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Range Value-at-Risk and Its Optimization in Vehicle Insurance

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Abstract. A popular risk measure is tail value-at-risk (TVaR), which is the mean of a random risk's losses above the value-at-risk (VaR). Moreover, TVaR is the most popular competitor of VaR. However, TVaR has some theoretical obstacles; for example, it does not exist when the risk distribution has a very heavy tail or has an infinite mean. In practice, this may compel financial institutions or insurance companies to deposit additional funds to fulfill requisites specified by regulators. Many authors suggested using a generalization of TVaR known as range value-at-risk (RVaR), which measures the actual risk of an aggregated risk. Furthermore, we suggest the range conditional tail variance (RCTV), a second conditional moment of the tail distribution with the RVaR at its center. RVaR and RCTV are significantly more flexible than TVaR and conditional tail variance (CTV) since they both have a contraction parameter. We also provide analytical formulations for the RVaR and RCTV of exponentially distributed risk. This article proposes an optimization method for the RVaR by applying the Newton method and a metaheuristic algorithm, spiral optimization (SpO). We use the Newton technique and SpO with RCTV and CTV to find the contraction parameter that optimizes RVaR. This study shows the use of RVaR optimization to forecast the RVaR of vehicle insurance claim amounts in Australia. We find that the SpO method produces the estimation result quite well as noticed by the quadratic form of objective function converging to zero. On the other hand, the Newton method produces not only similar results for the estimation but also has less RVaR at the same probability levels, which means better in lowering the magnitude of TVaR. However, the empirical results show that, compared with Newton's method, the SpO method captures the RVaR more successfully.

Key words and Phrases: Insurance claim, Newton method, RCTV, RVaR, Spiral optimization.

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1. INTRODUCTION

There have been numerous risk measures proposed in the subject of quantitative risk management, the two most well-known of which are the value-at-risk (VaR) and the tail value-at-risk. VaR is the greatest loss in a period for a given confidence level, whereas TVaR is the mean loss after VaR is exceeded. As VaR has been approved by the Basel Committee [1], it has emerged as the standard measurement of risk in the financial markets. Despite its simplicity and ease of application, VaR lacks coherence as a risk measure in the sense that [2] suggests. This is because VaR does not meet subadditivity, which means that the VaR of an aggregated risk is not necessarily less than the total of the VaRs of the individual risks that make up the aggregate, contrary to the principle of diversification. Furthermore, the VaR ignores the possibility of losses outside the considered quantile.

On the other hand, the TVaR, which accounts for losses beyond the VaR, is the most popular competitor for VaR. TVaR was proposed properly as a solution to VaR's limitations in terms of risk sensitivity. Furthermore, at some confidence level, TVaR equals the conditional expected value of losses that are greater than or equal to VaR. Since TVaR is a coherent measure of risk [2], it has been considerably implemented by financial institutions. However, TVaR has some theoretical hurdles, such as not existing for distributions with a very heavy tail or infinite mean [3], i.e., it relies heavily on the extreme tail of the return distribution, potentially overestimating the original risk [4]. Nonetheless, regulatory agencies apply both risk measures; VaR more in banking, while TVaR more in insurance [1]. Furthermore, Li *et al.* [5] argued that RVaR may be considered as a trestle between VaR and TVaR and is robust in the sense that it is continuous in relation to the weak convergence of random variables, whereas neither TVaR nor VaR is continuous generally.

In particular, Jiang *et al.* [6] implemented RVaR to illustrate the applicability of Pareto-optimal reinsurance policies that incorporate both the insurer's and reinsurer's risks and returns. Wang and Wei [7] identified and characterized Pareto-optimal allocations in quantile-based risk-sharing problems within the RVaR family. Li *et al.* [5] developed worst-case values and scenarios for RVaR under partial information assumption, for both single and aggregate risk models. Moreover, Fissler and Ziegel [8] statistically assessed, compared, and ranked the predictive performance of different RVaR models.

To overcome those limitations of VaR and TVaR simultaneously, we propose investigating the role of RVaR by finding a technique to optimize RVaR. Moreover, the existence of the contraction parameter $a > 0$ gives RVaR some flexibility. Instead of infinite, this parameter can be applied to provide an upper bound on the risk for values that exceed VaR. As a result, when we construct the RVaR forecast, it reduces the size of the loss beyond VaR, making it smaller than the TVaR forecast. This is a useful characteristic in risk modeling, particularly when dealing with high-variability portfolio returns or data with outliers. Nevertheless, Jadhav *et al.* [4] only set the values of a while computing RVaR rather than optimizing RVaR.

Furthermore, none of the previously listed authors ([5], [6], [8], [7], and [9]) offered a numerical optimization technique to ascertain parameter a directly or indirectly.

Besides TVaR, risk variability in the tail region of the distribution to the right of VaR was also noted by certain investigators. Essentially, the idea is that, regardless of the intended features and usefulness, TVaR only considers the mean losses on the tail and ignores variability, therefore it makes sense to include a second central moment or variance of the tail distribution. Thus, tail variance, or CTV, is a variability measure introduced by Furman *et al.* [10]. Following [10], we then suggest a variability measure surrounding RVaR that is subject to the same restriction. We refer to this measure as range conditional tail variance (RCTV). We consider this measure because we believe that RCTV provides a better comprehension of the behavior of a risk throughout the tail of its distribution.

The goal of this study is to present an optimization technique for indirectly estimating the value of parameter a that maximizes RVaR while being constrained by RCTV and CTV. The Lagrangian approach is used to convert the constrained maximization problem into a problem of solving a system of nonlinear equations. The transformed problem is now a problem of unconstrained minimization. In this study, two methods are used to solve the RVaR optimization problem: Newton and spiral optimization (SpO) methods. Recently, the Newton method as well as its variants have been extensively used in a variety of optimization problems [11, 12, 13, 14]. We choose Newton method because it can achieve convergence quickly. Of course, the method requires the selection of appropriate initial conditions as well as the calculation of the Jacobian matrix. Many people avoid Newton's method due to the difficulties of deriving the Jacobian matrix. Moreover, they do not want to face the Jacobian matrix becoming singular for some points and at some iteration. In our unconstrained problem, we can still derive the Jacobian matrix although it has a long expression. The elements of the Jacobian matrix are given in Appendix. On the other hand, the spiral phenomena in nature serves as the foundation for the SpO method, a metaheuristic algorithm. This method was first introduced in [15]. Many researchers have used the SpO method in tackling a range of optimization problems [16, 17, 18, 19] including an optimization problem in energy risk using copula-based extended value-at-risk [20]. We favor the SpO method over other metaheuristic methods because the updating points scheme is deterministic. The stochastic terms only appear in the initialization, not in the updating scheme. Therefore, this method is quite quicker than other metaheuristic methods.

The order of the remaining text is as follows. We provide a quick overview of RVaR in Section 2. The second conditional central moment of the distribution tail, with RVaR as the center, is given in Section 3. Section 4 explains the RVaR optimization approach in depth, whereas Section 5 applies it to an insurance claim. Section 6 offers a conclusion to the article. A proof, two analytical examples, and derivation of partial derivatives are relegated to Appendix.

2. RANGE VALUE-AT-RISK

Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space and define L as the set of random risks (with finite means) defined on $(\Omega, \mathcal{F}, \Pr)$. A risk measure is a functional $\varrho : L \to \mathbb{R},\ L \mapsto \varrho(L)$. Let L be a random risk whose marginal d.f. F_L is assumed to be a strictly increasing function. The value-at-risk (VaR) of L at probability level α is the quantile Q_α of F_L given a value $\alpha \in (0,1)$, typically close to one. Following is the definition of the VaR mathematically.

$$Q_\alpha = F_L^{-1}(\alpha). \tag{1}$$

In the formulation (2), the tail value-at-risk (TVaR) is defined as follows,

$$\mathrm{TVaR}_\alpha(L) = E[L | L \geq Q_\alpha(L)] = \frac{1}{1-\alpha} \int_\alpha^1 Q_p(L) \mathrm{d}p. \tag{2}$$

Formulation (3) defines the range value-at-risk (RVaR).

$$\mathrm{RVaR}_{(\alpha,a)}(L) = E\big[L | Q_\alpha \leq L \leq Q_{\alpha_1}\big] = \frac{\int_\alpha^{\alpha_1} Q_p(L)\,\mathrm{d}p}{(1-\alpha)^{a+1}}, \tag{3}$$

where $\alpha_1 = \alpha + (1-\alpha)^{1+a}$, $a \geq 0$. Here, α denotes the probability level, while a denotes the contraction parameter. In simpler terms, $\mathrm{RVaR}_{(\alpha,a)}$ is the mean of quantiles between probability levels α and α_1. This modified risk measure has long been projected to reduce the impact of exceptionally high-valued returns, making it a sort of truncated TVaR.

Remark 2.1. When we set $a = 0$, then RVaR collapses into TVaR given in (2).

Remark 2.2. RVaR can also be written as follows,

$$\mathrm{RVaR}_{(\alpha,a)}(L) = \frac{1}{(1-\alpha)^{a+1}} \Big[(1-\alpha)\mathrm{TVaR}_\alpha(L) - (1-\alpha_1)\mathrm{TVaR}_{\alpha_1}(L)\Big]. \tag{4}$$

3. RANGE CONDITIONAL TAIL VARIANCE

The RVaR, like the TVaR, determines mean losses in the tail of its distribution. Meanwhile, the second central moment or variance in the tail must be determined in order to generate a new risk measure involving variability. Valdez et al. [21] and Furman et al. [10] proposed a risk measure called tail variance (TV) or conditional tail variance (CTV) to study the variability of risk on the tail in relation to TVaR, which is given below,

$$\mathrm{CTV}_\alpha(L) = E\big[\big(L - \mathrm{TVaR}_\alpha(L)\big)^2 \big| L \geq Q_\alpha\big].$$

In this section, we introduce a new risk measure [1] called range conditional tail variance (RCTV), which is a rescaled version of CTV. CTV is closely connected to CTV, and so is RCTV to RVaR. The RCTV of L is defined as follows.

[1]We use the words "risk measure" and "variability measure" interchangeably when describing RCTV.

Definition 3.1. *Suppose L represents a risk. Assume Q_p represents the VaR of L at probability level p, and $\mathrm{RVaR}_{(\alpha,a)}$ represents the RVaR of L in (3). The range conditional tail variance (RCTV) of L can be calculated as follows,*

$$\mathrm{RCTV}_{(\alpha,a)}(L) = E\Big[(L - \mathrm{RVaR}_{(\alpha,a)}(L))^2 \mid Q_\alpha \le L \le Q_{\alpha_1}\Big]. \tag{5}$$

As a result, RCTV is a variability measure of risk L around RVaR that is trimmed by two quantiles. RCTV is a more generalized form of CTV. The RCTV formula for risk L is presented in the following lemma. The proof is omitted here.

Lemma 3.2. *Suppose that L denotes a random risk. Let $\alpha \in (0, 1)$ and $a > 0$. Then, the RCTV of L is given by,*

$$\mathrm{RCTV}_{(\alpha,a)}(L) = \frac{\int_\alpha^{\alpha_1} \big(Q_p(L)\big)^2 \,\mathrm{d}p}{(1-\alpha)^{a+1}} - \Big(\mathrm{RVaR}_{(\alpha,a)}(L)\Big)^2, \tag{6}$$

with $\mathrm{RVaR}_{(\alpha,a)}(L)$ is given in (4).

The following theorem provides properties that hold for the variability measure of RCTV.

Theorem 3.3. *The measure of RCTV fulfills the following properties:*

(1): *For any constant $k > 0$, we have $\mathrm{RCTV}_{(\alpha,a)}\big((k+L)\big) = k + \mathrm{RCTV}_{(\alpha,a)}(L)$.*
(2): *For any constant $k > 0$, we have $\mathrm{RCTV}_{(\alpha,a)}(k\,L) = k^2\,\mathrm{RCTV}_{(\alpha,a)}(L)$.*

Proof. **(1):** For any constant $k > 0$, we have,

$$\mathrm{RCTV}_{(\alpha,a)}(k+L) = E\Big[\big(k + L - \mathrm{RVaR}_{(\alpha,a)}(k+L)\big)^2\big|Q_\alpha(k+L) \le k+L \le Q_{\alpha_1}(k+L)\Big].$$

Since both VaR and RVaR satisfiy the property of translation invariance, then,

$$\begin{aligned}\mathrm{RCTV}_{(\alpha,a)}(k+L) &= E\Big[\big(k + L - k - \mathrm{RVaR}_{(\alpha,a)}(L)\big)^2\big|k + Q_\alpha(L) \le k+L \le k + Q_{\alpha_1}(L)\Big] \\ &= E\Big[\big(L - \mathrm{RVaR}_{(\alpha,a)}(L)\big)^2\big|Q_\alpha(L) \le L \le Q_{\alpha_1}(L)\Big] \\ &= \mathrm{RCTV}_{(\alpha,a)}(L).\end{aligned}$$

(2): For any constant $k > 0$, we have,

$$\mathrm{RCTV}_{(\alpha,a)}(kL) = E\Big[\big(kL - \mathrm{RVaR}_{(\alpha,a)}(kL)\big)^2\big|Q_\alpha(kL) \le kL \le Q_{\alpha_1}(kL)\Big].$$

Since both VaR and RVaR satisfy the property of positive homogeneity, then,

$$\begin{aligned}\mathrm{RCTV}_{(\alpha,a)}(kL) &= E\Big[\big(kL - k\mathrm{RVaR}_{(\alpha,a)}(L)\big)^2\big|kQ_\alpha(L) \le kL \le kQ_{\alpha_1}(L)\Big] \\ &= E\Big[k^2\big(L - \mathrm{RVaR}_{(\alpha,a)}(L)\big)^2\big|Q_\alpha(L) \le L \le Q_{\alpha_1}(L)\Big] \\ &= k^2\mathrm{RCTV}_{(\alpha,a)}(L).\end{aligned}$$

□

We illustrate RCTV for exponential distribution via Example 6.2 in the Appendix.

4. RVAR OPTIMIZATION USING RCTV AND CTV

Remember that RVaR is defined as follows,

$$\mathrm{RVaR}_{(\alpha,a)}(L) = E\big[L|Q_\alpha \leq L \leq Q_{\alpha_1}\big]. \tag{7}$$

This section explains the RVaR optimization issue under a constraint consisting of RCTV and CTV. Moreover, RVaR, RCTV, and CTV can be interpreted as functions of α and a. The optimization is done to get the values of contraction parameter a that maximize RVaR by introducing a constraint: the corresponding kRCTV must be equal to CTV for some multiplier $k \in \mathbb{R}^+$. The selection of this constraint is in line with Josaphat *et al.*[20], specifically in Eq. 10. They proposed a more complicated constraint, namely kDCTV-CTV, which involves dependence and copula. Since this research does not involve dependence, kDCTV-CTV becomes simply kRCTV-CTV. Thus, the RVaR optimization is provided as,

$$\begin{aligned} \mathrm{RVaR}_{(\alpha,a^*)}(L) = \max_{a>0} &\, \mathrm{RVaR}_{(\alpha,a)}(L) \\ s.t.\ & k\,\mathrm{RCTV}_{(\alpha,a)}(L) - \mathrm{CTV}_\alpha(L) = 0 \end{aligned} \tag{8}$$

However, determining a that maximizes (8) is not simple. To do this, we apply the transformation of a, which is $\alpha_1 = \alpha + (1-\alpha)^{1+a}$. Since $0 < \alpha < 1$ and $a > 0$, then $\alpha < \alpha_1 < 1$. Therefore, (8) can be written as follows,

$$\begin{aligned} \mathrm{RVaR}_{(\alpha,a^*)}(L) = \max_{\alpha<\alpha_1<1} &\, \mathrm{RVaR}_{(\alpha,a)}(L) \\ s.t.\ & k\,\mathrm{RCTV}_{(\alpha,a)}(L) - \mathrm{CTV}_\alpha(L) = 0, \end{aligned} \tag{9}$$

where $k > 0$. To simplify the notations, we rewrite (9) as,

$$\begin{aligned} \mathrm{RVaR}_{(\alpha,a^*)}(L) = \max_{\alpha<\alpha_1<1} &\, g(\alpha_1,k) \\ s.t.\ & h(\alpha_1,k) = 0, \end{aligned} \tag{10}$$

where $g(\alpha_1,k) = \mathrm{RVaR}_{(\alpha,a)}(L)$, $h(\alpha_1,k) = k\,\mathrm{RCTV}_{(\alpha,a)}(L) - \mathrm{CTV}_\alpha(L)$. Then, we apply the Lagrange multiplier method in [22] by defining a Lagrangian function,

$$\mathcal{L} = g(\alpha_1,k) - \mu\, h(\alpha_1,k).$$

We estimate the values of α_1, μ, and k by solving the following system of nonlinear equations,

$$G(\mathbf{x}) = 0, \tag{11}$$

where $\mathbf{x} = (\alpha_1,\mu,k)'$ and $G(\mathbf{x}) = (\partial\mathcal{L}/\partial\alpha_1,\ \partial\mathcal{L}/\partial\mu,\ \partial\mathcal{L}/\partial k)'$. In this article, we solve (11) in two ways, namely by using the Newton method and the spiral optimization (SpO) method. The Newton method is given by,

$$\mathbf{x}_{t+1} = \mathbf{x}_t - J(\mathbf{x}_t)^{-1}G(\mathbf{x}_t), \tag{12}$$

where $J(\mathbf{x})$ is a Jacobian matrix given by,

$$J(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial^2 \mathcal{L}}{\partial \alpha_1^2} & \dfrac{\partial^2 \mathcal{L}}{\partial \alpha_1 \partial \mu} & \dfrac{\partial^2 \mathcal{L}}{\partial \alpha_1 \partial k} \\ \dfrac{\partial^2 \mathcal{L}}{\partial \mu \partial \alpha_1} & \dfrac{\partial^2 \mathcal{L}}{\partial \mu^2} & \dfrac{\partial^2 \mathcal{L}}{\partial \mu \partial k} \\ \dfrac{\partial^2 \mathcal{L}}{\partial k \partial \alpha_1} & \dfrac{\partial^2 \mathcal{L}}{\partial k \partial \mu} & \dfrac{\partial^2 \mathcal{L}}{\partial k^2} \end{pmatrix}. \tag{13}$$

The calculation of each element of the Jacobian matrix (13) is given in the Appendix.

To perform the SpO method for solving (11), we need to change the problem into a minimization problem by defining an objective function as follows,

$$H(\mathbf{x}) = G(\mathbf{x})'G(\mathbf{x}) = \Big(\frac{\partial \mathcal{L}}{\partial \alpha_1}\Big)^2 + \Big(\frac{\partial \mathcal{L}}{\partial \mu}\Big)^2 + \Big(\frac{\partial \mathcal{L}}{\partial k}\Big)^2. \tag{14}$$

SpO is a metaheuristic algorithm based on logarithmic spiral phenomena like whirling water currents, nautilus shells, and spiral galaxies. SpO spins a number of points in $\mathbb{R}^n$ counterclockwise to a center point of rotation with a particular angle and radius scale. The center of rotation is the location where the objective function has the least value. In this article, our problem lies in $\mathbb{R}^3$, and hence the SpO method presented below is designed to solve the problem of minimization of H in (14) in $\mathbb{R}^3$.

The minimization problem $\min_{\mathbf{x}} H(\mathbf{x})$ is equivalent to the constrained maximization of $\mathrm{RVaR}_{\alpha,a}(L)$ because $H(\mathbf{x}) = G(\mathbf{x})'G(\mathbf{x})$ represents the squared norm of the KKT residuals of the original problem. Hence, minimizing $H(\mathbf{x})$ drives all first-order optimality conditions $\nabla \mathcal{L} = 0$ to be satisfied, and any point where $H(\mathbf{x}) = 0$ corresponds to a stationary point of the Lagrangian of the maximization problem $\max_a \mathrm{RVaR}_{\alpha,a}(L)$.

Before processing the SpO method, in the initialization, write $\mathbf{x} = (\alpha_1, \mu, k)$, define the search space $I \subset \mathbb{R}^3$, set m to be the number of points where $m \geq 2$, ω to be the angle of rotation where $0 < \omega \leq \pi$, r to be the rotation's scaling radius where $0 < r < 1$, and $t_{\max}$ to be the iteration maximum number. As for the starting process, choose random numbers $\mathbf{x}_i(0) \in I$ as the starting point, where $i = 1, 2, \dots, m$. Set $\mathbf{x}^{\mathrm{gbest}}(0) = \mathbf{x}^{\mathrm{g}0}$, where $H(\mathbf{x}^{\mathrm{g}0}) = \min_i H(\mathbf{x}_i(0))$ is the starting center of the rotation. Set $t = 0$.

(1): Update the point position for each i ($i = 1, 2, \dots, m$):

$$\mathbf{x}_i(t+1) = S_3(r,\omega)\mathbf{x}_i(t) - (S_3(r,\omega) - I_3)\mathbf{x}^{\mathrm{gbest}}(t),$$

where I_3 is the 3×3 identity matrix, $S_3(r,\omega) = r\Pi_{i=1}^2\big(\Pi_{j=1}^i R^{(3)}_{3-i,4-j}(\omega)\big)$, and $R^{(3)}_{i,j}(\omega)$ is the plane rotation matrix acting on the $(i,j)-$coordinates, that is equal to the identity matrix except for the four entries: the iith and jjth entries are $\cos\omega$, the jith entry is $\sin\omega$, and the ijth entry is $-\sin\omega$.

(2): Check to see if the updated position is still present in the search space: If $\mathbf{x}_i(t+1) \in I$, then calculate $H(\mathbf{x}_i(t+1))$. If $\mathbf{x}_i(t+1) \notin I$, then set $H(\mathbf{x}_i(t+1)) = \mathrm{NaN}$ [23].

(3): Update the rotation center: $\mathbf{x}^{\mathrm{gbest}}(t+1) = \mathbf{x}^{\mathrm{g}}$, where $H(\mathbf{x}^{\mathrm{g}}) = \min_i H(\mathbf{x}_i(t+1))$.

(4): If $t < t_{\max}$, then redo the processes 1 until 3 with setting $t = t + 1$. If $t = t_{\max}$, then stop the processes.

The output of the algorithm is $\mathbf{x}^{\mathrm{gbest}}(t_{\max})$, where the value of $H[\mathbf{x}^{\mathrm{gbest}}(t_{\max})]$ is nearly zero. In our numerical experiment, we run the SpO method using the following algorithm settings: $I = (\alpha, 1) \times (0, 1) \times (10, 1000)$, where $\alpha = 0.9$, $m = 5000$, $\omega = \pi/4$, $r = 0.95$, $t_{max} = 500$, while the number of runs is 20.

The following is an example of RVaR optimization with RCTV and CTV. More specifically, we construct RVaR for the exponential distribution. Application of this distribution encompasses a wide spectrum of fields, one of which is actuarial science.

Example 4.1. Refer to Examples 6.1 and 6.2 in the Appendix. Before performing the optimization, we compute the CTV of risk L. The TVaR of L at probability level α is given by,

$$\mathrm{TVaR}_\alpha(L) = \frac{1}{\lambda}\big[1 - \ln(1-\alpha)\big].$$

While $\int_\alpha^1 \big(Q_p(L)\big)^2\,\mathrm{d}p$ can be calculated as follows,

$$\int_\alpha^1 \big(Q_p(L)\big)^2\,\mathrm{d}p = \frac{1-\alpha}{\lambda^2}\big[\ln^2(1-\alpha) - 2\ln(1-\alpha) + 2\big].$$

Hence,

$$\begin{aligned}
\mathrm{CTV}_\alpha(L) &= \frac{1}{1-\alpha}\int_\alpha^1 \big(Q_p(L)\big)^2\,\mathrm{d}p - \big(\mathrm{TVaR}_\alpha(L)\big)^2 \\
&= \frac{1}{\lambda^2}\big[\ln^2(1-\alpha) - 2\ln(1-\alpha) + 2\big] - \frac{1}{\lambda^2}\big[\ln^2(1-\alpha) - 2\ln(1-\alpha) + 1\big] \\
&= \frac{1}{\lambda^2}.
\end{aligned} \tag{15}$$

Based on (17) in p. 15 and (15), we may construct the following analytical formula,

$$h(\alpha_1, k) = k\,\mathrm{RCTV}_{(\alpha,a)}(L) - \mathrm{CTV}_\alpha(L) = k\left(\frac{1}{\lambda^2(\alpha_1-\alpha)}\Lambda_2 - \frac{1}{\lambda^2(\alpha_1-\alpha)^2}\Lambda_1^2\right) - \frac{1}{\lambda^2},$$

where

$$\Lambda_1 = (1-\alpha)\big(1 - \ln(1-\alpha)\big) - (1-\alpha_1)\big(1 - \ln(1-\alpha_1)\big),$$

$$\Lambda_2 = (1-\alpha)\Big(\ln^2(1-\alpha) - 2\ln(1-\alpha) + 2\Big) - (1-\alpha_1)\Big(\ln^2(1-\alpha_1) - 2\ln(1-\alpha_1) + 2\Big).$$

Furthermore, we may extract the Lagrangian multiplier, which is provided by,

$$\mathcal{L} = g(\alpha_1, k) - \mu\, h(\alpha_1, k) = \frac{1}{\lambda\big(\alpha_1-\alpha\big)}\Lambda_1 - \frac{\mu k}{\lambda^2(\alpha_1-\alpha)}\Lambda_2 + \frac{\mu k}{\lambda^2(\alpha_1-\alpha)^2}\Lambda_1^2 + \frac{\mu}{\lambda^2}.$$

The first- and second-order partial derivatives are then calculated, as shown in the Appendix.

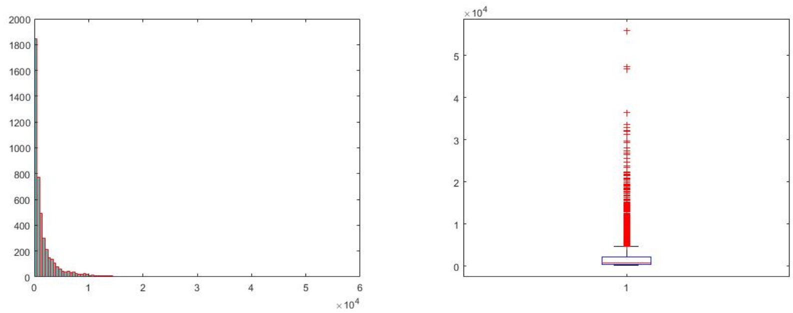
5. APPLICATION

This research aims to examine the performance of the SpO and Newton methods on RVaR optimization. Our data comes from 2004 and 2005 one-year auto insurance policies[2]; out of the 67,856 policies, 4,618 (6.8%) policies had at least one claim. Suppose the risk L is the claim amount.

Table 1 provides a statistical summary of the claim amount. It can be seen that the claim amount has positive skewness, which is 5.0470. Furthermore, the kurtosis of the claim amount is said to be significant when greater than 3. A kurtosis value above 3 (43.3102) indicates that, relative to the normal distribution, the probability of the observations tending to be far from the mean will be greater than the probability of observations tending to be close to the mean. The information in Figure 1(a) supports this finding. Moreover, a box plot of the data in Figure 1(b) indicates that the right tail contains 758 outliers. Take notice that every figure and calculation mentioned in this article is done using MATLAB programs.

TABLE 1. Descriptive statistic of claim amount

Statistic	Claim amount (L)
Sample size	4,618
Mean	2.0131×10^3
Standard deviation	3.5480×10^3
Skewness	5.0470
Kurtosis	43.3102



(a) (b)

FIGURE 1. (a) Histogram and (b) box plot of the claim amount.

We find that L is one-parameter exponentially distributed by applying the maximum likelihood estimation (MLE) approach, where the parameter estimate is $\hat{\lambda} = 1/2013.09$. The significant number of outliers is probably going to have an

[2]The data is retrieved from the Department of Applied Finance and Actuarial Studies, Macquarie University (accessed on March 24th, 2021).

impact on this estimate. We have in Figure 2 the comparison of five risk measures of L. Recall that, the CTV of L is the same for all α. On the other hand, the larger the α the smaller the RCTV. Moreover, RCTV is relatively much smaller than CTV. Note that RVaR is smaller than TVaR but larger than the quantile at the same probability level. It means that RVaR reduces the amount of TVaR.

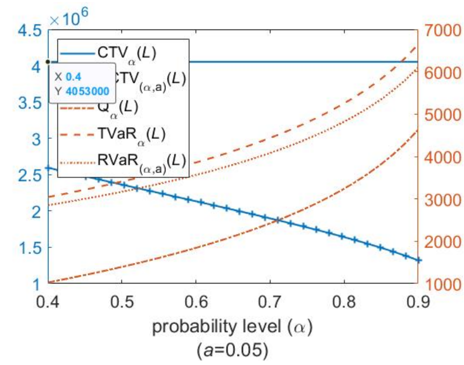


FIGURE 2. Risk measures for the claim amount.

TABLE 2. Values of α_1, μ, and k maximizing RVaR using SpO method, contraction parameter estimation, risk measures forecast, number and proportion of violations of the $\mathrm{RVaR}_{(\alpha,0)}$ forecasts.

α	sig. level	Trial	α_1	μ	k	H	a	$\mathrm{RVaR}_{(\alpha,a)}$	$\mathrm{TVaR}_{0.9}$	$\mathrm{VaR}_{0.9}$	no. violations	prop. violations[a]	difference[b]
0.9	0.1	1	0.9114	1.443×10^{-5}	820.523	0.0051	0.9433	4,754.6	6,648.4	4,635.3	488	0.1057	0.0057
		2	0.9110	1.392×10^{-5}	884.694	0.0041	0.9587	4,750.3			489	0.1059	0.0059
		3	0.9116	1.468×10^{-5}	791.380	0.0057	0.9359	4,756.8			488	0.1057	0.0057
		4	0.9116	1.468×10^{-5}	791.379	0.0057	0.9359	4,756.8			488	0.1057	0.0057
		5	0.9111	1.409×10^{-5}	862.415	0.0044	0.9535	4,751.7			489	0.1059	0.0059
0.95	0.05	1	0.9554	1.370×10^{-5}	913.964	0.0037	0.7420	6,143.8	8,043.8	6,030.7	358	0.0775	0.0275
		2	0.9552	1.319×10^{-5}	988.643	0.0029	0.7544	6,139.6			358	0.0775	0.0275
		3	0.9557	1.443×10^{-5}	820.490	0.0051	0.7250	6,150.0			358	0.0775	0.0275
		4	0.9560	1.509×10^{-5}	747.777	0.0067	0.7105	6,155.5			357	0.0773	0.0273
		5	0.9554	1.368×10^{-5}	916.729	0.0037	0.7425	6,143.7			358	0.0775	0.0275
0.98	0.02	1	0.9822	1.361×10^{-5}	926.728	0.0035	0.5699	7,987.7	9,888.3	7,875.3	235	0.0509	0.0309
		2	0.9831	1.969×10^{-5}	427.175	0.0349	0.4775	8,039.4			235	0.0509	0.0309
		3	0.9821	1.327×10^{-5}	975.900	0.0030	0.5761	7,984.8			236	0.0511	0.0311
		4	0.9821	1.319×10^{-5}	988.912	0.0029	0.5777	7,984.1			237	0.0513	0.0313
		5	0.9822	1.362×10^{-5}	925.179	0.0036	0.5697	7,987.7			235	0.0509	0.0309

[a]Prop. violations is the proportion of violations to the total number of observations (4,618 observations).
[b]Difference is the absolute difference between prop. violations and sig. level.

By utilizing the estimated parameter $\hat{\lambda}$ derived from the data, we employ SpO and Newton methods to resolve the RVaR optimization problem (10). Using the SpO approach, the outcomes are found in Table 2. The values of three variables α_1, μ, and k that maximize RVaR (10) for three distinct $\alpha = 0.9, 0.95, 0.98$ choices are shown in the Table 2. For each case of α, we run the SpO method in five trials. Table 2 column 6 shows that the values of the objective function $H = H(\alpha_1, \mu, k)$ are relatively near to zero which indicates that the method is quite successful for minimizing H in (14). The visualization example of H versus iterations for one trial with 20 runs is given in Figure 3. Since H is close to zero, we can say that the system of nonlinear equations (11) which consists of the first derivatives of the Lagrangian function $\mathcal{L}$ has been solved. From the estimated value of α_1, we calculate the value of contraction parameter a to forecast $\mathrm{RVaR}_{(\alpha,a)}$. Another attractive result is that the RVaR forecasts gained from the optimization (column 9 in Table 2) are comparatively significantly smaller than the TVaR forecast. This show that the relatively large TVaR forecast can be reduced by computing the RVaR forecast. Those relatively large TVaR forecasts, especially when compared to the VaR forecasts, are of course influenced by $\hat{\lambda}$ which is also affected by the number of outliers, namely 758 observations.

Table 2 presents the number of violations of the RVaR forecast using the SpO technique. It is the number of sample observations that are larger than the RVaR forecast and thus outside of the critical value. Then, by calculating the number of violations of the $\mathrm{RVaR}_{(0.9,a)}$, we obtain the proportion of violations (prop. violations) of 0.1057 (there are 488 violations out of 4,618 observations, therefore 488/4,618=0.1057). Table 2 also displays the significance levels and proportions of violations for $\alpha = 0.95$, 0.98. Table 2 shows that the probability level $\alpha = 0.9$ results in the smallest differences between the prop. violations and the significance level, 0.0057 and 0.0059, respectively. Therefore, in this case, the best RVaR forecast is obtained when $\alpha = 0.9$.

TABLE 3. Values of α_1, μ, and k maximizing RVaR using Newton method, contraction parameter estimation, risk measures forecast, number and proportion of violations of the $\mathrm{RVaR}_{(\alpha,0)}$ forecasts.

α	sig. level	Initial guess			α_1	μ	k	a	$\mathrm{RVaR}_{(\alpha,a)}$	$\mathrm{TVaR}_{0.9}$	$\mathrm{VaR}_{0.9}$	no. violations	prop. violations[a]	difference[b]
0.9	0.1	$\mu = 0.01$	$k = 10$	$\alpha_1 = 0.95$	0.9014	-4.981×10^{-11}	47,121	1.8440	4,649.8	6,648.4	4,635.3	500	0.1083	0.0083
				$\alpha_1 = 0.99$	0.9059	-3.620×10^{-11}	4,657.2	1.2290	4,695.9			492	0.1065	0.0065
			$k = 100$	$\alpha_1 = 0.95$	0.9016	-6.845×10^{-11}	38,097	1.7981	4,651.4			500	0.1083	0.0083
				$\alpha_1 = 0.99$	0.9102	-1.932×10^{-8}	854.560	0.9921	4,741.5			490	0.1061	0.0061
		$\mu = 0.001$	$k = 10$	$\alpha_1 = 0.95$	0.9015	-5.726×10^{-11}	42,927	1.8239	4,650.5			500	0.1083	0.0083
				$\alpha_1 = 0.99$	0.9049	-2.097×10^{-9}	3,838.5	1.3068	4,685.8			494	0.1070	0.0070
			$k = 100$	$\alpha_1 = 0.95$	0.9015	-5.923×10^{-11}	41,967	1.8190	4,650.7			500	0.1083	0.0083
0.95	0.05	$\mu = 0.01$	$k = 10$	$\alpha_1 = 0.98$	0.9510	-1.116×10^{-10}	24,180	1.3069	6,050.9	8,043.8	6,030.7	366	0.0793	0.0293
				$\alpha_1 = 0.99$	0.9518	-6.846×10^{-10}	7,168.4	1.1066	6,067.7			366	0.0793	0.0293
			$k = 100$	$\alpha_1 = 0.98$	0.9511	-1.670×10^{-10}	18,465	1.2623	6,053.8			366	0.0793	0.0293
				$\alpha_1 = 0.99$	0.9529	-2.835×10^{-9}	2,755.6	0.9506	6,090.2			364	0.0788	0.0288
		$\mu = 0.001$	$k = 10$	$\alpha_1 = 0.98$	0.9511	-1.321×10^{-10}	21,603	1.2882	6,052.1			366	0.0793	0.0293
				$\alpha_1 = 0.99$	0.9517	-5.176×10^{-10}	8,648.7	1.1374	6,064.4			366	0.0793	0.0293
			$k = 100$	$\alpha_1 = 0.98$	0.9510	-1.242×10^{-10}	22,513	1.2950	6,051.6			366	0.0793	0.0293
				$\alpha_1 = 0.99$	0.9532	-3.625×10^{-9}	2,322.4	0.9228	6,095.5			364	0.0788	0.0288
0.98	0.02	$\mu = 0.01$	$k = 10$	$\alpha_1 = 0.99$	0.9803	-3.627×10^{-11}	44,675	1.0786	7,890.1	9,888.3	7,875.3	238	0.0515	0.0315
			$k = 100$	$\alpha_1 = 0.99$	0.9803	-4.261×10^{-11}	40,113	1.0649	7,891.0			238	0.0515	0.0315
		$\mu = 0.001$	$k = 10$	$\alpha_1 = 0.99$	0.9803	-3.902×10^{-11}	42,546	1.0724	7,890.5			238	0.0515	0.0315
			$k = 100$	$\alpha_1 = 0.99$	0.9803	-3.861×10^{-11}	42,842	1.0733	7,890.4			238	0.0515	0.0315

[a]Prop. violations is the proportion of violations to the total number of observations (4,618 observations).
[b]Difference is the absolute difference between prop. violations and sig. level.

Furthermore, Table 3 shows the findings for the Newton approach (12). The table displays the values of three variables α_1, μ, and k that maximize RVaR (10) for three different chosen $\alpha = 0.9, 0.95, 0.98$. We run the Newton method for a number of combinations of initial guesses: $\mu = 0.01$, 0.001, $k = 10$, 100, and $\alpha_1 = 0.95$, 0.98, 0.99. From the table, we can see that the obtained values of α_1 are similar to the results of the SpO method for each different case of α. In addition, the running time of the Newton method is far less consuming than that of the SpO

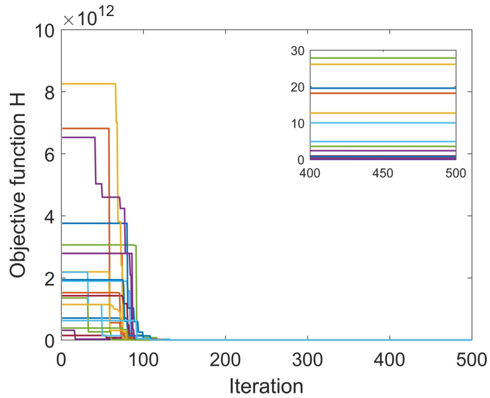


FIGURE 3. Objective function H of the SpO method over iteration.

method because the Newton method is a non-stochastic algorithm, and hence it is very fast. Moreover, from Table 3, we can notice that the RVaR forecasts are less than those in Table 2 at the same probability levels. The comparison between RVaR forecasts using SpO and Newton methods for different cases of α is visualized in Figure 4.

Moreover, the number of violations of the RVaR forecast using the Newton method is also presented in Table 3. Similar to the results in Table 2, the probability level $\alpha = 0.9$ in Table 3 results in the smallest differences between the prop. violations and the significance level. Therefore, for the SpO method, the best RVaR forecast is obtained when $\alpha = 0.9$. Surprisingly, we find the fact that the prop. violations for the SpO method are always smaller than those of the Newton method. In other words, the Newton method underestimates RVaR. That is, in spite of the fact that the Newton method is very fast, the SpO method is more appropriate for forecasting RVaR.

6. CONCLUDING REMARKS

This article presents an optimization approach for a generalized tail value-at-risk known as range value-at-risk (RVaR), by implementing Newton method and SpO method. The notion of the constraint function is to compare the second conditional central moment related to RVaR, known as RCTV, with the counterpart second conditional central moment connected to TVaR, known as CTV. We recognize that RCTV must be less than CTV, therefore we multiply RCTV by a positive multiplier k to create a new measure proportionate to CTV.

In the case of exponentially distributed risk, we demonstrate numerical findings for contraction parameter a and multiplier k using Newton and SpO methods. We gain that our objective function utilizing the SpO method converges to zero as the iteration increases, which implies that SpO is very appropriate for RVaR optimization. Moreover, the RVaR forecasts of the vehicle insurance claim amount

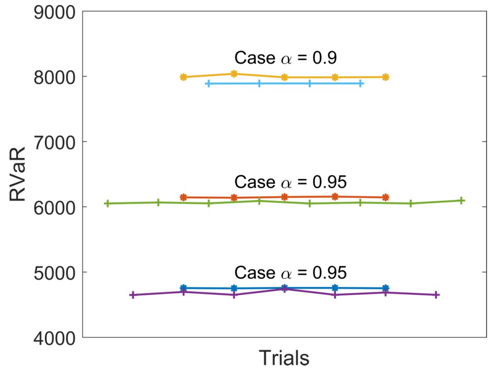


FIGURE 4. Comparison between RVaR forecasts obtained from SpO method (straight-star line) and Newton method (straight-plus line) for different cases of α.

obtained through optimization lower the magnitude of TVaR. This indicates that using the suggested RVaR optimization strategy will prevent unnecessary additional capital allocation while having no effect on the risk management policies imposed by regulators. However, the SpO method is more appropriate in forecasting RVaR since the differences between the proportion of violations and the corresponding significance level are lower than those of the Newton method. Thus, the suggested optimization method is effective in the risk management of insurance claims and beneficial for actuaries in insurance companies.

However, the data has a significant number of outliers (16.41%), so our study suggests that there should be a necessity to consider most of the worst-case observations that exceed VaR in order to forecast the actual risk of the claim amount. The number of outliers that appear in the data should be accommodated with heavy-tailed distributions, such as Pareto, Weibull, and Burr. In the next research, we will apply RVaR optimization to the vehicle insurance data using the suggested heavy-tailed distributions.

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REFERENCES

[1] A. Goel and A. Sharma, "Mixed value-at-risk and its numerical investigation," *Physica A: Statistical Mechanics and its Applications*, vol. 541, p. 123524, 2020. `https://doi.org/10.1016/j.physa.2019.123524`.

[2] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, "Coherent measures of risk," *Mathematical finance*, vol. 9, no. 3, pp. 203–228, 1999. `https://doi.org/10.1111/1467-9965.00068`.

[3] P. Embrechts and M. Hofert, "Statistics and quantitative risk management for banking and insurance," *Annual Review of Statistics and Its Application*, vol. 1, no. 1, pp. 493–514, 2014. `https://doi.org/10.1146/annurev-statistics-022513-115631`.

[4] D. Jadhav, T. V. Ramanathan, and U. Naik-Nimbalkar, "Modified expected shortfall: a new robust coherent risk measure," *The Journal of Risk*, vol. 16, no. 1, p. 69, 2013. `https://doi.org/10.21314/JOR.2013.269`.

[5] L. Li, H. Shao, R. Wang, and J. Yang, "Worst-case range value-at-risk with partial information," *SIAM Journal on Financial Mathematics*, vol. 9, no. 1, pp. 190–218, 2018. `https://doi.org/10.1137/17M1126138`.

[6] W. Jiang, H. Hong, and J. Ren, "On pareto-optimal reinsurance with constraints under distortion risk measures," *European Actuarial Journal*, vol. 8, no. 1, pp. 215–243, 2018. `https://doi.org/10.2139/ssrn.2955764`.

[7] R. Wang and Y. Wei, "Characterizing optimal allocations in quantile-based risk sharing," *Insurance: Mathematics and Economics*, vol. 93, pp. 288–300, 2020. `https://doi.org/10.1016/j.insmatheco.2020.06.001`.

[8] T. Fissler and J. F. Ziegel, "On the elicitability of range value at risk," *Statistics & risk modeling*, vol. 38, no. 1-2, pp. 25–46, 2021. `https://doi.org/10.1515/strm-2020-0037`.

[9] C. Bernard, R. Kazzi, and S. Vanduffel, "Range value-at-risk bounds for unimodal distributions under partial information," *Insurance: Mathematics and Economics*, vol. 94, pp. 9–24, 2020. `https://doi.org/10.1016/j.insmatheco.2020.05.013`.

[10] E. Furman and Z. Landsman, "Tail variance premium with applications for elliptical portfolio of risks," *ASTIN Bulletin: The Journal of the IAA*, vol. 36, no. 2, pp. 433–462, 2006. `https://doi.org/10.2143/AST.36.2.2017929`.

[11] D. J. Mavriplis, "A residual smoothing strategy for accelerating newton method continuation," *Computers & Fluids*, vol. 220, p. 104859, 2021. `https://doi.org/10.1016/j.compfluid.2021.104859`.

[12] J. de Jesús Rubio, M. A. Islas, G. Ochoa, D. R. Cruz, E. Garcia, and J. Pacheco, "Convergent newton method and neural network for the electric energy usage prediction," *Information Sciences*, vol. 585, pp. 89–112, 2022. `https://doi.org/10.1016/j.ins.2021.11.038`.

[13] G. Candelario, A. Cordero, J. R. Torregrosa, and M. P. Vassileva, "An optimal and low computational cost fractional newton-type method for solving nonlinear equations," *Applied Mathematics Letters*, vol. 124, p. 107650, 2022. `https://doi.org/10.1016/j.aml.2021.107650`.

[14] J. Zhang, K. You, and T. Başar, "Distributed adaptive newton methods with global superlinear convergence," *Automatica*, vol. 138, p. 110156, 2022. `https://doi.org/10.1016/j.automatica.2021.110156`.

[15] K. Tamura and K. Yasuda, "Spiral dynamics inspired optimization," *Journal of Advanced Computational Intelligence and Intelligent Informatics*, vol. 15, no. 8, pp. 1116–1122, 2011. `https://doi.org/10.20965/jaciii.2011.p1116`.

[16] L. Benasla, A. Belmadani, and M. Rahli, "Spiral optimization algorithm for solving combined economic and emission dispatch," *International Journal of Electrical Power & Energy Systems*, vol. 62, pp. 163–174, 2014. `https://doi.org/10.1016/j.ijepes.2014.04.037`.

[17] K. A. Sidarto and A. Kania, "Finding all solutions of systems of nonlinear equations using spiral dynamics inspired optimization with clustering," *Journal of Advanced Computational Intelligence and Intelligent Informatics*, vol. 19, no. 5, pp. 697–707, 2015. `https://doi.org/10.20965/jaciii.2015.p0697`.

[18] M. F. Ansori, K. A. Sidarto, and N. Sumarti, "Model of deposit and loan of a bank using spiral optimization algorithm," *Journal of the Indonesian Mathematical Society*, vol. 25, no. 3, pp. 292–301, 2019. `https://doi.org/10.22342/jims.25.3.826.292-301`.

[19] Y. Cao, H. N. Rad, D. H. Jamali, N. Hashemian, and A. Ghasemi, "A novel multi-objective spiral optimization algorithm for an innovative solar/biomass-based multi-generation energy system: 3e analyses, and optimization algorithms comparison," *Energy Conversion and Management*, vol. 219, p. 112961, 2020. `https://doi.org/10.1016/j.enconman.2020.112961`.

[20] B. P. Josaphat, M. F. Ansori, and K. Syuhada, "On optimization of copula-based extended tail value-at-risk and its application in energy risk," *IEEE Access*, vol. 9, pp. 122474–122485, 2021. `https://doi.org/10.1109/ACCESS.2021.3106715`.
[21] E. A. Valdez, "Tail conditional variance for elliptically contoured distributions," *Belgian Actuarial Bulletin*, vol. 5, no. 1, pp. 26–36, 2005. `https://doi.org/10.21314/JOR.2000.038`.
[22] J. Nocedal and S. J. Wright, *Numerical optimization*. Springer, 2006. `https://doi.org/10.1007/978-0-387-40065-5`.
[23] M. F. Ansori, K. A. Sidarto, N. Sumarti, and I. Gunadi, "Dynamics of bank's balance sheet: A system of deterministic and stochastic differential equations approach," *International Journal of Mathematics and Computer Science*, vol. 16, no. 3, pp. 871–884, 2021. `https://future-in-tech.net/16.3/R-Ansori-Sidarto-Sumarti-Gunadi.pdf`.

Appendix

In the following calculations, for reducing long-expression equation, we write

$$\Lambda_1 = (1-\alpha)\big(1-\ln(1-\alpha)\big) - (1-\alpha_1)\big(1-\ln(1-\alpha_1)\big),$$
$$\Lambda_2 = (1-\alpha)\Big(\ln^2(1-\alpha) - 2\ln(1-\alpha)+2\Big) - (1-\alpha_1)\Big(\ln^2(1-\alpha_1) - 2\ln(1-\alpha_1)+2\Big),$$
$$\Lambda_3 = \frac{1-\alpha}{1-\alpha_1}\big(1-\ln(1-\alpha)\big) + \ln(1-\alpha_1) + \ln^2(1-\alpha_1) - 1.$$

Example 6.1 (Forecast of RVaR of exponential risk)**.** Assume that L represents a risk with an exponential distribution and parameter λ. The distribution function, probability function, VaR, and TVaR of L are as follows,

$$F_L(l) = 1-e^{-\lambda l}, \qquad f_L(l) = \lambda e^{-\lambda l},$$
$$Q_\alpha(L) = -\frac{1}{\lambda}\ln(1-\alpha), \quad \mathrm{TVaR}_\alpha(L) = \frac{1}{\lambda}[1-\ln(1-\alpha)].$$

Then, by remembering that $\alpha_1 - \alpha = (1-\alpha)^{a+1}$, RVaR of L at level α $(0<\alpha<1)$ is given by,

$$\mathrm{RVaR}_{(\alpha,a)}(L) = \frac{1}{\lambda(1-\alpha)^{a+1}}\Lambda_1 = \frac{1}{\lambda(\alpha_1-\alpha)}\Lambda_1. \tag{16}$$

Example 6.2. Consider the previous Example 6.1. When calculating RCTV, keep in mind that,

$$\begin{aligned}\frac{1}{(1-\alpha)^{1+a}}\int_\alpha^{\alpha_1}\big(Q_p(L)\big)^2\mathrm{d}p &= \frac{1}{(1-\alpha)^{1+a}}\left(\int_\alpha^1\big(Q_p(L)\big)^2\mathrm{d}p - \int_{\alpha_1}^1\big(Q_p(L)\big)^2\mathrm{d}p\right)\\ &= \frac{1}{\lambda^2(1-\alpha)^{1+a}}\Lambda_2.\end{aligned}$$

Therefore,

$$\mathrm{RCTV}_{(\alpha,a)}(L) = \frac{1}{\lambda^2(1-\alpha)^{1+a}}\Lambda_2 - \frac{1}{\lambda^2(1-\alpha)^{2a+2}}\Lambda_1^2 = \frac{1}{\lambda^2(\alpha_1-\alpha)}\Lambda_2 - \frac{1}{\lambda^2(\alpha_1-\alpha)^2}\Lambda_1^2. \tag{17}$$

Partial derivatives. The following are the first and second-order partial derivatives of $\mathcal{L}$.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha_1} &= \frac{\lambda - 2(\mu k)\ln(1-\alpha_1)}{\lambda^2\big(\alpha_1-\alpha\big)^2}\Lambda_1 - \frac{2\mu k}{\lambda^2\big(\alpha_1-\alpha\big)^3}\Lambda_1^2 + \frac{\mu k}{\lambda^2(\alpha_1-\alpha)^2}\Lambda_2 \\
&\quad - \frac{\lambda\ln(1-\alpha_1) + (\mu k)\ln^2(1-\alpha_1)}{\lambda^2\big(\alpha_1-\alpha\big)}, \\
\frac{\partial \mathcal{L}}{\partial \mu} &= \frac{k}{\lambda^2\big(\alpha_1-\alpha\big)^2}\Lambda_1^2 - \frac{k}{\lambda^2\big(\alpha_1-\alpha\big)}\Lambda_2 + \frac{1}{\lambda^2}, \\
\frac{\partial \mathcal{L}}{\partial k} &= \frac{\mu}{\lambda^2\big(\alpha_1-\alpha\big)^2}\Lambda_1^2 - \frac{\mu}{\lambda^2\big(\alpha_1-\alpha\big)}\Lambda_2, \\
\frac{\partial^2 \mathcal{L}}{\partial \alpha_1^2} &= \frac{2\lambda + 8\mu k\ln(1-\alpha_1)}{\lambda^2\big(\alpha_1-\alpha\big)^3}\Lambda_1 + \frac{6\mu k}{\lambda^2\big(\alpha_1-\alpha\big)^4}\Lambda_1^2 - \frac{2\mu k}{\lambda^2(\alpha_1-\alpha)^3}\Lambda_2 + \frac{2\mu k}{\lambda^2(\alpha_1-\alpha)^2}\Lambda_3 \\
&\quad + \frac{2\lambda\ln(1-\alpha_1) + 2\mu k\ln^2(1-\alpha_1)}{\lambda^2\big(\alpha_1-\alpha\big)^2} + \frac{\lambda + 2\mu k\ln(1-\alpha_1)}{\lambda^2(\alpha_1-\alpha)(1-\alpha_1)}, \\
\frac{\partial^2 \mathcal{L}}{\partial \mu \partial \alpha_1} &= \frac{\partial^2 \mathcal{L}}{\partial \alpha_1 \partial \mu} = -\frac{2k\ln(1-\alpha_1)}{\lambda^2\big(\alpha_1-\alpha\big)^2}\Lambda_1 - \frac{2k}{\lambda^2\big(\alpha_1-\alpha\big)^3}\Lambda_1^2 + \frac{k}{\lambda^2(\alpha_1-\alpha)^2}\Lambda_2 - \frac{k\ln^2(1-\alpha_1)}{\lambda^2\big(\alpha_1-\alpha\big)}, \\
\frac{\partial^2 \mathcal{L}}{\partial k \partial \alpha_1} &= \frac{\partial^2 \mathcal{L}}{\partial \alpha_1 \partial k} = -\frac{2\mu\ln(1-\alpha_1)}{\lambda^2\big(\alpha_1-\alpha\big)^2}\Lambda_1 - \frac{2\mu}{\lambda^2\big(\alpha_1-\alpha\big)^3}\Lambda_1^2 + \frac{\mu}{\lambda^2(\alpha_1-\alpha)^2}\Lambda_2 - \frac{\mu\ln^2(1-\alpha_1)}{\lambda^2\big(\alpha_1-\alpha\big)}, \\
\frac{\partial^2 \mathcal{L}}{\partial k \partial \mu} &= \frac{\partial^2 \mathcal{L}}{\partial \mu \partial k} = \frac{1}{\lambda^2\big(\alpha_1-\alpha\big)^2}\Lambda_1^2 - \frac{1}{\lambda^2\big(\alpha_1-\alpha\big)}\Lambda_2, \\
\frac{\partial^2 \mathcal{L}}{\partial \mu^2} &= \frac{\partial^2 \mathcal{L}}{\partial k^2} = 0.
\end{aligned}$$