THE GOODNESS OF LONG PATH WITH RESPECT TO MULTIPLE COPIES OF COMPLETE GRAPHS

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Abstract. Let $H$ be a graph with the chromatic number $\chi(H)$ and the chromatic surplus $s(H)$. A connected graph $G$ of order $n$ is called good with respect to $H$, $H$-good, if $R(G, H) = (n - 1)(\chi(H) - 1) + s(H)$. The notation $tK_m$ represents a graph with $t$ identical copies of complete graphs on $m$ vertices, $K_m$. In this note, we discuss the goodness of path $P_n$ with respect to $tK_m$. It is obtained that the path $P_n$ is $tK_m$-good for $m, t \geq 2$ and sufficiently large $n$. Furthermore, it is also obtained the Ramsey number $R(G, tK_m)$, where $G$ is a disjoint union of paths.

Key words and Phrases: $(G, H)$-free, $H$-good, complete graph, path, Ramsey number.

1. Introduction

All graphs in this paper are finite, undirected and simple. Let $G$ and $H$ be two graphs, where $H$ is a subgraph of $G$, we define $G - H$ as a graph obtained from $G$ by deleting the vertices of $H$ and all edges incident to them. Let $t$ be a
natural number and \( G_i \) be a connected graph with the vertex set \( V_i \) and the edge set \( E_i \) for every \( i = 1, 2, \ldots, t \). The disjoint union of graphs, \( \bigcup_{i=1}^{t} G_i \), has the vertex set \( \bigcup_{i=1}^{t} V_i \) and the edge set \( \bigcup_{i=1}^{t} E_i \). Furthermore, if each \( G_i \) is isomorphic to a connected graph \( G \) then we denote by \( tG \) the disjoint union of \( t \) copies of \( G \).

For graphs \( G \) and \( H \), the Ramsey number \( R(G, H) \) is the minimum \( n \) such that in every coloring of the edges of the complete graph \( K_n \) with two colors, say red and blue, there is a red copy of \( G \) or a blue copy of \( H \). A graph \( F \) is called \((G, H)\)-free if \( F \) contains no subgraph isomorphic to \( G \) and its complement \( \overline{F} \) contains no subgraph isomorphic to \( H \). The Ramsey number \( R(G, H) \) can be equivalently defined as the smallest natural number \( n \) such that no \((G, H)\)-free graph on \( n \) vertices exists.

Determining \( R(G, H) \) is a notoriously hard problem. Burr [4] showed that the problem of determining whether \( R(G, H) \leq n \) for a given \( n \) is NP-hard. Furthermore in Shaeffer [8] one can find a rare natural example of a problem higher than NP-hard in the polynomial hierarchy of computational complexity theory, that is, Ramsey arrowing is \( \Pi^p_1 \)-complete. The few known values of \( R(G, H) \) are collected in the dynamic survey of Radziszowski [7].

Burr [3] proved the general lower bound
\[
R(G, H) \geq (n - 1)(\chi(H) - 1) + s(H),
\]
where \( G \) is a connected graph of order \( n \), \( \chi(H) \) denotes the chromatic number of \( H \) and \( s(H) \) is its chromatic surplus, namely, the minimum cardinality of a color class taken over all proper colorings of \( H \) with \( \chi(H) \) colors. Motivated by this inequality, the graph \( G \) is said to be \( H \)-good if equality holds in (1). Chvátal [5] proved that trees are \( K_m \)-good graphs. Sudarsana et al. [10] showed that path is a good graph with respect to \( 2K_2 \), and \( P_n \) is also \( tW_4 \)-good in [12]. Other result concerning the goodness of graphs with the chromatic surplus one can be found in Lin et al. [6]. However, the goodness of path \( P_n \) with respect to \( tK_m \) for \( t \geq 2 \) is still open. In this paper, we establish that \( P_n \) is \( tK_m \)-good for \( t \geq 2 \) and sufficiently large \( n \).

2. Known Results

For the proof of our new result, Theorem 3.1, we use the following results.

**Theorem 2.1** (Chvátal [5]). Let \( n, m \geq 2 \) be integers and \( T_n \) is a tree of order \( n \). Then, \( R(T_n, K_m) = (n - 1)(m - 1) + 1 \).

Note that the chromatic surplus of \( K_m \), \( s(K_m) \), is equal to one and path \( P_n \) is a tree of order \( n \). Therefore, \( R(P_n, K_m) = (n - 1)(m - 1) + 1 \).

**Theorem 2.2** (Sudarsana et al. [10]). Let \( m \geq 2 \) and \( n \geq 3 \) be integers. Then, \( R(P_n, 2K_2) = (n - 1)(m - 1) + 2 \).

**Lemma 2.3** (Sudarsana et al. [10]). Let \( n \) and \( t \) be positive integers. Then,
\[
R(P_n, tK_2) = \begin{cases} 
  n + t - 1, & t \leq \left\lfloor \frac{n}{2} \right\rfloor; \\
  2t + \left\lfloor \frac{n}{2} \right\rfloor - 1, & t > \left\lfloor \frac{n}{2} \right\rfloor.
\end{cases}
\]
3. The Main Result

The following theorem deals with the goodness of path $P_n$ with respect to $t$ identical copies of complete graphs, $tK_m$.

**Theorem 3.1.** Let $m, t \geq 2$ be integers and $g(t, m) = (t-2)((tm-2)(m-1)+1)+3$. If $n \geq g(t, m)$ then $R(P_n, tK_m) = (n-1)(m-1)+t$.

**Proof of Theorem 3.1:** The lower bound $R(P_n, tK_m) \geq (n-1)(m-1)+t$ follows from the fact that $(m-1)K_{n-1} \cup K_{t-1}$ is a $(P_n, tK_m)$-free graph of order $(n-1)(m-1)+t-1$.

To prove the upper bound $R(C_n, tK_m) \leq (n-1)(m-1)+t$ we use inductions on $t$ and $m$. For $t = 2$, we have $g(2, m) = 3$ and therefore Theorem 2.2 implies that $R(P_n, 2K_m) = (n-1)(m-1)+2$ for $n \geq g(2, m) = 3$. Hence, the assertion holds for $n \geq g(2, m) = 3$. Assume that the theorem is true for $n \geq g(t-1, m)$, that is $R(P_n, (t-1)K_m) \leq (n-1)(m-1)+t-1$.

From Lemma 2.3, we have $R(P_n, tK_2) = n+t-1$ for $n \geq 2t$. Note that if $t \geq 2$ then $n \geq g(t, 2) > 2t$. Therefore, the theorem holds for $m = 2$. Assume that $m \geq 3$ and the theorem is true for $n \geq g(t, m-1)$, that is $R(P_n, tK_{m-1}) \leq (n-1)(m-2)+t$.

Now we will show that the theorem is also valid for $n \geq g(t, m)$. Let $F$ be an arbitrary graph on $(n-1)(m-1)+t$ vertices. We shall show that $F$ contains $P_n$ or $\overline{F}$ contains $tK_m$. Note that Theorem 2.1 guarantees that $F$ contains $P_n$ or $\overline{F}$ contains $K_m$. If $F$ contains $P_n$ then we are done. Thus we may assume that $\overline{F}$ contains $K_m$. Since the subgraph $F - \overline{K_m}$ of $F$ has $(n-2)(m-1)+t-1$ vertices and $n-1 \geq g(t, m) - 1 > g(t-1, m)$, by the induction hypothesis on $t$ we know that $F - \overline{K_m}$ contains $P_{n-1}$ or the complement of $F - \overline{K_m}$ contains $(t-1)K_m$. If the complement of $F - \overline{K_m}$ contains $(t-1)K_m$ then by companying with the first ones we have a $tK_m$ in $\overline{F}$ and hence the proof is done. Thus, $F$ has a path $P_{n-1}$. Therefore, the subgraph $F - P_{n-1}$ of $F$ has $(n-1)(m-2)+t$ vertices. Note that $n \geq g(t, m) > g(t, m-1)$. By the induction hypothesis on $m$, we know that $F - P_{n-1}$ contains $P_n$ or the complement of $F - P_{n-1}$ contains $tK_{m-1}$. If $F - P_{n-1}$ contains $P_n$ then we are done. Hence we may assume that $P$ contains a path $P_{n-1}$ with vertex set, say $p_1, p_2, \ldots, p_{n-1}$ and edges $p_ip_{i+1}$ (subscripts modulo $(n-1)$), and that $F$ contains $t$ disjoint copies $K_{m-1}, K_{m-1}, \ldots, K_{m-1}$ of the complete graph with $m-1$ vertices. It is clear that the subgraphs $P_{n-1}$ and $tK_m$ have no vertices in common.

Assume that $F$ contains no $P_n$. We will show that $\overline{F}$ contains $tK_m$. Thus, the end vertices $p_1$ and $p_{n-1}$ of path $P_{n-1}$ must not be adjacent to any vertices in $K_{m-1}, K_{m-1}, \ldots, K_{m-1}$. Therefore, the set $D = \{p_1 \cup V(K_{m-1})\} \cup \{p_{n-1} \cup V(K_{m-1})\}$ forms a $2K_m$ in $\overline{F}$. Let us now consider the relation between the vertices in $A' = \{p_2, p_3, \ldots, p_{n-2}\}$ and in $B' = V(K_{m-1}) \cup V(K_{m-1}) \cup \ldots \cup V(K_{m-1})$.

Since there is no $P_n$ in $F$, it follows that every two consecutive vertices $p_i, p_{i+1}$ in $A'$ can not be adjacent to any vertices in $B'$ for every $i \in \{2, 3, \ldots, n-2\}$. Suppose that the neighborhood $N_{A'}(u)$ in $A'$ of a vertex $u \in B'$ satisfies $|N_{A'}(u) \cap V(P_{n-1})| \geq tm-1$. Let $p_i, p_j \in N_{A'}(u) \cap V(P_{n-1})$ with $i < j$. Note that $j-i > 1$ since otherwise
we can extend $P_{n-1}$ to a path of order $n$ containing $u$. If $p_{i+1}p_{j+1}$ is an edge in $F$ then we also have a new path $\{p_1p_2...p_ip_{j-1}p_{j-2}...p_{i+1}p_{j+1}p_{j+2}...p_{n-1}\}$ of length $n-1$ in $F$. If $p_{i+1}p_{j+1}$ is not an edge for every pair $p_i, p_j \in N_{A'}(u) \cap V(P_{n-1})$ then $\{p_i : p_i \in N_{A'}(u) \cap V(P_{n-1})\}$ is a set of $tm$ independent vertices in $F$ and we obtain that $F$ contains $tK_m$. Hence, for each $u \in B'$ we have $|N_{A'}(u) \cap V(P_{n-1})| \leq tm - 2$. Therefore,

$$\left| A \setminus \bigcup_{u \in B'} N_{A'}(u) \right| \geq n - 3 - (t-2)(tm - 2)(m-1). \quad (2)$$

Since $n \geq g(t, m)$, it follows that there are at least $t-2$ vertices in $A'$ which are adjacent to no vertex in $B'$ and hence together with $D$ we have that $F$ contains $tK_m$. This concludes the proof of Theorem 3.1. □

By extending previous results of Baskoro et al. [1] and Stahl [9], Bielak [2] and Sudarsana et al. [11] independently proved a formula for $R(G, H)$ when every connected component of $G$ is an $H$-good graph. This result motivates the study of general families of $H$-good graphs. In particular, Theorem 3.1 provides the following computation of $R(G, tK_m)$, if $G$ is a set of disjoint paths (linear forest).

**Corollary 3.2.** Let $m, t \geq 2$ be integers and $g(t, m) = (t-2)((tm - 2)(m-1)+1)+3$. Let $G \simeq \bigcup_{i=1}^k l_iP_{n_i}$, where $l_i \geq 1$ and each $P_{n_i}$ is a path of order $n_i$.

If $n_1 \geq n_2 \geq ... \geq n_k \geq g(t, m)$ then

$$R(G, tK_m) = \max_{1 \leq i \leq k} \left\{ (n_i - 1)(m-2) + \sum_{j=1}^i l_in_j \right\} + t - 1. \quad (3)$$

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**References**


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