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CONSTRUCTION OF THE BINO-TRINOMIAL METHOD USING THE FUZZY SET APPROACH FOR OPTION PRICING

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Abstract. An option is a financial instrument that investors often use for speculation or hedging purposes. Calculating the profit in the investment using options also considers its price, so the investor needs to know the proper value of the option's price or at least the range of these values. This paper aims to improve the Bino-Trinomial tree model for determining the price of a European call option with a volatility parameter in the form of a triangular Fuzzy number. The Bino-Trinomial tree model is a combination of the Binomial and Trinomial trees that aims to control the values of its branches. Due to the involvement of the Fuzzy number, the obtained value of the option price is in a range or interval, so the investor could use it appropriately in arranging investment strategies. In the proposed model, the Fuzzy volatility parameter is utilized to capture the uncertainty of the estimated volatility in the financial market which can fluctuate from time to time. This parameter is expected to provide reasonable ranges and appropriate Fuzzy membership functions for option pricing so that investors can expect different optimal values for different risk preferences. We also adjusted the formulation of the increase and decrease factors in the Fuzzy Binomial tree to model stock price movements. Using different values of the volatility's sensitivity level and the option period, the results of numerical simulations show that prices of European call options given by the market are always within the option price range of the proposed model's result. Likewise, the results of the defuzzification of options prices in our Fuzzy Bino-Trinomial tree model are not much different from the prices given by the market. This shows that the Fuzzy Bino-Trinomial tree model performs better in determining the price of European call options than the Fuzzy Binomial tree and Fuzzy Trinomial models.

Key words and Phrases : Fuzzy volatility, option pricing, Bino-Trinomial tree model, Binomial tree model, Trinomial tree model

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1. INTRODUCTION

As a financial instrument, an option is a contract between a seller and a buyer that gives the buyer the right, not the obligation, to buy or sell an underlying asset at or before maturity time for a specified price. The option buyer must pay a fee to the option seller to obtain the right, which is called the option price. Currently, the global Over-The-Counter (OTC) trading on all contracts exceeds 632.238 trillion with a gross market value of 18.438 trillion [1]. Both the buyer and seller must be aware of the appropriate values of the option's price, or at least its range, to calculate the profit on their investment utilizing options. Many methods have been developed for determining the option prices. In many cases, they resulted in theoretical prices that unfortunately are not close to the market's selling prices. Therefore, the study of option pricing is essential for traders and investors in theory and practice.

The option pricing formula, especially for the European type, was first developed by Black and Scholes by deriving a closed-form solution of the partial differential equation known as the Black-Scholes formula for European options [2]. Several years later, the method of European option pricing using a discrete approach was developed by Cox, Ros, and Rubenstein, where it was discovered that the Black-Scholes continuous model formula was a particular case of the Cox-Ross-Rubinstein (CRR) Binomial tree discrete model[3]. Since the development of the CRR Binomial tree model, many researchers have developed discrete option pricing models for vanilla options ([4], [5], [6], [7]) and exotic options ([8], [9], [10], [11], [12], [13], [14], [15]).

The focus of option price modeling studies that have developed over the past two decades is related to the relaxation of the assumptions in the Black-Scholes continuous model. These established assumptions include: (1) price fluctuations of the underlying asset should follow a log-normal distribution; (2) the short-term risk-free interest rate remains unchanged; and (3) stock volatility remains constant [3]. Some input parameters in the Black-Scholes model or Binomial CRR model cannot always be determined precisely and realistically, because financial market conditions fluctuate from time to time [16]. One theory that can be used to describe the uncertainty conditions of input parameter values in models of mathematical finance is the Fuzzy set theory, which was introduced by Zadeh [17] in 1965. Applying Fuzzy parameters in option price modeling is expected to provide a reasonable range. Thus, investors may interpret different optimal values of the obtained option prices for particular risk preferences.

The study of Fuzzy Binomial modeling for option pricing initiated by Muzzioli and Torricelli [18], where the generation of underlying asset's prices using a one-period Binomial tree is in Fuzzy numbers with forms of triangular and L-R Fuzzy numbers. The principle of no arbitrage was applied as in the standard Binomial method of option pricing [3], and the obtained prices of the standard method became a special case of the Fuzzy method for $\alpha = 0$, where the α refers to the α-cut of the Fuzzy numbers.

In [19], the previous authors developed the multi-period Binomial Fuzzy method and improved it by defining the increase and decrease factors of stock prices, which are u and d, as Fuzzy triangular numbers. Appadoo and Bector [20] improved the method in [19] by defining those factors in the trapezoidal type. Liu et al. [21] modeled the increase and decrease factors of the price of basic assets as a Fuzzy variable in the credibility space. Yu et al. [22] incorporated an element of uncertainty in the volatility parameter, whose form was Fuzzy triangular numbers. Yoshida [23] analyzed both American option pricing in the continuous and discrete time-parameter indexes. In particular, the study discussed the problem of determining the optimal downtime or maximizing the price of American options in an environment that contains randomness and uncertainty. Muzzioli and Reynaerts [24] constructed the American option pricing using the Fuzzy Binomial method in [19], where the Fuzzy number forms were triangular and trapezoidal. Xu et al. [25] developed a Fuzzy Binomial method for pricing Vulnerable options by assuming the instantaneous volatility of the underlying stock and company values, and the rate of recovery as Fuzzy variables. Muzzioli and De Baets [26] conducted a literature review on option pricing modeling using a Fuzzy approach. They stated that although there were many contributions in discrete option price models, there is still room for developing other applications.

Some implementations of the Fuzzy Binomial method in investments have been conducted. Liao an Ho [27] used the method for a project valuation under uncertainty that was embedded with real options. Sumarti and Nadya [28] utilized the obtained fuzzy numbers of the American option price for developing the dynamic portfolio that was adjusted periodically, and the strategies of buying or selling the options were determined by their membership level of the Fuzzy numbers. Future research development can be carried out by investigating the usefulness of the Fuzzy Binomial method to determine the price of exotic options with American features. In addition, Trinomial trees and other non-recombination trees can be explored in the Fuzzy parameter setting to enlarge the possible states at each tree level.

Prior to this work, we had constructed A Fuzzy Binomial method for pricing European call options with constant and exponential barriers [29]. In this paper, we utilize the Bino-Trinomial method, which was firstly proposed by Dai and Lyuu [12] for solving some barrier option problems. This type of model fits the challenge to define the volatility as a Fuzzy number, which means it is in a range as it happens in real world. Overall, this paper is structured as follows. An overview of Fuzzy set theory is presented in Section 2. In Section 3, we explain the Fuzzy Binomial, Trinomial, and Bino-Trinomial tree methods for pricing European call options. We define volatility levels and risk-free interest rates in the form of triangular Fuzzy numbers. The numerical analysis are shown in Section 4. The conclusions are stated in Section 5. m

2. FUZZY SET THEORY

This section presents some of the basic Fuzzy properties of Fuzzy numbers used in this paper. The Fuzzy set $\tilde{A}$ is the set of ordered pairs $(x, \mu(x))$, where $x \in X$, $X \subset \mathbb{R}$. Here $\mu(x)$ is the membership function of $x \in X$ that maps x into the real number interval $[0, 1]$. The Fuzzy subset $\tilde{A}$ is defined by its membership function $\tilde{A} : \mathbb{R} \to [0, 1]$. Throughout this paper, the universal set X is assumed to be the set of all real numbers having a regular topology. Suppose f is a real-valued function defined on X, then f is said upper semi-continuous if $\{x|f(x) \geq \alpha\}$ is a closed set for every α. Generally, a Fuzzy number is a subset defined over real numbers. The Fuzzy number $\tilde{A}$ that corresponds to a real number A can be interpreted as values around A, and $\mu_{\tilde{A}} = 1$.

Definition 2.1. [16] *The Fuzzy set $\tilde{A}_\alpha$ in $\mathbb{R}^n$ is called a convex Fuzzy set, if and only if for any $x_1, x_2 \in \mathbb{R}^n$ dan $0 \leq \alpha \leq 1$,*

$$\mu_{\tilde{A}}\left(\alpha x_1 + (1-\alpha)x_2\right) \geq min\{\mu_{\tilde{A}(x_1)}, \mu_{\tilde{A}(x_2)}\}.$$

Definition 2.2. [17] *The following conditions must be met for A to be defined as a Fuzzy number :*

(1) $\tilde{A}$ is a normal and convex Fuzzy set.
(2) The membership function $\mu_{\tilde{A}}$ is upper semi-continuous.
(3) The $\alpha - cut$ set of $\tilde{A}$ is constrained to all $\alpha \in [0, 1]$.

Proposition 2.3. *(Identity Resolution)* [17] *Suppose $\tilde{A}$ is a Fuzzy set with a membership function $\mu_{\tilde{A}}$ and $\alpha - cut$ set of $\tilde{A}$, that is $\tilde{A}_\alpha = \{x|\mu_{\tilde{A}(x)\geq\alpha}\}$, then*

$$\mu_{\tilde{A}}(x) = \sup_{\alpha\in[0,1]} \alpha 1_{(\tilde{A})_\alpha}(x), \tag{1}$$

where $1_{\tilde{A}}$ the indicator function of the set $\tilde{A}$, which is defined as follows: $1_{\tilde{A}}(x) = 1$ if $x \in \tilde{A}$, and $1_{\tilde{A}}(x) = 0$ if $x \notin \tilde{A}$.

Note that $\alpha - cut$ set of $\tilde{A}$ is a set of crisp numbers. Zadeh [17], proves that if $\tilde{A}$ is a Fuzzy number, then its $\alpha - cut$ is a convex and compact set, which is denoted by $\tilde{A}_\alpha = \left[\tilde{A}^L_\alpha, \tilde{A}^U_\alpha\right]$.

Definition 2.4. [16] *$\tilde{A}$ is called a crisp number with the value m if its membership function is*

$$\mu_{\tilde{A}} = \begin{cases} 1, & x = m, \\ 0, & x \neq m, \end{cases} \tag{2}$$

which is denoted by $\tilde{A} = \tilde{1}_{\{m\}}$.

It is easily to follow that $\left(\tilde{1}_{\{m\}}\right)^L_\alpha = \left(\tilde{1}_{\{m\}}\right)^U_\alpha = m$, $\forall\alpha \in [0, 1]$, so any real number can be considered a crips number.

Let we denote $\mathbb{F}$ be the set of all Fuzzy subsets of $\mathbb{R}$, $f\left(x_1, x_2, \cdots, x_n\right)$ be real-valued function from $\mathbb{R}^n$ to $\mathbb{R}$, and $\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n$ are $n-$Fuzzy subsets in

$\mathbb{R}$. Using the extension of principle [30], we can induce a Fuzzy-valued function $\tilde{f}: F^n \to F$ according to the real-valued function $f(x_1, x_2, \cdots, x_n)$. In other words, $\tilde{f}\left(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n\right)$ is a Fuzzy subset of $\mathbb{R}$ with the degree of membership defined as follows [31].

$$\mu_{\tilde{f}\left(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n\right)}(r) = \sup_{\{(x_1, x_2, \cdots, x_n) | r = f(x_1, x_2, \cdots, x_n)\}} min\{\mu_{\tilde{A}_1}(x_1), \cdots, \mu_{\tilde{A}_n}(x_n)\}. \tag{3}$$

Proposition 2.5. [31] *Let $f(x_1, x_2, \cdots, x_n)$ be a real-valued function defined on $\mathbb{R}^n$ and $\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n$ is n Fuzzy subsets of $\mathbb{R}$. Let $\tilde{f}: \mathbb{F}^n \to \mathbb{F}$be a fuzzy-valued function induced by $f(x_1, x_2, \cdots, x_n)$ via the extension principle defined in (3). Suppose that each membership function μ_{A_i} is the upper semi-continuous in $\mathbb{R}$ for $i = 1, 2, \cdots, n$ and each $(x_1, x_2, \cdots, x_n) | r = f(x_1, x_2, \cdots, x_n)$ is a compact subset of $\mathbb{R}^n$ (it will be a closed and bounded set in $\mathbb{R}^n$) for r in the range of f. Then the $\alpha-$level set of $\tilde{f}\left(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_1\right)$ is*

$$\left(\tilde{f}\left(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n\right)\right)_\alpha = \left\{f(x_1, x_2, \cdots, x_n) | x_1 \in (\tilde{A}_1)_\alpha, \cdots, x_n \in (\tilde{A}_n)_\alpha\right\}$$

Definition 2.6. [32] *$\tilde{A}$ is the Fuzzy set in the set of real numbers $\mathbb{R}$ with its reference function $\mu_{\tilde{A}}(x)$ satisfies the listed below conditions for $a, b, c, d \in \mathbb{R}$, and $(a \leq b)$ is known as a generalized L-R type fuzzy number,*

(1) $\mu_{\tilde{A}}(x)$ is a piece-wise continuous mapping from the real line $\mathbb{R}$ onto the interval $[0, \omega]$ where ω is a constant lying in the unit interval $[0, 1]$,
(2) $\mu_{\tilde{A}}(x) = 0$, for all $x \in (\infty, a]$,
(3) $\mu_{\tilde{A}}(x)$ is continuously growing on $[a, b)$,
(4) $\mu_{\tilde{A}}(x) = \omega$, for all $xin[b, c]$,
(5) $\mu_{\tilde{A}}(x)$ is continuously diminishing on $(c, b + \gamma]$,
(6) $\mu_{\tilde{A}(x)} = 0$, for all $x \in [b + \gamma, \infty[$.

Definition 2.7. [32] *The generalized L-R Fuzzy number in definition 2.6 is represented by $\tilde{A} = (a, b, \beta, \gamma; \omega)$ and its membership function $\mu_{\tilde{A}}(x)$ is stated as*

$$\mu_{\tilde{A}} = \begin{cases} \mu^L_{\tilde{A}}(x), & x \in [a - \gamma, a], \\ \omega, & x \in [a, b], \\ \mu^R_{\tilde{A}}(x), & x \in [b, b + \gamma] \\ 0, & otherwise \end{cases}. \tag{4}$$

where $\mu^L_{\tilde{A}}(x) : [a - \beta, a] \to [0, \omega]$ is continuously increasing, called the left membership function and $\mu^R_{\tilde{A}}(x) : [b, b + \gamma] \to [0, \omega]$ is continuously decreasing, called the left membership functions of $\tilde{A} = (a, b, \beta, \gamma; \omega)$

Definition 2.8. *If the membership function of $\tilde{A} = (a,\beta,\gamma)_n$ has the following form:*

$$\mu_{\tilde{A}} = \begin{cases} \left(\frac{x-(a-\beta)}{\beta}\right)^n, & x \in [a-\beta, a] \\ 1, & x = a, \\ \left(\frac{(a+\gamma)-x}{\gamma}\right)^n, & x \in [a, a+\gamma] \\ 0, & otherwise, \end{cases}, \tag{5}$$

then $\tilde{A} = (a,\beta,\gamma)_n$ is called generalized triangular Fuzzy number, which has the core is a, the left width is $\beta = \rho_1 * a$, and the right width is $\gamma = \rho_2 * a$, $0 \le \rho_1, \rho_2 \le 1$. The value of ρ_1 and ρ_2 can be equal or different. If $n = 1$, $\tilde{A} = (a,\beta,\gamma)$ which is known as a triangular Fuzzy number.

Let $\tilde{A} = (a,\beta,\gamma)_n$ express generalized triangular Fuzzy number, then $\alpha-$ cut set of $\tilde{A}$ for all $\alpha \in [0,1]$ has the following form:

$$\tilde{A}_\alpha = \left(\tilde{A}^L_\alpha, \tilde{A}^U_\alpha\right) = \left[(a-\beta) + \alpha^{\frac{1}{n}} \times \beta, (a+\gamma) - \alpha^{\frac{1}{n}} \times \gamma\right]. \tag{6}$$

Cardinality of a Fuzzy number $\tilde{A}$ described by (5) is the value of the integral

$$Card\tilde{A} = \int_{a-\beta}^{a+\gamma} \mu_{\tilde{A}}(x)dx = \frac{\gamma+\beta}{n+1} \tag{7}$$

Proposition 2.9. *The median value of Fuzzy number $\tilde{A}$ described by (5) denoted by $m_{\tilde{A}}$ is the real number from the support of $\tilde{A}$ such that*

$$\int_{a-\beta}^{m_{\tilde{A}}} \mu_{\tilde{A}}(x)dx = \int_{m_{\tilde{A}}}^{a+\gamma} \mu_{\tilde{A}}(x)dx \tag{8}$$

For practical purpose expressionwe can rewritten equation (8) as [33]:

$$\int_{a-\beta}^{m_{\tilde{A}}} \mu_{\tilde{A}}(x)dx = 0.5 \times Card\tilde{A} \tag{9}$$

We can classify Fuzzy numbers with respect to the "distribution" of their cardinality as follows: a Fuzzy number $\tilde{A} = (a,\beta,\gamma)_n$ described by (5) is called

(1) a Fuzzy number with equally heavy tails if $\int_{a-\beta}^{a} \mu_{\tilde{A}}(x)dx = \int_{a}^{a+\gamma} \mu_{\tilde{A}}(x)dx$
(2) a Fuzzy number with a heavy left tail $\int_{a-\beta}^{a} \mu_{\tilde{A}}(x)dx > 0.5 \times \int_{a-\beta}^{a+\gamma} \mu_{\tilde{A}}(x)dx$
(3) a Fuzzy number with a heavy right tail $\int_{a}^{a+\gamma} \mu_{\tilde{A}}(x)dx > 0.5 \times \int_{a-\beta}^{a+\gamma} \mu_{\tilde{A}}(x)dx$

Proposition 2.10. *If $\tilde{A} = (a,\beta,\gamma)_n$ described by (5) is a Fuzzy number with equally heavy tails, then*

$$m_{\tilde{A}} = a \tag{10}$$

and $\mu_{\tilde{A}}(m_{\tilde{A}}) = 1$

Proof. Because $\int_{a-\beta}^{a} \mu_{\tilde{A}}(x)dx = \int_{a}^{a+\gamma} \mu_{\tilde{A}}(x)dx$, the median value that is $m_{\tilde{A}}$ must be in the core of $\tilde{A}$. Then

$$\int_{a-\beta}^{m_{\tilde{A}}} \mu_{\tilde{A}}(x)dx + \int_{m_{\tilde{A}}}^{a} \mu_{\tilde{A}}(x)dx = \int_{a}^{m_{\tilde{A}}} \mu_{\tilde{A}}(x)dx + \int_{m_{\tilde{A}}}^{a+\gamma} \mu_{\tilde{A}}(x)dx$$

$$\int_{a-\beta}^{m_{\tilde{A}}} \mu_{\tilde{A}}(x)dx + (a - m_{\tilde{A}}) = (m_{\tilde{A}} - a) + \int_{m_{\tilde{A}}}^{a+\gamma} \mu_{\tilde{A}}(x)dx$$

$$m_{\tilde{A}} = a$$

Because a is the core of $\tilde{A}$, then $\mu_{\tilde{A}}(m_{\tilde{A}}) = 1$ □

Proposition 2.11. *If $\tilde{A} = (a, \beta, \gamma)_n$ described by (5) is a Fuzzy number, then*

$$m_{\tilde{A}} = (a - \beta) + \left(\frac{1}{2}(\gamma + \beta)(\beta)^n\right)^{\frac{1}{n+1}} \tag{11}$$

if $\tilde{A}$ has a heavy left tail, and

$$m_{\tilde{A}} = (a + \gamma) + \left(\frac{1}{2}(\gamma + \beta)(\gamma)^n\right)^{\frac{1}{n+1}} \tag{12}$$

if $\tilde{A}$ has a heavy right tail.

Proposition 2.12. *Let $\tilde{A} = (a, \beta, \gamma)_n$ described by (5) is a Fuzzy number with heavy tail, then*

$$\left(\frac{1}{2}\right)^{\frac{n}{n+1}} < \mu_{\tilde{A}}(m_{\tilde{A}}) = \left(\frac{\gamma + \beta}{s}\right)^{\frac{n}{n+1}} < 1 \tag{13}$$

where $s = \beta$ if $\tilde{A}$ has a heavy left tail, and $s = \gamma$ if $\tilde{A}$ has a heavy right tail.

Definition 2.13. [33] *The center of gravity $g_{\tilde{A}}$ of the support of a Fuzzy number $\tilde{A}$ weighted by the membership grade is given by*

$$g_{\tilde{A}} = \frac{\int_{a-\beta}^{a+\gamma} x\mu_{\tilde{A}}(x)dx}{\int_{a-\beta}^{a+\gamma} \mu_{\tilde{A}}(x)dx} \tag{14}$$

and the center of the core of $\tilde{A}$ denoted with $MO_{\tilde{A}}$ and given by

$$MO_{\tilde{A}} = a \tag{15}$$

Definition 2.14. [33] *Let $\tilde{A}$ be a Fuzzy number. Let $g_{\tilde{A}}, MO_{\tilde{A}}$, and $m_{\tilde{A}}$ be the center of gravity, the center of core, and the median value of $\tilde{A}$, respectively. Then the central value of $\tilde{A}$ is given by*

$$c_{\tilde{A}} = \frac{g_{\tilde{A}}\mu_{\tilde{A}}(g_{\tilde{A}}) + MO_{\tilde{A}}\mu_{\tilde{A}}(MO_{\tilde{A}}) + m_{\tilde{A}}\mu_{\tilde{A}}(m_{\tilde{A}})}{\mu_{\tilde{A}}(g_{\tilde{A}}) + \mu_{\tilde{A}}(MO_{\tilde{A}}) + \mu_{\tilde{A}}(m_{\tilde{A}})} \tag{16}$$

We can use the central value as a crisp approximation of a Fuzzy number.

3. BINO-TRINOMIAL TREE FUZZY OPTION PRICING MODEL

Option pricing can be done using analytical and numerical approaches. However, not all existing options in the financial market can be priced analytically. One of the numerical methods that can be used to determine option prices is the Lattice model. The commonly used Lattice models are the Binomial and Trinomial tree models [34], who have their advantages and disadvantages. The Binomial tree model requires calculations that are not too heavy but require a reasonably large time interval to achieve a convergence to the result of the Black-Scholes model. The Trinomial tree model is more flexible with real situations, where different possible scenarios of stock price movements could converge more quickly to the result of the Black-Scholes model than the previous one, but it requires heavier calculations.

In general, the Binomial and Trinomial tree models assume that the volatility parameters are known and have fixed values. In these lattice models, the volatility parameter is closely related to the increase (u) and decrease (d) factors of stock price movements. In reality, the financial market prices fluctuate from time to time, so the value of the volatility parameter cannot always be determined precisely and realistically. This volatility parameter contains uncertainty conditions. One way to describe uncertainty conditions in volatility parameters is by representing volatility parameters in the form of Fuzzy numbers. The Fuzzy numbers used in this research are in the form of triangular Fuzzy numbers.

For pricing European options, we propose to fuzzify the combining Binomial and Trinomial tree models. For developing these models, note that there are assumptions needed to be satisfied: (1) the price of the underlying asset follows a log-normal distribution, (2) there are no transaction costs, no taxes, no restriction of sales short, no-arbitrage opportunity in the market, and the asset can be divided infinitely, (3) the underlying asset does not pay dividends over the life of the derivative, and (4) for all maturities, the riskless interest rate is fixed [2],[3]. In the following subsections, we explain the fuzzification of Binomial, Trinomial, and Bino-Trinomial tree models for pricing European options, and the implementation of these models will be compared.

3.1. Fuzzy Binomial Tree Model.

This section discusses the construction of a Binomial tree model to determine European call option prices using the Fuzzy theory developed by Lee et al. [35] and Yu et al. [36]. Using stock's price movements of up and down in each period, one of the well-known Binomial tree models is the CRR (Cox-Ross-Rubinstein) [3]. The fuzzification process is made on the volatility parameter and risk-free interest rates, as in [35] and [36], where they are represented in the form of triangular. The results of Fuzzy stock price inference and Fuzzy call option prices are also discussed.

The Fuzzy volatility parameters are expressed as follows:

$$\tilde{\sigma} = ((1-\rho_1)\sigma,\ \sigma,\ (1+\rho_2)\sigma),\ 0 \leq \rho_1, \rho_2 \leq 1. \tag{17}$$

so $\sigma_L = (1-\rho_1)\sigma, \sigma_M = \sigma$ and $\sigma_H = (1+\rho_2)\sigma$ represent the low, moderate, and high volatilities respectively. Here ρ_1, rho_2 is defined as the volatility sensitivity index. The value of ρ_1 and ρ_2 can be equal or different.

In the CRR Binomial tree model for stock prices [3], the increase factor is determined by $u_b = e^{\sigma\sqrt{\Delta t}}$, and the decrease factor is determined by $d_b = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u}$. The Fuzzy increase factor in the form of a triangular Fuzzy number is $\tilde{u} = (u_{L_b}, u_{M_b}, u_{H_b})$ with the membership degrees $\mu_(u_{H_b}) = \mu_(u_{L_b}) = 0.1$ and $\mu_(u_{M_b}) = 1$. Likewise, the Fuzzy decrease factor in the form of a triangular Fuzzy number is $\tilde{d} = (d_{L_b}, d_{M_b}, d_{H_b})$ with the membership degrees $\mu_(d_{H_b}) = \mu_(d_{L_b}) = 0.1$ and $\mu_(d_{M_b}) = 1$ ([36, 35]). The increase factors u_{L_b}, u_{M_b}, and u_{H_b} are related to prices with low, moderate, and high volatility conditions respectively. On the other hand, the decrease factors d_{L_b}, d_{M_b}, and d_{H_b} are related to high, moderate, and low volatility conditions respectively. Those factors are defined as follows:

$$u_{H_b} = e^{(1+\rho_2)\sigma\sqrt{dt}},\ u_{M_b} = e^{\sigma\sqrt{dt}},\ u_{L_b} = e^{(1-\rho_1)\sigma\sqrt{dt}}, \tag{18}$$

$$d_{L_b} = e^{-(1+\rho_2)\sigma\sqrt{dt}},\ d_{M_b} = e^{-\sigma\sqrt{dt}},\ d_{H_b} = e^{-(1-\rho_1)\sigma\sqrt{dt}}. \tag{19}$$

Let S be the initial stock price at $t = 1$. The stock price can move up or down, and is impacted by three volatility conditions at time $t = 2$. Therefore, there are three possible prices for the upward movement, namely Su_{H_b}, Su_{M_b}, and Su_{L_b}, and there are three possible prices for the downward movement, namely Sd_{H_b}, Sd_{M_b}, and Sd_{L_b}. Thus, the stock price S becomes six possible stock prices, which is also happening at each price movement for one interval period, as follows. The stock price moves:

(1) Up in the condition of the highest volatility : $S_{u_{H_b}} = u_{H_b}S$,
(2) Up in the condition of moderate volatility : $S_{u_{M_b}} = u_{M_b}S$,
(3) Up in the condition of the lowest volatility : $S_{u_{L_b}} = u_{L_b}S$,
(4) Down at the condition of the lowest volatility : $S_{d_{H_b}} = d_{H_b}S$,
(5) Down in the condition of moderate volatility : $S_{d_{M_b}} = d_{M_b}S$,
(6) Down in the condition of the highest volatility : $S_{d_{L_b}} = d_{L_b}S$,

At time $t = 3$, each stock price at time $t = 2$ moves with six possible stock prices corresponding to the volatility conditions, so that there are 36 possible stock prices. An illustration of stock price movements from $t = 1$ to $t = 2$, and from $t = 2$ to $t = 3$ in the Fuzzy Binomial tree model is presented in Figure 1. We also show the membership degree of each obtained stock price.

Using various combinations of $u_{H_b}, u_{M_b}, u_{L_b}, d_{H_b}, d_{M_b}$, and d_{L_b}, a list of all possible stock prices at time $t = n$ can be created, using the following formulation [36]:

$$u_{H_b}^a \times u_{M_b}^b \times u_{L_b}^c \times d_{H_b}^x \times d_{M_b}^y \times d_{L_b}^z \times S, \tag{20}$$

where a, b, c, x, y, and z are integers and can be any combination provided with $a+b+c+x+y+z = n-1$, and n is the total periods of time. Thus, suppose there are n periods, then the number of stock prices at $t = n$ is 6^{n-1}. The formula of membership degree for each stock price in Equation (20) is defined as follows:

$$\mu_S = \mu^a_{u_{H_b}} \times \mu^b_{u_{M_b}} \times \mu^c_{u_{L_b}} \times \mu^x_{d_{H_b}} \times \mu^y_{d_{M_b}} \times \mu^z_{d_{L_b}}. \tag{21}$$

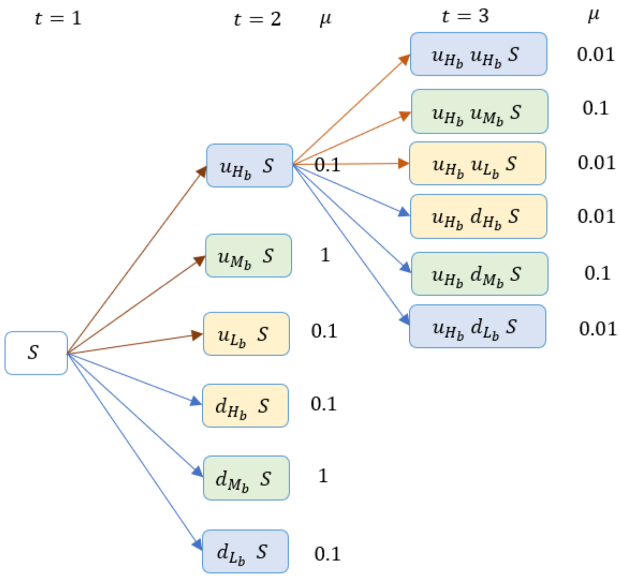

FIGURE 1. Stock price movement two periods under the Fuzzy Binomial tree

Proposition 3.1. [22] *Let (C_L, C_M, C_H) be a triangular Fuzzy number for the current European call option price in the one-period Binomial Fuzzy tree model. C_L, C_M, and C_H are the European call option prices respectively at the highest, moderate, and lowest volatility conditions, which are determined with the following formulas:*

$$C_H = e^{-r\Delta t}\left[p_{uH} C_{u_{H_b}} + p_{dH} C_{d_{L_b}}\right], \tag{22}$$

$$C_M = e^{-r\Delta t}\left[p_{uM} C_{u_{M_b}} + p_{dM} C_{d_{M_b}}\right], \tag{23}$$

$$C_L = e^{-r\Delta t}\left[p_{uL} C_{u_{L_b}} + p_{dL} C_{d_{H_b}}\right]. \tag{24}$$

Variables p_{uH} and p_{dH} are the probability for the stock price to move up and down, respectively, in the high volatility condition, which are defined as follows:

$$p_{uH} = \frac{e^{r\Delta t} - d_{L_b}}{e^{r\Delta t}(u_{H_b} - d_{L_b})}, \tag{25}$$

$$p_{dH} = \frac{u_{H_b} - e^{r\Delta t}}{e^{r\Delta t}(u_{H_b} - d_{L_b})}. \tag{26}$$

Variables p_{uM} and p_{dM} are the probability for the stock price to move up and down, respectively, in the moderate volatility condition, which are defined as follows:

$$p_{uM} = \frac{e^{r\Delta t} - d_{M_b}}{e^{r\Delta t}(u_{M_b} - d_{M_b})}, \tag{27}$$

$$p_{dM} = \frac{u_{M_b} - e^{r\Delta t}}{e^{r\Delta t}(u_{M_b} - d_{M_b})}. \tag{28}$$

Variables p_{uL} and p_{dL} are the probability for the stock price to move up and down, respectively, in the low volatility condition, which are defined as follows:

$$p_{uL} = \frac{e^{r\Delta t} - d_{H_b}}{e^{r\Delta t}(u_{L_b} - d_{H_b})}, \tag{29}$$

$$p_{dL} = \frac{u_{L_b} - e^{r\Delta t}}{e^{r\Delta t}(u_{L_b} - d_{H_b})}. \tag{30}$$

Proof. The process of inferring European call option prices in the Fuzzy Binomial tree model starts from a one-period Fuzzy Binomial tree. It is generalized into an n-period Binomial Fuzzy tree. In a one-period Fuzzy Binomial tree, the payoffs of a European call option with Strike Price K at expiration time $t = 2$ are the following.

$$C_{uH_b} = max(u_{H_b}S - K, 0), C_{dH_b} = max(d_{H_b}S - K, 0),$$
$$C_{uM_b} = max(u_{M_b}S - K, 0), C_{dM_b} = max(d_{M_b}S - K, 0),$$
$$C_{uL_b} = max(u_{L_b}S - K, 0), C_{dL_b} = max(d_{L_b}S - K, 0).$$

The price determination of European call options can be obtained by the strategy of combining stocks and riskless assets, for instant bonds. We show an illustration of the high volatility condition. The other conditions will follow similarly.

$$C_H = \Delta_H S + B_H, \tag{31}$$

where Δ_H represents the number of stocks purchased in the high volatility condition, and B_H represents the amount of funds invested in bonds in the high volatility condition. In this condition, initially, an investor buys Δ_H stocks with price S and invests B_H bonds with an interest rate r. After one period, there are two possible situations: (1) The stock price increases to Su_{H_b}, so the portfolio value becomes $\Delta_H Su_{H_b} + e^{r\Delta_t}B_H$ and (2) The stock price decreases to Sd_{H_b}, so the portfolio value becomes $\Delta_H Sd_{L_b} + e^{r\Delta_t}B_H$. Based on the Option Delta Method [37], the number of stocks Δ_H can replicate the value of the corresponding option both in upward or downward stock prices, defined as follows.

$$C_{u_{H_b}} = \Delta_H Su_{H_b} + e^{r\Delta t}B_H, \tag{32}$$

$$C_{d_{L_b}} = \Delta_H Sd_{L_b} + e^{r\Delta t}B_H. \tag{33}$$

We can find the number of stocks Δ_H and the bond's value B_H by solving the system of equations $(32 - 33)$.

$$\Delta_H = \frac{C_{u_{H_b}} - C_{d_{L_b}}}{S(u_{H_b} - d_{L_b})}, \tag{34}$$

$$B_H = \frac{u_{H_b} C_{d_{L_b}} - d_{L_b} C_{u_{H_b}}}{e^{r\Delta t}(u_{H_b} - d_{L_b})}. \quad (35)$$

By substituting equation (34) and equation (35) into equation (31), we obtain the option price with high volatility condition.

$$\begin{aligned} C_H &= \frac{e^{r\Delta t} - d_{L_b}}{e^{r\Delta t}(u_{H_b} - d_{L_b})} C_{u_{H_b}} + \frac{u_{H_b} - e^{r\Delta t}}{e^{r\Delta t}(u_{H_b} - d_{L_b})} C_{d_{L_b}}, \\ &= e^{-r\Delta t}\left[p_{uH} C_{u_{H_b}} + p_{dH} C_{d_{L_b}}\right], \end{aligned}$$

where p_{uH} and p_{dH} are in Equations (25) and (26).

By doing a similar procedure, we obtain the European Call Option prices for the moderate and low volatility conditions as follows.

$$C_M = \frac{e^{r\Delta t} - d_{M_b}}{e^{r\Delta t}(u_{M_b} - d_{M_b})} C_{u_{M_b}} + \frac{u_{M_b} - e^{r\Delta t}}{e^{r\Delta t}(u_{M_b} - d_{M_b})} C_{d_{M_b}},$$

$$C_M = e^{-r\Delta t}\left[p_{uM} C_{u_{M_b}} + p_{dM} C_{d_{M_b}}\right],$$

where p_{uM} and p_{dM} are in (27) and (28).

$$C_L = \frac{e^{r\Delta t} - d_{H_b}}{e^{r\Delta t}(u_{L_b} - d_{H_b})} C_{u_{L_b}} + \frac{u_{L_b} - e^{r\Delta t}}{e^{r\Delta t}(u_{L_b} - d_{H_b})} C_{d_{H_b}}$$

$$C_L = e^{-r\Delta t}\left[p_{uL} C_{u_{L_b}} + p_{dL} C_{d_{H_b}}\right]$$

where p_{uL} and p_{dL} are in (29) and (30). □

Now we construct the general formulas of the option prices as in [28]. Let $h(n)$ represent the combination of $u_{H_b}, u_{M_b}, u_{L_b}, d_{H_b}, d_{M_b}$, and d_{L_b} at time $t = n$. $S_{h(n)}$ represents the stock price at time $t = n$. For example for $n = 2$, when the stock price move up in the high volatility condition we will have six possible movement that is:

$$\begin{aligned} h(2) &= u_{H_b}^2 \times u_{M_b}^b \times u_{L_b}^c \times d_{H_b}^x \times d_{M_b}^y \times d_{L_b}^z = u_{H_b} \times h(1) \\ h(2) &= u_{H_b}^1 \times u_{M_b}^1 \times u_{L_b}^c \times d_{H_b}^x \times d_{M_b}^y \times d_{L_b}^z = u_{M_b} \times h(1) \\ h(2) &= u_{H_b}^1 \times u_{M_b}^b \times u_{L_b}^1 \times d_{H_b}^x \times d_{M_b}^y \times d_{L_b}^z = u_{L_b} \times h(1) \\ h(2) &= u_{H_b}^1 \times u_{M_b}^b \times u_{L_b}^c \times d_{H_b}^1 \times d_{M_b}^y \times d_{L_b}^z = d_{H_b} \times h(1) \\ h(2) &= u_{H_b}^1 \times u_{M_b}^b \times u_{L_b}^c \times d_{H_b}^x \times d_{M_b}^1 \times d_{L_b}^z = d_{M_b} \times h(1) \\ h(2) &= u_{H_b}^1 \times u_{M_b}^b \times u_{L_b}^c \times d_{H_b}^x \times d_{M_b}^y \times d_{L_b}^1 = d_{L_b} \times h(1) \end{aligned}$$

By doing similar procedure, we will obtain $h(2)$ for the stock move up in the moderate and low volatility conditions and also for the stock move down in the high, moderate, and low volatility conditions.

Then we obtain the stock price at time $t = 2$ when the stock price move up in the high volatility condition, that is:

$$S_{h(2)} = u_{H_b}^2 \times u_{M_b}^b \times u_{L_b}^c \times d_{H_b}^x \times d_{M_b}^y \times d_{L_b}^z \times S = u_{H_b} \times S \times h(1) = u_{H_b} \times S_{h(1)}$$
$$S_{h(2)} = u_{H_b}^1 \times u_{M_b}^1 \times u_{L_b}^c \times d_{H_b}^x \times d_{M_b}^y \times d_{L_b}^z \times S = u_{M_b} \times S \times h(1) = u_{M_b} \times S_{h(1)}$$
$$S_{h(2)} = u_{H_b}^1 \times u_{M_b}^b \times u_{L_b}^1 \times d_{H_b}^x \times d_{M_b}^y \times d_{L_b}^z \times S = u_{L_b} \times S \times h(1) = u_{L_b} \times S_{h(1)}$$
$$S_{h(2)} = u_{H_b}^1 \times u_{M_b}^b \times u_{L_b}^c \times d_{H_b}^1 \times d_{M_b}^y \times d_{L_b}^z \times S = d_{H_b} \times S \times h(1) = d_{H_b} \times S_{h(1)}$$
$$S_{h(2)} = u_{H_b}^1 \times u_{M_b}^b \times u_{L_b}^c \times d_{H_b}^x \times d_{M_b}^1 \times d_{L_b}^z \times S = d_{M_b} \times S \times h(1) = d_{M_b} \times S_{h(1)}$$
$$S_{h(2)} = u_{H_b}^1 \times u_{M_b}^b \times u_{L_b}^c \times d_{H_b}^x \times d_{M_b}^y \times d_{L_b}^1 \times S = d_{L_b} \times S \times h(1) = d_{L_b} \times S_{h(1)}$$

By doing similar procedure, we will obtain $S_{h(2)}$ for the stock price at time $t = 2$ when the stock move up in the moderate and low volatility conditions and also when the stock move down in the high, moderate, and low vollatility conditions.

$C_{h(n)}$ represents the payoff or option value with Strike Price K at expiration time $t = n$, which is determined by $C_{h(n)} = max(S_{h(n)} - K, 0)$. Its degree of membership is the same as of $S_{h(n)}$, or $\mu_{C_{h(n)}} = \mu_{S_{h(n)}}$. The payoffs of option at expiration time $t = n$ can be determined in detail as follows :

$$\begin{aligned}
C_{u_H h(n-1)} &= max(u_{H_b} S_{h(n-1)} - K),\\
C_{u_M h(n-1)} &= max(u_{M_b} S_{h(n-1)} - K),\\
C_{u_L h(n-1)} &= max(u_{L_b} S_{h(n-1)} - K),\\
C_{d_H h(n-1)} &= max(d_{H_b} S_{h(n-1)} - K),\\
C_{d_M h(n-1)} &= max(d_{M_b} S_{h(n-1)} - K),\\
C_{d_L h(n-1)} &= max(d_{L_b} S_{h(n-1)} - K).
\end{aligned}$$

We do backward iteration from $t = n$ to $t = 1$ for obtaining the Fuzzy option price. When $t = n - 1$, each option value is a Fuzzy number in the form of $(C_{Lh(n-1)}, C_{Mh(n-1)}, C_{Hh(n-1)})$, which is determined by the following formulas:

$$C_{Hh(n-1)} = e^{-r\Delta t}\left[p_{uH} C_{u_{H_b} h(n-1)} + p_{dH} C_{d_{L_b} h(n-1)}\right], \tag{36}$$

$$C_{Mh(n-1)} = e^{-r\Delta t}\left[p_{uM} C_{u_{M_b} h(n-1)} + p_{dM} C_{d_{M_b} h(n-1)}\right], \tag{37}$$

$$C_{Lh(n-1)} = e^{-r\Delta t}\left[p_{uL} C_{u_{L_b} h(n-1)} + p_{dL} C_{d_{H_b} h(n-1)}\right]. \tag{38}$$

The backward iteration is continued for $t = n - 2$ until $t = 1$ using the similar process as in equations (36), (37), and (38). Finally we obtain the Fuzzy European Call price (C_L, C_M, C_H).

3.2. **Fuzzy Trinomial Tree Model.**

The Trinomial model is a stock price dynamics model with three possible scenarios; upward, constant, or downward stock price movements in each period. Joheski and Apostolov [34] studied the Binomial and Trinomial tree models for determining option prices along with their convergence to the Black-Scholes model.

One of the results states that the most efficient Lattice model for determining option prices is the Kamrad-Ritchken (KR) Trinomial tree model. In this section, we construct a KR Trinomial tree model to determine the price of European call options by considering uncertainty conditions in the volatility parameters, which are expressed in the form of triangular Fuzzy numbers as in equation (17).

As in [5], the increase and decrease factors in the KR trinomial tree model are similar to the CRR Binomial tree model as in Equations (18)-(19), but there is a modification by adding a new parameter λ. So the increase and decrease factors are respectively

$$u_t = e^{\lambda\sigma\sqrt{\Delta t}}, \text{ and } d_t = e^{-\lambda\sigma\sqrt{\Delta t}} = \frac{1}{u_t}. \tag{39}$$

The chosen value of λ is $\sqrt{\frac{3}{2}} \approx 1.2247$ based on a heuristic reason. The volatility parameter's values are expressed in the form of a triangular Fuzzy number corresponding to high, moderate, and low conditions, so consequently, the increase and decrease factors are respectively in the form of $\tilde{u}_t = (u_{L_t}, u_{M_t}, u_{H_t})$ and $\tilde{d}_t = (d_{L_t}, d_{M_t}, d_{H_t})$, with their membership degrees are $\mu_{u_{H_t}} = \mu_{u_{L_t}} = 0.1$ and $\mu_{u_{M_t}} = 1$, and $\mu_{d_{H_t}} = \mu_{d_{L_t}} = 0.1$ and $\mu_{d_{M_t}} = 1$. The Fuzzy increase and decrease factors are defined as follows,

$$u_{H_t} = e^{(1+\rho_2)\lambda\sigma\sqrt{dt}}; u_{M_t} = e^{\lambda\sigma\sqrt{dt}}; u_{L_t} = e^{(1-\rho_1)\lambda\sigma\sqrt{dt}} \tag{40}$$

$$d_{L_t} = e^{-(1+\rho_2)\lambda\sigma\sqrt{dt}}, d_{M_t} = e^{-\lambda\sigma\sqrt{dt}}, d_{H_t} = e^{-(1-rho_1)\lambda\sigma\sqrt{dt}}. \tag{41}$$

Besides increasing and decreasing stock price movement in the Trinomial tree model, the stock price can remain constant, which is defined as S_m, and its membership degree is $\mu_m = 1$.

Let S be the initial stock price at $t = 1$. At time $t = 2$, the KR Trinomial Fuzzy tree model has seven possible stock price movement scenarios. The stock price will move:

(1) Up in conditions of the highest volatility: $S_{u_{H_t}} = u_{H_t}S$,
(2) Up in conditions of moderate volatility : $S_{u_{M_t}} = u_{M_t}S$,
(3) Up in conditions of the lowest volatility : $S_{u_{L_t}} = u_{L_t}S$,
(4) Constant, the same as before: $S_m = S$,
(5) Down at conditions of the lowest volatility is determined by $S_{d_{H_t}} = d_{H_t}S$,
(6) Down in conditions of moderate volatility : $S_{d_{M_t}} = d_{M_t}S$,
(7) Down in conditions of the highest volatility : $S_{d_{L_t}} = d_{L_t}S$.

At time $t = 3$, each stock price at time $t = 2$ also moves with seven possible stock price movements. Figure 2 illustrates the structure of stock price movements in the Fuzzy Trinomial tree model for two periods, with their defined degree of memberships. From $t = 1$ to $t = 2$, there are seven possible stock price movements, and from $t = 2$ to $t = 3$, there are 49 possible stock price movements.

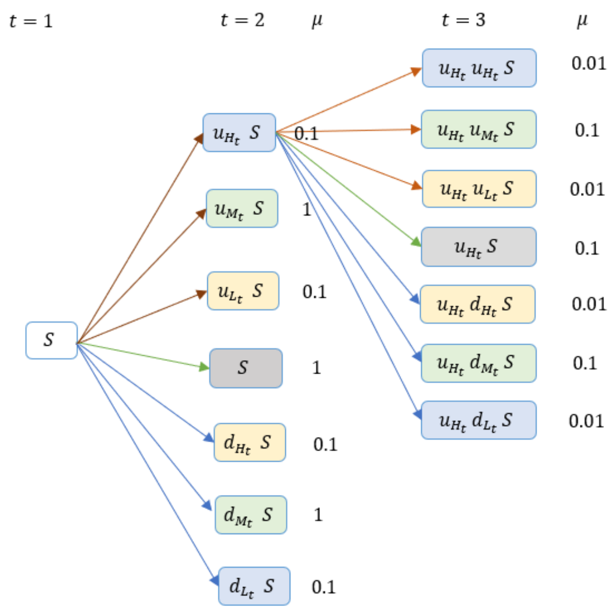

FIGURE 2. Stock price movement two periods under the Fuzzy trinomial tree

By using various combinations of $u_{H_t}, u_{M_t}, u_{L_t}, m, d_{H_t}, d_{M_t}$, and d_{L_t}, a list of all possible stock prices at time $t = n$ can be created, by using the following formula:

$$u_{H_t}^a \times u_{M_t}^b \times u_{L_t}^c \times m^w \times d_{H_t}^x \times d_{M_t}^y \times d_{L_t}^z \times S, \tag{42}$$

where a, b, c, w, x, y, and z are integers and can be any combination provided with $a+b+c+w+x+y+z = n-1$, and n is the total period time. If there is n periods, then the number of stock prices at $t = n$ is 7^{n-1}. The degree of membership with the stock price formulation in equation (42) is the following.

$$\mu_S = \mu_{u_{H_t}}^a \times \mu_{u_{M_t}}^b \times \mu_{u_{L_t}}^c \times \mu_m^w \times \mu_{d_{H_t}}^x \times \mu_{d_{M_t}}^y \times \mu_{d_{L_t}}^z. \tag{43}$$

Proposition 3.2. *Let (C_L, C_M, C_H) be a triangular Fuzzy number for the European call option price in the one-period Trinomial Fuzzy tree model. C_L, C_M, and C_H are respectively the European call option prices at the highest, moderate, and lowest volatility conditions, which are determined with the following formulas.*

$$C_H = e^{-r\Delta t}\left[p_{uH} C_{u_{H_t}} + p_{mH} C_m + p_{d_H} C_{d_{L_t}}\right], \tag{44}$$

$$C_M = e^{-r\Delta t}\left[p_{uM} C_{u_{M_t}} + p_{mM} C_m + p_{d_M} C_{d_{M_t}}\right], \tag{45}$$

$$C_L = e^{-r\Delta t}\left[p_{uL} C_{u_{L_t}} + p_{mL} C_m + p_{d_L} C_{d_{H_t}}\right]. \tag{46}$$

The probabilities for stock prices to increase, remain constant, and decrease in the high volatility condition are respectively p_{uH},p_{mH}, *and* p_{dH},*which are determined as follows.*

$$p_{uH} = \frac{e^{r\Delta t}(2u_{H_t} - d_{L_t} - 1) + (d_{L_t}^2 - u_{H_t})}{2(d_{L_t}^2 - d_{L_t} + u_{H_t}^2 - u_{H_t})}, \tag{47}$$

$$p_{mH} = \frac{e^{r\Delta t}(2 - u_{H_t} - d_{L_t}) + (d_{L_t}^2 + u_{H_t}^2 - d_{L_t} - u_{H_t})}{2(d_{L_t}^2 - d_{L_t} + u_{H_t}^2 - u_{H_t})}, \tag{48}$$

$$p_{dH} = \frac{e^{r\Delta t}(2d_{L_t} - u_{H_t} - 1) + (u_{H_t}^2 - d_{L_t})}{2(d_{L_t}^2 - d_{L_t} + u_{H_t}^2 - u_{H_t})}. \tag{49}$$

The probabilities for stock prices to increase, remain constant, and decrease in the moderate volatility condition are respectively p_{uM}, p_{mM}, *and* p_{dM}, *which are defined as the following.*

$$p_{uM} = \frac{e^{r\Delta t}(2u_{M_t} - d_{M_t} - 1) + (d_{M_t}^2 - u_{M_t})}{2(d_{M_t}^2 - d_{M_t} + u_{M_t}^2 - u_{M_t})}, \tag{50}$$

$$p_{mM} = \frac{e^{r\Delta t}(2 - u_{M_t} - d_{M_t}) + (d_{M_t}^2 + u_{M_t}^2 - d_{M_t} - u_{M_t})}{2(d_{M_t}^2 - d_{M_t} + u_{M_t}^2 - u_{M_t})}, \tag{51}$$

$$p_{dM} = \frac{e^{r\Delta t}(2d_{M_t} - u_{M_t} - 1) + (u_{M_t}^2 - d_{M_t})}{2(d_{M_t}^2 - d_{M_t} + u_{M_t}^2 - u_{M_t})}. \tag{52}$$

Finally, the probabilities for stock prices to increase, remain constant, and decrease in the low volatility condition are respectively p_{uL}, p_{mL}, *and* p_{dL}, *which are determined by the following formulas.*

$$p_{uL} = \frac{e^{r\Delta t}(2u_{L_t} - d_{H_t} - 1) + (d_{H_t}^2 - u_{L_t})}{2(d_{H_t}^2 - d_{H_t} + u_{L_t}^2 - u_{L_t})} \tag{53}$$

$$p_{mL} = \frac{e^{r\Delta t}(2 - u_{L_t} - d_{H_t}) + (d_{H_t}^2 + u_{L_t}^2 - d_{H_t} - u_{L_t})}{2(d_{H_t}^2 - d_{H_t} + u_{L_t}^2 - u_{L_t})} \tag{54}$$

$$p_{dL} = \frac{e^{r\Delta t}(2d_{H_t} - u_{L_t} - 1) + (u_{L_t}^2 - d_{H_t})}{2(d_{H_t}^2 - d_{H_t} + u_{L_t}^2 - u_{L_t})} \tag{55}$$

Proof. Similar to the process in the previous subsection, it starts with a one-period KR Trinomial Fuzzy tree, then it is generalized into an n-period KR Trinomial Fuzzy tree. In a one-period, the prices of a European call option at expiration $t = 2$ are

$$\begin{aligned}
C_{u_{H_t}} &= max(u_{H_t}S - K, 0),\ C_{d_{H_t}} = max(d_{H_t}S - K, 0),\\
C_{u_{M_t}} &= max(u_{M_t}S - K, 0),\ C_{d_{M_t}} = max(d_{M_t}S - K, 0),\\
C_{u_{L_t}} &= max(u_{L_t}S - K, 0),\ C_{d_{L_t}} = max(d_{L_t}S - K, 0),\\
C_m &= max(S - K, 0).
\end{aligned}$$

Similar to the procedure in the previous subsection, we define the strategy of combining stocks and bonds in in equation (31). An illustration for the high volatility condition is shown below. After one period, there are three possible values of portfolio based on the movement of the stock price:

(1) $\Delta_H S u_{H_t} + e^{r\Delta t} B_H$ (Upward price),
(2) $\Delta_H S + e^{r\Delta t} B_H$ (Constant price),
(3) $\Delta_H S d_{L_t} + e^{r\Delta t} B_H$ (Downward price).

Using the Option Delta Method, we find the number of stocks Δ_H that can replicate the value of the corresponding options. We have

$$\begin{cases} C_{u_{H_t}} = \Delta_H S u_{H_t} + e^{r\Delta t} B_H, \\ C_{u_{m_t}} = \Delta_H S + e^{r\Delta t} B_H, \\ C_{d_{L_t}} = \Delta_H S d_{L_t} + e^{r\Delta t} B_H \end{cases} \tag{56}$$

The portfolio replication presented in equation (56) is an over-determined system of linear equations consisting of three equations and two variables, so the solution is not straightforward. Based on [38], under this condition, equation (56) is an incomplete description of the market model. Solving equations (56) can be done using the pseudoinverse matrix concept [39]. Using the concept of pseudoinverse matrix, we have:

$$\Delta_H = \frac{C_{u_{H_t}}(2u_{H_t} - d_{L_t} - 1) + C_m(2 - u_{H_t} - d_{L_t}) + C_{d_{L_t}}(2d_{L_t} - u_{H_t} - 1)}{2S(d_{L_t}^2 - d_{L_t} + u_{H_t}^2 - u_{H_t})}, \tag{57}$$

$$B_H = \frac{C_{u_{H_t}}(d_{L_t}^2 - u_{H_t}) + C_m(d_{L_t}^2 + u_{H_t}^2 - d_{L_t} - u_{H_t}) + C_{d_{L_t}}(u_{H_t}^2 - d_{L_t})}{2e^{r\Delta t}(d_{L_t}^2 - d_{L_t} + u_{H_t}^2 - u_{H_t})} \tag{58}$$

By substituting equations (57) and (58) into equation (31), we can obtain:

$$C_H = e^{-r\Delta t}\left[p_{uH} C_{u_{H_t}} + p_{mH} C_m + p_{d_H} C_{d_{L_t}}\right]$$

where the probabilities for stock prices to increase, remain constant, and decrease in the high volatility condition are respectively written in equations (47)-(49).

Similarly, we obtained the option prices for the moderate volatility condition, as follows.

$$C_M = e^{-r\Delta t}\left[p_{uM} C_{u_{M_t}} + p_{mM} C_m + p_{d_M} C_{d_{M_t}}\right],$$

where the corresponding probabilities are respectively written in equations (50)-(52). For the low volatility condition, we obtain:

$$C_L = e^{-r\Delta t}\left[p_{uL} C_{u_{L_t}} + p_{mL} C_m + p_{d_L} C_{d_{H_t}}\right],$$

where the corresponding probabilities are respectively written in equations (53)-(55). □

Now we generalize the formulation. At $t = n$, let $h(n)$ represent the combination of $u_{H_t}, u_{M_t}, u_{L_t}, m, d_{H_t}, d_{M_t}$, and d_{L_t}. $S_{h(n)}$ and $C_{h(n)}$ represent the stock prices and the option value, respectively, at time when $t = n$. The latter is determined by $C_{h(n)} = max(S_{h(n)} - K, 0)$ and its degree of membership is

$\mu_{C_{h(n)}} = \mu_{S_{h(n)}}$. In detail, the option values are as follows:

$$\begin{aligned} C_{u_H h(n-1)} &= max(u_{H_t} S_{h(n-1)} - K), \\ C_{u_M h(n-1)} &= max(u_{M_t} S_{h(n-1)} - K), \\ C_{u_L h(n-1)} &= max(u_{L_t} S_{h(n-1)} - K), \\ C_{mh(n-1)} &= max(m S_{h(n-1)} - K), \\ C_{d_H h(n-1)} &= max(d_{H_t} S_{h(n-1)} - K), \\ C_{d_M h(n-1)} &= max(d_{M_t} S_{h(n-1)} - K), \\ C_{d_L h(n-1)} &= max(d_{L_t} S_{h(n-1)} - K), \end{aligned}$$

By doing backward iteration at $t = n - 1$, each option price is a Fuzzy number $(C_{Lh(n-1)}, C_{Mh(n-1)}, C_{Hh(n-1)})$ which is determined by the following formulas:

$$C_{Hh(n-1)} = e^{-r\Delta t} \left[p_{uH} C_{u_{H_t} h(n-1)} + p_{mH} C_{mh(n-1)} + p_{dH} C_{d_{L_t} h(n-1)} \right], \quad (59)$$

$$C_{Mh(n-1)} = e^{-r\Delta t} \left[p_{uM} C_{u_{M_t} h(n-1)} + p_{mM} C_{mh(n-1)} + p_{dM} C_{d_{M_t} h(n-1)} \right], \quad (60)$$

$$C_{Lh(n-1)} = e^{-r\Delta t} \left[p_{uH} C_{u_{L_t} h(n-1)} + p_{mL} C_{mh(n-1)} + p_{dL} C_{d_{H_t} h(n-1)} \right]. \quad (61)$$

Having continued the iteration, the values of option at $t = n - 1$ are determined using equations (59)-(61), and so on. Finally, at $t = 1$ we will obtain the Fuzzy European Call option price (C_L, C_M, C_H).

3.3. **Fuzzy Bino-Trinomial Tree Model.**

A combination of the Binomial and trinomial tree models is the Bino-Trinomial tree model. Day and Lyuu [12] developed this model to overcome distribution and nonlinearity errors in determining barrier option prices. We improve this model with the Fuzzy approach to obtain the Fuzzy European Call option and compare the numerical results from all models.

In [12], the basic idea of constructing a Bino-Trinomial tree model is to replace the first two steps of the Binomial tree model with one step of the Trinomial tree model, with the aim that the resulting nodes can adjust the location of barriers and to increase flexibility. Dai and Lyuu have proven that the Bino-Trinomial tree method performs better than the Binomial and Trinomial models in determining barrier option prices.

In this paper, the first two steps in the Fuzzy CRR Binomial tree are replaced with one step of the Fuzzy KR Trinomial tree. The increase and decrease factors of stock price movements for the Fuzzy Binomial tree part are as in (39) which is the same with the factors for the Fuzzy Trinomial tree part. The value of λ for the Fuzzy Binomial tree is set to be $1 < \lambda < \sqrt{\frac{3}{2}}$. The volatility parameter in this model is expressed in the form of a triangular Fuzzy number (17).

At $t = 1$, the initial stock price is S. At time $t = 2$, the price movement can be up, remain the same, or down, which are Su_{H_t}, Su_{M_t}, and Su_{L_t}, for up movement, and Sd_{H_t}, Sd_{M_t}, and Sd_{L_t}, for down movement. So, at time $t = 2$ there are seven possible stock prices. From $t = 3$, stock price movements can increase with factors u_{H_b}, u_{M_b}, and Su_{L_b}, or decrease with factors d_{H_b}, d_{M_b}, and d_{L_b}. So, starting from $t = 3$, each stock price becomes six possible stock prices. Suppose there are n periods for the Bino-Trinomial tree, then the number of possible stock prices at $t = n$ is $6^n + 6^{n-1}$.

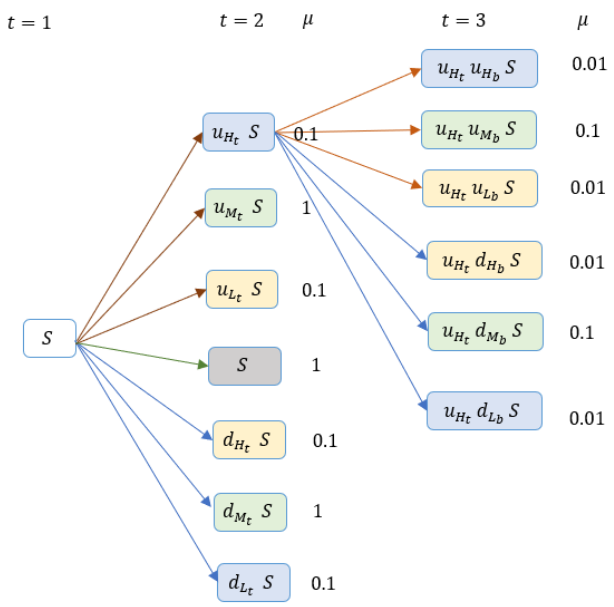

FIGURE 3. Stock price movement two periods under the Fuzzy Bino-Trinomial tree

Figure 3 illustrates the structure of stock price movements in the Fuzzy Bino-Trinomial tree model for two periods. In this model, there are seven possible stock prices from $t = 1$ to $t = 2$, and 42 possible stock prices from $t = 2$ to $t = 3$.

At time $t = n$, the payoffs of an Eropean Call option with Strike Price K and Maturity Time n are the following.

$$C_{u_H h(n-1)} = max(u_{H_b} S_{h(n-1)} - K),$$
$$C_{u_M h(n-1)} = max(u_{M_b} S_{h(n-1)} - K),$$
$$C_{u_L h(n-1)} = max(u_{L_b} S_{h(n-1)} - K),$$
$$C_{d_H h(n-1)} = max(d_{H_b} S_{h(n-1)} - K),$$
$$C_{d_M h(n-1)} = max(d_{M_b} S_{h(n-1)} - K),$$
$$C_{d_L h(n-1)} = max(d_{L_b} S_{h(n-1)} - K).$$

By doing the backward iteration from $t = n - 1$ to $t = n - 2$, each option value is a Fuzzy number $(C_{Lh(n-1)}, C_{Mh(n-1)}, C_{Hh(n-1)})$ which is determined by Equations (36),(37), and(38). The process is continued from time $t = n - 2$ until $t = 2$ using the same formula. Finally, we have the option price at time $t = 1$ as in Equations (59), (60), and (61).

4. NUMERICAL SIMULATION

In this numerical simulation, we use data from Lee et al.'s research [35] as a benchmark, for determining prices for European call options. The underlying asset used was the S & P 500 stock index. The stock price index and call option price on the issuance date were USD 996.52 and USD 16.4, respectively. The strike price of the S & P 500 index is 1100.

TABLE 1. European call option prices (USD) from Binomial and Trinomial Fuzzy tree models with different ρ_1, ρ_2 and n

			Fuzzy Binomial			Fuzzy Trinomial		
n	ρ_1	ρ_2	Left	Middle	Right	Left	Middle	Right
3	0.05	0.05	13.3702	14.7499	16.1201	13.1301	14.6318	16.1229
	0.1	0.1	11.9793	14.7499	17.4819	11.6160	14.6318	17.6048
	0.4	0.6	3.2321	14.7499	33.4908	2.3615	14.6318	39.3468
	0.8	0.5	0.0000	14.7499	28.2148	0.0000	14.6318	34.5711
6	0.05	0.05	12.0726	13.2916	14.5022	12.9250	14.8911	16.8436
	0.1	0.1	10.8438	13.2916	16.6747	10.9432	14.8911	18.7844
	0.4	0.6	3.1194	13.2916	40.2460	2.9136	14.8911	37.7793
	0.8	0.5	0.0000	13.2916	35.5842	0.0000	14.8911	34.0240

TABLE 2. European call option prices (USD) from Bino-Trinomial Fuzzy tree model with different ρ_1, ρ_2 and n

	λ			Fuzzy Bino-Trinomial		
n	for Binomial Tree	ρ_1	ρ_2	Left	Middle	Right
3	1.0717	0.05	0.05	14.6283	16.3055	17.9722
	1.0717	0.1	0.1	12.9388	16.3055	19.6299
	0.9798	0.4	0.6	3.0053	14.0654	37.8903
	1.0410	0.8	0.5	0.0000	15.5605	35.5970
6	1.0717	0.05	0.05	14.3397	16.4193	17.2401
	1.0717	0.1	0.1	12.5321	16.4193	19.1863
	1.0410	0.4	0.6	3.2322	14.6080	35.7468
	1.0717	0.8	0.5	0.0000	16.4193	34.4974

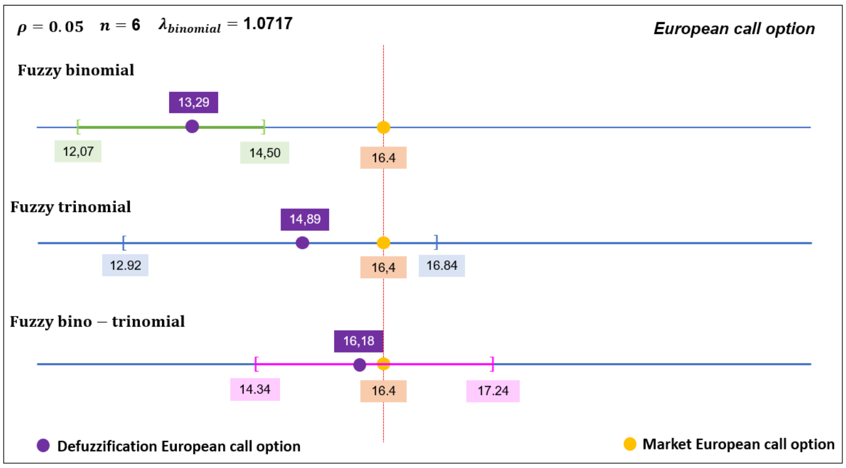

FIGURE 4. European call option price intervals from all Fuzzy tree models with $\rho = 0.05$

Note that in [35], the risk-free interest rate is assumed to be constant. The risk-free interest rate used is 0.00915. The Volatility value is estimated using historical closing price data from daily data for the last 30 to 60 days, and the volatility estimate obtained from historical data is 0.0114181. In our research, the 3-month US Treasury Interest Rate is used to have the riskless interest rate represented in the form of a triangular Fuzzy number. The tables and the figures show the obtained prices of European call options obtained using the Fuzzy models of CRR Binomial tree model, KR trinomial, and Bino-Trinomial Fuzzy trees with different periods and different sensitivity levels of volatility.

Based on Tables 1 and 2, the Fuzzy Bino-Trinomial model provides more moderate ranges of the option prices than of the Fuzzy trinomial and Fuzzy Binomial models. With the observed market price USD 16.4, at the volatility sensitivity level $\rho = 0.05$, the observed market price is within the option price range of the Fuzzy Bino-Trinomial and the Fuzzy Trinomial models. In contrast, for the Fuzzy Binomial model, the market price is not always within the obtained option price range. On the other hand, for the volatility sensitivity level $\rho = 0.1$, the observed market price is within the range of obtained option prices for the three Fuzzy models. The level of volatility sensitivity influences the length or shortness of the option price range produced by the three Fuzzy models. The level of volatility sensitivity in determining high and low volatility conditions does not always have to be the same; in other words, the Fuzzy volatility parameter can be written in an asymmetrical triangular Fuzzy number.

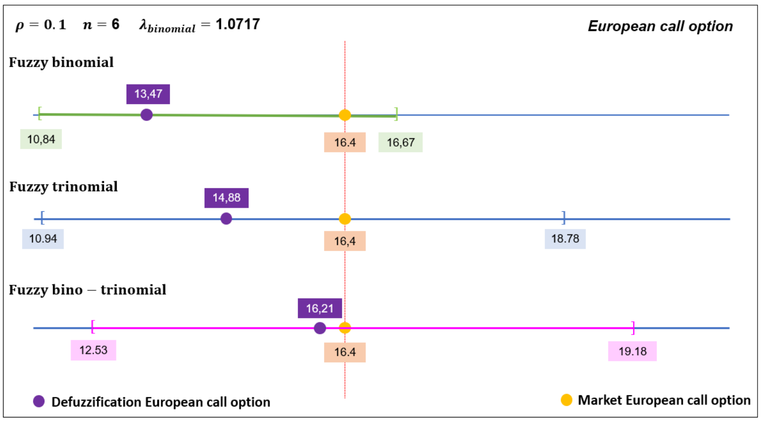

Figure 5. European call option price intervals from all Fuzzy tree model with $\rho = 0.1$

Figures 4 and 5 show illustrations of the intervals made by the obtained prices in Tables 1 and 2. The Fuzzy Binomial model has the shortest price interval compared to the other two Fuzzy models, and for a symmetrical triangular Fuzzy number, the Bino-Trinomial Fuzzy model has the longest option price interval compared to the other two Fuzzy models. It can also be seen in those Figures that the results of the defuzzification of European option prices from the Bino-Trinomial Fuzzy tree model can be said to be quite close to the prices given by the market compared to the results of the defuzzification of European option prices from the other two Fuzzy models.

In practice, financial analysts (or investors) may also be interested in the crisp numbers of option prices rather than the Fuzzy numbers. In other words, this kind of problem is known in the literature as a "defuzzification procedure." This research uses a "defuzzification procedure" of the central value in definition (??) as a crisp approximation of a Fuzzy number. The results of option price defuzzification for the Fuzzy Binomial model, Fuzzy trinomial model, and Fuzzy Bino-Trinomial model for different periods and levels of sensitivity can be seen in Table 3.

TABLE 3. Defuzzification of option prices of the Fuzzy Binomial model, Fuzzy trinomial model, and Fuzzy Bino-Trinomial model for different periods and levels of sensitivity

			Fuzzy Models		
n	ρ_1	ρ_2	Binomial	Trinomial	Bino-Trinomial
3	0.05	0.05	14.75	14.63	16.30
	0.1	0.1	14.74	14.62	16.29
	0.4	0.6	16.12	16.98	16.47
	0.8	0.5	14.50	15.64	16.42
6	0.05	0.05	13.29	14.89	16.18
	0.1	0.1	13.47	14.88	16.21
	0.4	0.6	16.43	16.96	16.46
	0.8	0.5	15.00	15.70	16.74

Based on Table 3, the results of the defuzzification of the prices from the Bino-Trinomial Fuzzy tree model show values that are pretty close to the observed market prices compared to the results of the defuzzification of the Binomial Fuzzy tree model, and the Trinomial Fuzzy tree model.

The Bino-Trinomial Fuzzy tree model requires slightly heavier calculations than the Fuzzy Binomial model, but is lighter when compared to the Fuzzy Trinomial model calculations. This is because the number of possible stock prices for Fuzzy Bino-Trinomial trees is more significantly larger than for Fuzzy Binomial trees, but smaller than for the Fuzzy trinomial trees. Like the Fuzzy Trinomial tree model, the Bino-Trinomial Fuzzy tree model is more flexible with real situations where the possible scenarios differ from stock price movements. By determining the appropriate value of lambda for the Fuzzy Binomial tree in the Fuzzy Bino-Trinomial tree, the result of defuzzification of the European call option price is closer to the given market price.

Now we discuss the benefit of our research in practices. Many investors face trade-offs and uncertainty regarding risk and return when making investment decisions. Determining the price of a European call option with a volatility parameter in a triangular Fuzzy number will produce a price in the form of a triangular Fuzzy number. Investors who exhibit risk-loving behavior tend to prioritize profits above risk considerations. Such investors tend to choose the right value of the European call option price triangle Fuzzy number to generate greater profits when the call option price shows higher volatility at a high price point. Conversely, risk-averse investors will prioritize mitigating dangers over pursuing profits. Investors with risk-averse behavior tend to choose the left value of the triangular Fuzzy number to minimize costs and reduce potential losses because it considers the concept of hedging. Investors who demonstrate a neutral attitude towards risk will consider the level of profitability and stability when determining appropriate prices. Investors with neutral behavior will choose a price between the range determined by the right and left values of the triangular Fuzzy number, thereby balancing these two factors.

Thus, it is clear that investors with different risk preferences will choose different market prices from the Fuzzy numbers of the option price triangle. Therefore, applying Fuzzy set theory, in this case using triangular Fuzzy numbers to represent volatility parameters, can expand the choices available to investors and provide insight into investors' risk preferences.

5. CONCLUSION

This paper proposes a new framework for option pricing with a Bino-Trinomial tree model for handling unpredictable conditions. Fluctuating market conditions from time to time, lack of information, different subjective management, and many other factors make option pricing decisions vary, resulting in uncertainty. Fuzzy set theory applied to the option pricing model Bino-Trinomial tree model to represent input parameters can provide a reasonable range of option prices so that many investors can use it for arbitrage or hedging. We apply the Fuzzy theory to represent the uncertainty conditions of the volatility parameters in the Bino-Trinomial tree model. In the numerical simulation for pricing an European Call option, the results are more satisfactory than other two tree models. It should be noted that the main limitation of the Bino-Trinomial tree model is determining the appropriate and optimal value λ in the Fuzzy Binomial tree part of the Fuzzy Bino-Trinomial tree model and is the relatively slow calculation speed even with a computer when the period increases.

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