+y).
\end{aligned}
$$

(5)

$$
\begin{aligned}
&\sin J(x)\cos J(-y) - \sin J(x)\sin J(-y) + \cos J(x)\sin J(-y) \\
&= \left(\frac{e^{2x}-e^{-x}}{3}\right)\left(\frac{2e^{-2y}-e^{y}}{3}\right) - \left(\frac{e^{2x}-e^{-x}}{3}\right)\left(\frac{e^{-2y}-e^{y}}{3}\right) \\
&\ + \left(\frac{2e^{2x}+e^{-x}}{3}\right)\left(\frac{e^{-2y}-e^{y}}{3}\right) \\
&= \frac{1}{9}\left[\left(2e^{2(x-y)} + e^{2x+y} - 2e^{-2y-x} - e^{-x+y}\right) - \left(e^{2(x-y)} - e^{2x+y} - e^{-2y-x} + e^{-x+y}\right)\right. \\
&\ \left. + \left(e^{2(x-y)} - 2e^{2x+y} + e^{-2y-x} - e^{-x+y}\right)\right] \\
&= \frac{e^{2(x-y)} - e^{-(x-y)}}{3} \\
&= \sin J(x-y).
\end{aligned}
$$

(6)

$$
\begin{aligned}
&\cos J(x)\cos J(y) + 2\sin J(x)\sin J(y) \\
&= \left(\frac{2e^{2x}-e^{-x}}{3}\right)\left(\frac{2e^{2y}-e^{-y}}{3}\right) + 2\left(\frac{e^{2x}-e^{-x}}{3}\right)\left(\frac{e^{2y}-e^{-y}}{3}\right) \\
&= \frac{1}{9}\left[4e^{2x+2y} + 2e^{2x-y} + 2e^{-x+2y} + e^{-x-y} + 2e^{2x+2y} - 2e^{2x-y}\right. \\
&\ \left. -2e^{-x+2y} + 2e^{-x-y}\right] \\
&= \frac{1}{9}\left[6e^{2x+2y} + 3e^{-x-y}\right] \\
&= \frac{2e^{2(x+y)} + e^{-(x+y)}}{3} \\
&= \cos J(x+y).
\end{aligned}
$$

(7)

$$
\begin{aligned}
&\cos J(x)\cos J(-y) + 2\sin J(x)\sin J(-y)\\
&= \left(\frac{2e^{2x}-e^{-x}}{3}\right)\left(\frac{2e^{-2y}-e^{+y}}{3}\right) + 2\left(\frac{e^{2x}-e^{-x}}{3}\right)\left(\frac{e^{-2y}-e^{y}}{3}\right)\\
&= \frac{1}{9}\left[4e^{2x-2y} + 2e^{2x+y} + 2e^{-x-2y} + e^{-x+y} + 2e^{2x-2y} - 2e^{2x+y}\right.\\
&\left.-2e^{-x-2y} + 2e^{-x+y}\right]\\
&= \frac{1}{9}\left[6e^{2x-2y} + 3e^{-x+y}\right]\\
&= \frac{2e^{2(x-y)} + e^{-(x-y)}}{3}\\
&= \cos J(x-y).
\end{aligned}
$$

(8) From (4), we have

$$
\begin{aligned}
\sin J(x+x) &= \sin J(x)\cos J(x) - \sin J(x)\sin J(x) + \cos J(x)\sin J(x)\\
&= 2\sin J(x)\cos J(x) - \sin J^2(x).
\end{aligned}
$$

(9) From (6), we have

$$
\begin{aligned}
\cos J(x+x) &= \cos J(x)\cos J(x) + 2\sin J(x)\sin J(x)\\
&= \cos J^2(x) + 2\sin J^2(x)
\end{aligned}
$$

$\square$

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