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EIGENVALUES AND EIGENVECTORS OF LATIN SQUARES IN MAX-PLUS ALGEBRA

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Abstract. A Latin square of order n is a square matrix with n different numbers such that numbers in each column and each row are distinct. Max-plus Algebra is algebra that uses two operations, $\oplus$ and $\otimes$. In this paper, we solve the eigenproblem for Latin squares in Max-plus Algebra by considering the permutations determined by the numbers in the Latin squares.

Key words and Phrases: Latin squares, Max-plus Algebra, Eigenproblems, Permutation.

Abstrak. Latin square order n merupakan matriks persegi dengan n angka berbeda sehingga angka-angka pada tiap baris dan kolom semuanya berbeda. Aljabar max-plus merupakan aljabar yang menggunakan dua operasi, $\oplus$ dan $\otimes$. Pada paper ini, diselesaikan permasalahan eigen dari Latin square pada aljabar max-plus dengan memperhatikan permutasi dari angka-angka pada Latin square tersebut.

Kata kunci: Latin square, Aljabar max-plus, Permasalahan eigen, Permutasi.

1. Introduction

In this paper we consider eigenproblems. From a square matrix A, eigenproblems are the problems of finding a scalar λ and corresponding vector v that satisfy $Av = \lambda v$ and we apply this problems into max-plus algebra. The problems can be solved by algorithm in [6]. The purpose of this paper is to solve eigenproblems in max-plus algebra for Latin squares by considering the permutations of symbol (or numbers) in Latin squares.

A reason for studying eigenproblems of Latin square in max-plus algebra is

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that such problems have been studied for other matrices, for example Monge matrix [2], inverse Monge matrix [4] and circulant matrix [10, 11]. Eigenproblems are more simple to solve for that special matrices. For instance, eigenvalue of circulant matrices is equal to maximal number of that ones [10, 11].

The outline of this paper is as follows. In Section 2, we introduce Latin squares and permutations in the context of Latin squares. In Section 3, we introduce max-plus algebra and some theories about graph representation in max-plus algebra. Next in Section 4 we give theory of eigenproblems in max-plus algebra and some conditions to solve it. In Section 5 we give analyses to solve eigenproblems in max-plus algebra. In Section 6 we give an illustration of our problems. We give some remarks and conclusion in Section 7.

2. Latin Square and Permutation

A Latin square of order n is a matrix of size $n \times n$ with n different numbers such that in each row and each column filled by the permutation of those numbers [3], in other words the entries in each row and in each column are distinct [5]. Latin squares were firstly studied by Swiss mathematician, Leonhard Euler. The study of Latin square has long tradition in combinatorics [1], for example the enumeration of Latin squares. The method or formula to enumerate the number of Latin squares can be found in [3, 12, 13]. An example of Latin square of order 4 is shown in below

Example 1

$$L = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 4 & 2 & 3 \\ 4 & 1 & 3 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}.$$

The notion of permutation is related to the act of rearranging objects or values. A permutation of n objects is an arrangement of this objects into a particular order. For example there are six permutations of numbers $1, 2, 3$, that is (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2) and (3,2,1). For simplicity, we write a permutation without parentheses and commas. So we will write 123 rather than (1, 2, 3). In this paper, we define $\underline{n} = \{1, 2, \ldots, n\}$ as set of the n first natural numbers.

In algebra, especially group theory, permutation is a bijective mapping on set X. A family of all permutations on X is called the symmetric group S_X [9], we write S_n rather than S_X for $X = \underline{n}$. From rearrangement $i_1 i_2 \ldots i_n$ of $\underline{n}$ we can define a function $\alpha : \underline{n} \to \underline{n}$ as $\alpha(1) = i_1, \alpha(2) = i_2, \ldots, \alpha(n) = i_n$. If $\alpha(i) = i$ for $i \in \underline{n}$, then i is fixed by α. For example, the rearrangement 321 determines the function α with $\alpha(1) = 3$, $\alpha(2) = 2$, $\alpha(3) = 1$ and 2 fixed by α.

We can write permutation in cycle form i.e. $(a_1\ a_2\ \ldots\ a_r)$ if $\alpha(a_1) = a_2, \alpha(a_2) = a_3, \ldots, \alpha(a_{r-1}) = a_r, \alpha(a_r) = a_1$ and called by r-cycle (cycle of length r). A complete factorization of a permutation α is a factorization of α into disjoint cycles that contains exactly one 1-cycle of i for every i fixed by α [9]. For example, the complete factorization of the 3-cycle $\alpha = (1\ 3\ 5) \in S_5$ is $\alpha = (1\ 3\ 5)(2)(4)$.

Suppose Latin square $L = (l_{i,j})$ has order n. We can get n permutations that represent of each number of L. Let $s \in \underline{n}$, we define *permutation symbol* of number s by σ_s such that $\sigma_s(i)$ equal to j for which $l_{i,j} = s$ [12]. For example, from Latin square L in Example 1, we get $\sigma_1, \sigma_2, \sigma_2, \sigma_4 \in S_4$ as permutation symbol of number 1, 2, 3, 4 in L respectively where $\sigma_1 = (1\ 3\ 2)(4), \sigma_2 = (1)(2\ 3\ 4), \sigma_3 = (1\ 2\ 4)(3), \sigma_4 = (1\ 4\ 3)(2)$.

3. Max-Plus Algebra

In max-plus algebra we define algebraic structure $(\mathbb{R}_\varepsilon, \otimes, \oplus)$, where $\mathbb{R}_\varepsilon$ is the set of all real numbers $\mathbb{R}$ extended by an infinite element $\varepsilon = -\infty$ and operation $\otimes, \oplus$ defined by

$$x \oplus y = \max\{x, y\} \text{ and } x \otimes y = x + y \tag{1}$$

respectively. It is easy to show that both operation $\oplus$ and $\otimes$ are associative and commutative. Because $x \oplus \varepsilon = \varepsilon \oplus x = x$ and $x \otimes 0 = 0 \otimes x = x$ for all $x \in \mathbb{R}_\varepsilon$ then the null and unit element in max-plus algebra is ε and 0 respectively.

For all $x \in \mathbb{R}_\varepsilon$ and non-negative integer n, we define

$$x^{\otimes n} = \begin{cases} 0, & \text{for } n = 0 \\ \underbrace{x \otimes x \otimes x \otimes \ldots \otimes x}_{n}, & \text{for } n > 0 \end{cases} \tag{2}$$

We can write $x^{\otimes n}$ in conventional algebra

$$x^{\otimes n} = \underbrace{x \otimes x \otimes x \otimes \ldots \otimes x}_{n} = n \times x$$

or generally for all $\beta \in \mathbb{R}$

$$x^{\otimes \beta} = \beta \times x$$

The set of all square matrices of order n in max-plus algebra are defined by $\mathbb{R}_\varepsilon^{n\times n}$. Let $A \in \mathbb{R}_\varepsilon^{n\times n}$, the entry of A at i^{th} row and j^{th} column is defined by $a_{i,j}$ and sometime we write $[A]_{i,j}$. For $A, B \in \mathbb{R}_\varepsilon^{n\times n}$, addition of matrix, $A \oplus B$, is defined by

$$\begin{aligned} [A \oplus B]_{i,j} &= a_{i,j} \oplus b_{i,j} \\ &= \max\{a_{i,j}, b_{i,j}\} \end{aligned} \tag{3}$$

and multiplication of matrix, $A \otimes B$, is defined by

$$\begin{aligned} [A \otimes B]_{i,j} &= \bigoplus_{k=1}^{n} a_{i,k} \otimes b_{k,j} \\ &= \max_{k \in \underline{n}}\{a_{i,k} + b_{k,j}\} \end{aligned} \tag{4}$$

For square matrix A, similar to scalar in max-plus algebra, we denote

$$A^{\otimes k} = \underbrace{A \otimes A \otimes A \otimes \ldots \otimes A}_{k}$$

as k^{th} power of A.

From $L \in \mathbb{R}_\varepsilon^{n\times n}$, we can get directed graph (digraph) $\mathcal{G}(L) = \mathcal{G}(V, E)$, where V is set of vertices and E is set of edges. In $\mathcal{G}(L)$, there are n vertices labelled by $1, 2, \ldots, n$ respectively. There is an edge from vertex i to vertex j if $a_{j,i} \neq \varepsilon$

denoted by (i, j). The weight of (i, j)-edge is denoted by $w(j, i)$ and equal to $a_{j,i}$, if $a_{ji} = \varepsilon$ then there is no (i, j)-edge. Graph representation of matrix L in Example 1 is shown in Fig. 1.

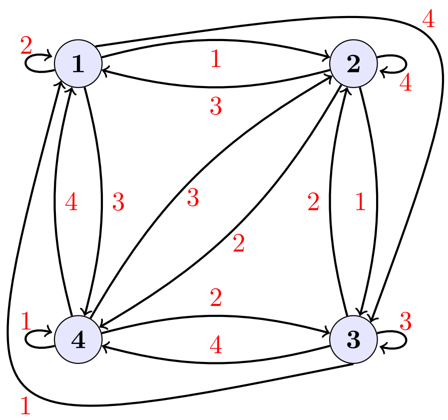

FIGURE 1. Graph representation of matrix L

A sequence of edges $(j_1, j_2), (j_2, j_3), \ldots, (j_{k-1}, j_k)$ is called *path* and if all vertices $j_1, j_2, \ldots, j_{k-1}$ are different then called *elementary path*. *Circuit* is an elementary closed path, i.e. $(j_1, j_2), (j_2, j_3), \ldots, (j_{k-1}, j_1)$. A circuit consists of a single edge, from a vertex to itself, is called a *loop*. Weight of a path $p = (j_1, j_2), (j_2, j_3), \ldots, (j_{k-1}, j_k)$ is denoted by $|p|_w$ and equal to sum of all weight each edge i.e. $|p|_w = a_{j_2 j_1} + a_{j_3 j_2} + \ldots + a_{j_k j_{k-1}}$ and length of path is denoted by $|p|_l$ and equal to the number of edges in path p. The average weight of path p defined by weight of p divide by length of path p,

$$\frac{|p|_w}{|p|_l} = \frac{a_{j_2 j_1} + a_{j_3 j_2} + \ldots + a_{j_k j_{k-1}}}{k-1} \tag{5}$$

Any circuit with maximum average weight is called a *critical circuit*. A graph called *strongly connected* if there is a path for any vertex i to any vertex j. If graph $\mathcal{G}(L)$ is strongly connected, then matrix L is *irreducible*. We can infer that $[L]_{i,j}$ is equal to the weight of path with length 1 from j to i, $[L^{\otimes 2}]_{i,j}$ is equal to the maximal weight of path with length 2 from j to i or generally for positive integer k, $[L^{\otimes k}]_{i,j}$ is equal to the maximal weight of path with length k form j to i.

There is relation between $\sigma_i \in S_n$ and a circuit in $\mathcal{G}(L)$. Every r-cycle in σ_i represented circuit of length r with each edge have weight i. Let graph representation in Fig. 1. We get $\sigma_2 = (1)(2\ 3\ 4)$ and there are two cycles of $(1)(2\ 3\ 4)$, 1-cycle (1) and 3-cycle $(2\ 3\ 4)$. As we can see in Fig. 1 there are two circuit with all edges have weight 2, a loop in vertex 1 and a circuit with length 3 $(4, 3), (3, 2), (2, 4)$.

Let $A \in \mathbb{R}_{\varepsilon}^{n\times n}$, we define the matrix A^+ as follow

$$A^+ \overset{def}{=} \bigoplus_{i=1}^{\infty} A^{\otimes i} = A \oplus A^{\otimes 2} \oplus \ldots \oplus A^{\otimes n} \oplus \ldots \tag{6}$$

Because $[A^{\otimes k}]_{i,j}$ is equal to maximal weight of all paths with length k from vertex j to vertex i then $[A^+]_{i,j}$ is equal to maximal weight of any path with any length from vertex j to vertex i.

If $B \in \mathbb{R}^{n\times n}$ such that all circuits in $\mathcal{G}(B)$ have average weight less than or equal to 0 then B^+ is equal to the summation (in max-plus) of $B^{\otimes k}$ for $k = 1, 2, \ldots, n$, or in other words

$$B^+ = B \oplus B^{\otimes 2} \oplus \ldots \oplus B^{\otimes n}$$

4. Eigenproblems

Eigenproblems are common problem in mathematics especially in linear algebra. In linear algebra, eigenproblems are the problems of finding $\lambda \in \mathbb{R}$ and vectors $v \in \mathbb{R}^n$ from matrix A of size $n \times n$ that satisfy $Av = \lambda v$ and then λ is called by *eigenvalue* while vector v is called by *eigenvector*. In max-plus algebra, similar to linear algebra, eigenproblems are formulated as $A \otimes v = \lambda \otimes v$ for given matrix $A \in \mathbb{R}_{\varepsilon}^{n\times n}$, where $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$. The method to solve eigenproblems in max-plus algebra is quite different in linear algebra.

Methods to solve eigenproblems in max-plus algebra were handled by several authors for ordinary matrices [6, 7, 8], as well as for special matrices such as circulant matrix [10, 11], Monge matrix [2] and inverse Monge matrix [4]. Special case for irreducible matrices, problem to get an eigenvalue related to problem to get critical circuits because the eigenvalue of A is equal to the weight of critical circuits in $\mathcal{G}(A)$ [8]. If the eigenvalue exist for irreducible matrix A then there is unique eigenvalue [8].

In this paper we define $\lambda(A)$ as eigenvalue of matrix A and A_λ be a matrix such that $[A_\lambda]_{i,j} = [A]_{i,j} - \lambda(A)$ or in other word $A_\lambda = (-\lambda(A)) \otimes A$. It is clear that the maximum average weight of any circuit in $\mathcal{G}(A_\lambda^+)$ is less than or equal 0. Consequently, we can derived as follow

$$A_\lambda^+ = A_\lambda \oplus A_\lambda^{\otimes 2} \oplus \ldots \oplus A_\lambda^{\otimes n}$$

and the i^{th} column of A_λ^+ is eigenvector of A if $[A_\lambda^+]_{i,i} = 0$ [8]. There is an algorithm to obtain eigenvalue and eigenvector that called *Power Algorithm* [6, 7].

5. Discussion, Analyses and Results

We will discuss Latin squares in max-plus algebra, so it is allowed to use infinite element $\varepsilon = -\infty$ as a symbol of a Latin square. Thus, we consider two cases of Latin squares.

- Case 1.
  Latin square without infinite element that use $\underline{n} = \{1, 2, \ldots, n\}$ as elements of Latin square.

- Case 2.
  Latin square with infinite element that use $\underline{n}_\varepsilon = \{\varepsilon, 1, 2, \ldots, n-1\}$ as elements of Latin square

We denote $\mathcal{L}^n$ and $\mathcal{L}^n_\varepsilon$ be the set of all Latin squares of order n without and with infinite element, respectively.

We begin the observation from graph representation of Latin square. Let $L_1 \in \mathcal{L}^n$, because all numbers in L_1 are finite then $[L_1]_{i,j} \neq \varepsilon$ for all $i, j \in \underline{n}$ and it is clear that $\mathcal{G}(L_1)$ is strongly connected, consequently L_1 is irreducible. It can be concluded that all Latin squares without infinite element are irreducible matrix.

Let $L_2 \in \mathcal{L}^n_\varepsilon$, because in each row and each column of L_2 there is exactly one ε then $[L_2^{\otimes 2}]_{i,j} = \max_{k \in \underline{n}}\{a_{i,k} + a_{k,j}\}$ is finite. Consequently, there is a path length 2 from any vertex i to any vertex j and L_2 also irreducible. It can be concluded that all Latin squares with infinite element are irreducible matrix. Because both L_1 and L_2 are irreducible matrix then to find eigenvalue of L_1 and L_2 we need to find the critical circuit of graph representation of each matrix.

In next discussion we will solve eigenproblems of Latin squares in max-plus algebra and given the result about eigenvalue, eigenvector and the number of linearly independent eigenvectors also derive some theorems about them. See Section 6 for examples.

Theorem 5.1. *Let $L_1 \in \mathcal{L}^n$ and $L_2 \in \mathcal{L}^n_\varepsilon$. The average weight of critical circuits of $\mathcal{G}(L_1)$ and $\mathcal{G}(L_2)$ is equal to n and $n-1$ respectively.*

Proof. We only need to consider permutation of the largest number in L_1 and L_2. It is clear that $\max \underline{n} = n$ and $\max \underline{n}_\varepsilon = n-1$. Let σ_n be permutation symbol of number n in L, from σ_n we get circuit with the weight of all edges are n. Because all edges have weight n, then the average weight of circuit is n and there is no circuit with average weight more than n. Thus, all circuits based on σ_n are critical circuit in $\mathcal{G}(L_1)$ and the average weight of those critical circuit in $\mathcal{G}(L_1)$ is equal to n.

By the same argument, we get the average weight of critical circuits in $\mathcal{G}(L_2)$ is equal to $n-1$.

■

Theorem 5.2. *Let $L_1 \in \mathcal{L}^n$ and $L_2 \in \mathcal{L}^n_\varepsilon$. Eigenvalue of L_1 and L_2 is equal to n and $n-1$ respectively or generally eigenvalue of Latin square L is equal to the maximal number in L.*

Proof. The proof of this theorem is from direct result of Theorem 5.1

■

Let L be Latin square of order n that has eigenvalue λ. To get eigenvalue of Latin square in max-plus algebra we consider the matrix L^+_λ. We know that the i^{th} column of L^+_λ is eigenvector of L if $[L^+_\lambda]_{i,i} = 0$. Number $i \in \underline{n}$ satisfies $[L^+_\lambda]_{i,i} = 0$ if and only if in graph $\mathcal{G}(L)$ there is critical circuit from vertex i.

If L is Latin square then λ is equal to the maximal number in L i.e. $\lambda(A) =$

max(A) and λ appears exactly once in each row and column of L, consequently there is always critical circuit that every edge has weight λ from any vertex i for all $i \in \underline{n}$. Consequently, for Latin square L all column of L_λ^+ are eigenvector of L with eigenvalue λ.

We say that two vectors v_1, v_2 are linearly independent (in max-plus algebra) if there is no $c \in \mathbb{R}$ such that $v_1 = c \otimes v_2$. In max-plus algebra, it is possible that any matrix L has two or more linearly independent eigenvectors.

We know that each critical circuit in $\mathcal{G}(L)$ represents eigenvector of L. If there are m different critical circuits then there are m linearly independent eigenvectors or we can say that the number of linearly independent eigenvectors is equal to the number of different critical circuit in $\mathcal{G}(L)$.

Theorem 5.3. *Let L a be Latin square with eigenvalue λ. The number of linearly independent eigenvectors of L with respect to eigenvalue λ is equal to the number of cycle in permutation symbol σ_λ.*

Proof. Because L is a Latin square with eigenvalue λ then in graph $\mathcal{G}(L)$ there are critical circuits with average weight equal to λ where each edge has weight λ. And because λ appears exactly once in each row and column of L then we can always make critical circuit based on permutation symbol of λ i.e. σ_λ.

We know that every r-cycle in σ_λ represented a critical circuit length r where each edge have weight λ then the number critical circuit is equal to the number of cycle in σ_λ and this completes the proof.

■

6. Example

We give two examples of Latin square, without and with infinite element $\varepsilon = -\infty$.

Example I.

$$A = \begin{bmatrix} 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

By Theorem 5.2 eigenvalue of A is maximal number in A i.e. $\lambda(A) = \max(A) = 4$. From A we get permutation symbol $\sigma_\lambda = \sigma_4 = (2\ 4) = (1)(2\ 4)(3) \in S_4$ and there are three cycles in σ_λ. Next we get

$$A_\lambda = \begin{bmatrix} 0 & -3 & -2 & -1 \\ -3 & -2 & -1 & 0 \\ -2 & -1 & 0 & -3 \\ -1 & 0 & -3 & -2 \end{bmatrix} \quad A_\lambda^{\otimes 2} = \begin{bmatrix} 0 & -1 & -2 & -1 \\ -1 & 0 & -1 & -2 \\ -2 & -1 & 0 & -1 \\ -1 & -2 & -1 & 0 \end{bmatrix}$$

$$A_\lambda^{\otimes 3} = \begin{bmatrix} 0 & -1 & -2 & -1 \\ -1 & -2 & -1 & 0 \\ -2 & -1 & 0 & -1 \\ -1 & 0 & -1 & -2 \end{bmatrix} \quad A_\lambda^{\otimes 4} = \begin{bmatrix} 0 & -1 & -2 & -1 \\ -1 & 0 & -1 & -2 \\ -2 & -1 & 0 & -1 \\ -1 & -2 & -1 & 0 \end{bmatrix}$$

and

$$\begin{aligned} A_\lambda^+ &= A_\lambda \oplus A_\lambda^{\otimes 2} \oplus A_\lambda^{\otimes 3} \oplus A_\lambda^{\otimes 4} \\ &= \begin{bmatrix} 0 & -1 & -2 & -1 \\ -1 & 0 & -1 & 0 \\ -2 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix} \end{aligned}$$

By Theorem 5.3, the number of linearly independent eigenvectors is equal to the number of cycle in σ_λ and from A_λ^+, we can get three different column vectors

$$\begin{bmatrix} 0 \\ -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

There are three linearly independent eigenvectors of A with eigenvalue $\lambda = 4$ and the number of cycle in σ_λ is also 3.

Example II.

$$B = \begin{bmatrix} 2 & 3 & 1 & -\infty \\ 3 & -\infty & 2 & 1 \\ 1 & 2 & -\infty & 3 \\ -\infty & 1 & 3 & 2 \end{bmatrix}$$

By Theorem 5.2, the eigenvalue of B is maximal number in B i.e. $\lambda(B) = \max(B) = 3$. From B we get permutation symbol $\sigma_\lambda = (1\ 2)(3\ 4) \in S_4$ and there are two cycles in σ_λ. Next we get

$$B_\lambda = \begin{bmatrix} -1 & 0 & -2 & -\infty \\ 0 & -\infty & -1 & -2 \\ -2 & -1 & -\infty & 0 \\ -\infty & -2 & 0 & -1 \end{bmatrix} \quad B_\lambda^{\otimes 2} = \begin{bmatrix} 0 & -1 & -1 & -2 \\ -1 & 0 & -2 & -1 \\ -1 & -2 & 0 & -1 \\ -2 & -1 & -1 & 0 \end{bmatrix}$$

$$B_\lambda^{\otimes 3} = \begin{bmatrix} -1 & 0 & -2 & -1 \\ 0 & -1 & -1 & -2 \\ -2 & -1 & -1 & 0 \\ -1 & -2 & 0 & -1 \end{bmatrix} \quad B_\lambda^{\otimes 4} = \begin{bmatrix} 0 & -1 & -1 & -2 \\ -1 & 0 & -2 & -1 \\ -1 & -2 & 0 & -1 \\ -2 & -1 & -1 & 0 \end{bmatrix}$$

and

$$\begin{aligned} B_\lambda^+ &= B_\lambda \oplus B_\lambda^{\otimes 2} \oplus B_\lambda^{\otimes 3} \oplus B_\lambda^{\otimes 4} \\ &= \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \end{aligned}$$

By Theorem 5.3, the number of linearly independent eigenvectors is equal to the number of cycle in σ_λ and from B_λ^+, we can get two different column vectors

$$\begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

There are two linearly independent eigenvectors of B with eigenvalue $\lambda = 3$ and the number of cycle in σ_λ is also 2.

7. Conclusion

Eigenproblems for any Latin square L can be solved by considering the permutation symbol of maximal number in L. Moreover, eigenvalue is equal to the maximal number in L and the number of linearly independent eigenvectors is equal to the number of cycle in permutation symbol of those maximal number.

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