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Representation Matrices of Coprime Graph of Generalized Quaternion Group

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Abstract. This study discusses the representation matrices of the coprime graph of the generalized quaternion group. The representation matrices are adjacency matrix, anti adjacency matrix, Laplacian matrix, and signless Laplacian matrix. Furthermore, the eigenvalues of each representation matrix are determined. As a result, we obtained the construction of the four representation matrices and their eigenvalues. The matrix determinant is zero based on the matrix form, so the matrices have zero eigenvalues except for the signless Laplacian matrix. As for the non-zero eigenvalues, the values depend on the type of representation matrices, the order of the graph, and its algebraic multiplicity.

Key words and Phrases: coprime graph, adjacency-antiadjacency matrices, Laplacian-signless, Laplacian matrices, eigen values

1. INTRODUCTION

A finite group can be represented as a graph. Some research on representation graphs of finite groups have been done, for instance, undirected power graphs of semigroups by Chakrabarty et al. [1], conjugate graphs of finite groups by Erfanian et al. [2], commuting graph of the dihedral group by Ali et al. [3], non-commuting graph of dihedral group by Khasraw et al. [4], twin g-noncommuting graph of a finite group by Zahidah et al. [5], and coprime graph of the generalized quaternion group by Zahidah et al. [6]. Meanwhile, a graph can be represented as a matrix. Research on graphs representation matrices that have been published are the adjacency matrix of circulant graphs on cyclic groups by So [7], adjacency and

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antiadjacency matrix of cyclic directed wheel graphs by Widiastuti et al. [8]. More specifically, some research studied on eigenvalues of representation matrices such as directed prism circle graph by Stin et al. [9], directed dumbbell circle graph by Budiyanto et al. [10] and Cayley graphs of group $\mathbb{Z}_n$ by Daniel et al. [11].

Based on these research developments, we want to continue the study from Zahidah et al. [6]. We determine the construction of representation matrices of obtained graphs on Zahidah et al. [6]. Here, we study four representation matrices, i.e., adjacency matrix, antiadjacency matrix, Laplacian matrix, and signless Laplacian matrix. Furthermore, we determine the eigenvalues of each representation matrix. Notations and terminologies related to groups can be found in Dummit and Fraleigh [12, 13], for graphs can be found in Chartrand et al. and Ma et al. [14, 15], meanwhile for matrices can be found in Anton et al. and Bapat [16, 17]. We remind definitions of the generalized quaternion group Q_{4n} and the coprime graph of a group as follows.

Definition 1.1. *A generalized quaternion group (Q_{4n}) is a finite group of order $4n$, generated by two elements a and b with the properties $a^{2n} = b^4 = e$ and $ab = ba^{-1}$, where e is the identity element of Q_{4n}.*

Definition 1.2. *The coprime graph of a finite group G is a graph with the vertices as elements of G and two vertices are adjacent if and only if its order as group elements is relatively prime.*

The following theorem is a result from Zahidah et al. [6] gives structure of coprime graph of Q_{4n}.

Theorem 1.3. *The coprime graph of the generalized quaternion group Q_{4n} are*

i. a star graph if n is a power of 2, and
ii. a tripartite graph if n is an odd prime.

According to the above objective and previous results, we divide our discussion into four sections based on the type of representation matrices. Each section includes two studies based on the type of graphs used. For simplicity, we denote graph in Theorem 1.3 (i) as G_1 and G_2 for Theorem 1.3 (ii).

2. ADJACENCY MATRIX OF COPRIME GRAPH OF Q_{4n}

In this section, firstly, we determine the construction of the adjacency matrix of G_1 and G_2. As we know, the elements of the adjacency matrix are binary numbers, one if two vertices are adjacent and zero if they are not. Referring to the definition of adjacency matrix and Theorem 1.3, we create the construction of the matrix as shown in the following theorems.

Theorem 2.1. *Let G_1 be the coprime graph of generalized quaternion group with the vertex set $V(G_1) = \{e, a, a^2, \ldots, a^{2n-1}, b, ab, a^2b, \ldots, a^{2n-1}b\}$. Then the adjacency matrix of graph G_1 is*

$$A(G_1) = \begin{bmatrix} [a_j]_{1\times 1} \\ [b_j]_{(4n-1)\times 1} \end{bmatrix}$$

where $[a_j]$ and $[b_j]$ are row vectors of order $1 \times 4n$ and

$$a_j = \begin{cases} 0, & j = 1 \\ 1, & otherwise \end{cases} ; b_j = \begin{cases} 1, & j = 1 \\ 0, & otherwise. \end{cases}$$

Proof. According to Theorem 1.3, graph G_1 is a star graph

$$S_{4n-1} \cong K_{1,4n-1} \cong K_{|V_1|,|V_2|}$$

where $V_1 = \{e\}$ and $V_2 = \{a, a^2, \ldots, a^{2n-1}, b, ab, a^2b, \ldots, a^{2n-1}b\}$. Therefore, vertex e (identity element) is adjacent to every vertex in V_2. Thus, the adjacency matrix of G_1 is

$$A(G_1) = \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

□

Theorem 2.2. *Let G_2 be the coprime graph of generalized quaternion group with the vertex set $V(G_2) = \{a, \ldots, a^{2k+1}, a^n, b, ab, \ldots, a^{2n-1}b, a^2, ..., a^{2k}, e : 1 \leq k \leq n-1, 2k+1 \neq n\}$. Then the adjacency matrix of graph G_2 is*

$$A(G_2) = \begin{bmatrix} [a_j]_{(n-1)\times 1} \\ [b_j]_{(2n+1)\times 1} \\ [c_j]_{(n-1)\times 1} \\ [d_j]_{1\times 1} \end{bmatrix}$$

where $[a_j], [b_j], [c_j]$ and $[d_j]$ are row vectors of order $1 \times 4n$ and

$$a_j = \begin{cases} 1, & j = 4n \\ 0, & otherwise \end{cases} ; b_j = \begin{cases} 1, & j = 3n+1, 3n+2, ..., 4n \\ 0, & otherwise \end{cases} ;$$

$$c_j = \begin{cases} 0, & j = 1, 2, ..., n-1, 3n+1, 3n+2, ..., 4n-1 \\ 1, & otherwise. \end{cases} ; d_j = \begin{cases} 0, & j = 4n \\ 1, & otherwise. \end{cases}$$

Proof. According to Theorem 1.3, graph G_2 is a 3-partite graph with partition set $V_1 = \{a^{2m+1}, a^i b : 0 \leq m \leq n-1, 0 \leq i \leq 2n-1\}, V_2 = \{a^{2m} : 1 \leq m \leq n-1\}$ and $V_3 = \{e\}$, thus $|V_2| = n-1$ and $|V_3| = 1$. Then we divide partition V_1 into two subpartitions, i.e., $S_1 = \{a^{2m+1} : 0 \leq m \leq n-1, 2m+1 \neq n\}$ and $S_2 = \{a^n, a^i b : 0 \leq i < 2n-1\}$, thus $|S_1| = n-1$ and $|S_2| = 2n+1$. Therefore, every vertex in S_1 is only adjacent to every vertice in V_3, while for every vertex in S_2 is adjacent to every vertice in V_3 and V_2. For other adjacencies, every vertex in V_2 is adjacent to every vertex in V_3 and S_2, and lastly, every vertex in V_3 is adjacent to every vertex in V_2 and V_1. Hence, the adjacency matrix of G_2 is

$$A(G_2) = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

□

Based on Theorem 2.1 and Theorem 2.2, it can be seen that the adjacency matrix of both graphs G_1 and G_2 have some rows with the same entries. Hence, the vectors are not linearly independent and we get the following corollary.

Corollary 2.3. *Determinant of the adjacency matrix of graphs G_1 and G_2 are zero. Thus, zero is one of its eigenvalues.*

In the following discussion, we determine the eigenvalues of the adjacency matrix using MATLAB that refer to construction in Theorem 2.1 and Theorem 2.2. The program's output can be seen in Table 1 and Table 2 for the matrix of graphs G_1 and G_2, respectively. We use the result to determine the pattern of the eigenvalues and then prove it theoretically.

Based on the results in Table 1, we get the eigenvalues of the adjacency matrix as follows.

Theorem 2.4. *Let A be the adjacency matrix of graph G_1. Then the eigenvalues of A are 0 and $\pm\sqrt{4n-1}$ where the algebraic multiplicity concerning each eigenvalue are $4n-1$ and 1, respectively.*

TABLE 1. Eigenvalues of The Adjacency Matrix of G_1

k	n	λ_1	λ_2	λ_3
1	2	0	$-2.6458 = -\sqrt{7}$	$2.6458 = \sqrt{7}$
2	4	0	$-3.8730 = -\sqrt{15}$	$3.8730 = \sqrt{15}$
6	64	0	$-15.9687 = -\sqrt{255}$	$15.9687 = \sqrt{255}$
9	512	0	$45.2438 = -\sqrt{2047}$	$45.2438 = \sqrt{2047}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	2^k	0	$-\sqrt{4n-1}$	$\sqrt{4n-1}$
Algebraic multiplicity		$4n-2$	1	1

Proof. Let A be the adjacency matrix of graph G_1. According to Theorem 2.4, matrix A can be written as

$$A = \begin{bmatrix} 0 & a_2 & a_3 & \cdots & a_{4n} \\ a_2 & 0 & 0 & \cdots & 0 \\ a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{4n} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $a_2, a_3, \ldots, a_{4n} = 1$, hence the polynomial characteristic of A is

$$|\lambda I - A| = \lambda^{4n-2}(\lambda^2 - 4n + 1). \tag{1}$$

We will prove Equation 1 by mathematical induction through the second row expansion. For $n = 2$, it is easy to get $|\lambda I - A| = \lambda^6(\lambda^2 - 7)$. Suppose for $n = p$ we have

$$|\lambda I - A| = \begin{bmatrix} \lambda & -a_2 & -a_3 & \cdots & -a_{4p} \\ -a_2 & \lambda & 0 & \cdots & 0 \\ -a_3 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{4p} & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

where $a_2, a_3, \ldots, a_{4p} = 1$ and

$$|\lambda I - A| = \lambda^{4p-2}(\lambda^2 - 4p + 1). \tag{2}$$

Now, observe for $n = p + 1$, then we have

$$|\lambda I - A| = \begin{bmatrix} \lambda & -a_2 & -a_3 & \cdots & -a_{4(p+1)} \\ -a_2 & \lambda & 0 & \cdots & 0 \\ -a_3 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{4(p+1)} & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

where $a_2, a_3, \ldots, a_{4(p+1)} = 1$. We can see that the order of matrix $(\lambda I - A)$ above is $(4p + 4) \times (4p + 4)$. Thus, to determine its determinant, firstly we need to determine the determinant of three matrices with the same construction of order

$(4p+1)\times(4p+1), (4p+2)\times(4p+2),$ and $(4p+3)\times(4p+3)$ which are explained as follows.

i.: Suppose X is a matrix of order $(4p+1)\times(4p+1)$ defined as

$$X = \begin{bmatrix} \lambda & -a_2 & -a_3 & \cdots & -a_{4p+1} \\ -a_2 & \lambda & 0 & \cdots & 0 \\ -a_3 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{4p+1} & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

where $a_2, a_3, \ldots, a_{4p+1} = 1$. By the second row expansion and based on Equation 2 we get

$$\det(X) = (-1)(-1)(-a_2)\lambda^{4p-1} + \lambda(\lambda^{4p-2}(\lambda^2 - 4p + 1)) = \lambda^{4p-1}(\lambda^2 - 4p). \quad (3)$$

ii.: Suppose Y is a matrix of order $(4p+2)\times(4p+2)$. Using similar way and Equation 3, then we get

$$\det(Y) = (-1)(-1)(-a_2)\lambda^{4p} + \lambda(\lambda^{4p-1}(\lambda^2 - 4p)) = \lambda^{4p}(\lambda^2 - 4p - 1). \quad (4)$$

iii.: Suppose Z is a matrix of order $(4p+3)\times(4p+3)$. Using similar way and Equation 4, then we get

$$\det(Z) = (-1)(-1)(-a_2)\lambda^{4p+1} + \lambda(\lambda^{4p}(\lambda^2 - 4p - 1)) = \lambda^{4p+1}(\lambda^2 - 4p - 2). \quad (5)$$

Therefore, to determine the determinant of matrix $|\lambda I - A|$ of order $(4p+4)\times(4p+4)$, we use similar way and Equation 5. Therefore

$$|\lambda I - A| = (-1)(-1)(b_2)\lambda^{4p+2} + \lambda(\lambda^{4p+1}(\lambda^2 - 4p - 2)) = \lambda^{4(p+1)-2}(\lambda^2(p+1)+1). \quad (6)$$

It means that we proved Equation 1, hence we get the eigenvalues of A are the roots of $\lambda^{4n-2}(\lambda^2 - 4n + 1) = 0$ i.e., $\lambda = 0$ or $\lambda = \pm\sqrt{4n-1}$. Furthermore, the algebraic multiplicity corresponding to the eigenvalue $\lambda = 0, \lambda = -\sqrt{4n-1}$, and $\lambda = \sqrt{4n-1}$ are $4n-2, 1$, and 1, respectively. □

Corollary 2.5. *The adjacency matrix of graph G_1 has two non-zero eigenvalues.*

Corollary 2.6. *The energy of graph G_1 is $E(G_1) = \sum_{i=1}^{4n} |\lambda_i| = 2\sqrt{4n-1}$.*

Now, we discuss the eigenvalues of the adjacency matrix of graph G_2. The output of the program in MATLAB can be seen in Table 2.

TABLE 2. Eigenvalues of The Adjacency Matrix of G_2

n	λ_1	λ_2	λ_3	λ_4	λ_5
3	-4.0458	-2	0.6401	5.4057	0
5	-6.8012	-2.7429	1.1199	8.4241	0
7	-9.6144	-3.2517	1.5277	11.3434	0
61	-85.9733	-8.6996	6.7321	87.9408	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
523	-739.3407	-23.8346	21.8384	741.3369	0
Algebraic multiplicity	1	1	1	1	$4n-4$

As we can see at the Table 2, there is no pattern of nonzero eigenvalues of the adjacency matrix of graph G_2. Therefore, in this case we only discuss on the number of zero and non-zero eigenvalues.

Theorem 2.7. *Let A be the adjacency matrix of graph G_2. Then A has four non-zero eigenvalues with each algebraic multiplicity is one and zero eigenvalues with algebraic multiplicity is $4n-4$.*

Proof. Suppose A is the adjacency matrix of graph G_2, then based on Theorem 2.2, we have four vectors, which are

$$\begin{aligned}[a_j] &= [0\ \ 0\cdots0\ \ 0\cdots0\ \ 0\ \ 1]\\ [b_j] &= [0\ \ 0\cdots0\ \ 0\cdots1\ \ 1\ \ 1]\\ [c_j] &= [0\ \ 0\cdots1\ \ 1\cdots0\ \ 0\ \ 1]\\ [d_j] &= [1\ \ 1\cdots1\ \ 1\cdots1\ \ 1\ \ 0].\end{aligned}$$

We will show that the vectors are linearly independent. Observe that

$$\alpha_1[a_j]+\alpha_2[b_j]+\alpha_3[c_j]+\alpha_4[d_j]=\mathbf{0}=[0\ \ 0\cdots0\ \ 0\cdots0\ \ 0\ \ 0].$$

Thus, we get $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$, that implies the set of vectors

$$\{[a_j],[b_j],[c_j],[d_j]\}$$

is linearly independent. Since the vectors are linearly independent, then the $rank(A)$ is four. Furthermore, since the adjacency matrix is a symmetry matrix, then matrix A has four non-zero eigenvalues with each algebraic multiplicity is one. Consequently, the algebraic multiplicity of zero eigenvalues of A is $4n-4$. □

3. ANTIADJACENCY MATRIX OF COPRIME GRAPH OF Q_{4n}

In this section, we determine the construction of antiadjacency matrix of graphs G_1 and G_2 and its eigenvalues. Antiadjacency matrix is constructed from adjacency matrix by exchanging the elements of matrix, i.e., the elements 1 into 0 and vice versa. Thus, we also have the following corollary.

Corollary 3.1. *The determinant of the antiadjacency matrix of graphs G_1 and G_2 are zero. Thus, zero is one of its eigenvalues.*

Referring to the construction, we do the computation using MATLAB to determine the eigenvalues of the antiadjacency matrix. The output of the program can be seen in Table 3 and Table 4 for matrix of graphs G_1 and G_2 respectively. We use the result to determine the pattern of the eigenvalues then prove it theoretically. Based on the results in Table 3, we create Theorem 3.4 that shows the eigenvalues of the antiadjacency matrix of graph G_1. However, firstly we need to introduce Lemma 3.2 that will be used for proving the theorem.

Table 3. Eigenvalues of The Antiadjacency Matrix of G_1

k	n	λ_1	λ_2	λ_3
1	2	0	1	7
2	4	0	1	15
6	64	0	1	255
9	512	0	1	2047
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	2^k	0	1	$4n-1$
Algebraic multiplicity		$4n-2$	1	1

Lemma 3.2. *Let $A = [a_{ij}]$ be a matrix of order $n \times n$ defined as*

$$a_{ij} = \begin{cases} 0, & i = 1 \ \textit{and} \ j = 2, 3, ..., n \\ 0, & j = 1 \ \textit{and} \ i = 2, 3, ..., n \\ \lambda - 1, & i = j = 1, 3, 4, ..., n \\ -1, & \textit{otherwise} \end{cases}$$

$$\textit{then } \det(A) = \begin{cases} -\lambda^{n-1} + \lambda^{n-2}, & n \textit{ is an odd number} \\ \lambda^{n-1} - \lambda^{n-2}, & n \textit{ is an even number.} \end{cases}$$

Proof. Based on the definition in Lemma 3.2, matrix A can be written as

$$A = \begin{bmatrix} \lambda - 1 & 0 & 0 & & 0 \cdots & 0 \\ 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & -1 & \lambda - 1 & -1 & \cdots & -1 \\ 0 & -1 & -1 & \lambda - 1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & -1 & -1 & -1 & \cdots & \lambda - 1 \end{bmatrix}$$

By doing elementary row operations $R_{23}(-1), R_{34}(-1), \ldots, R_{(n-1)n}(-1)$, respectively on the matrix A, we get matrix B

$$B = \begin{bmatrix} \lambda - 1 & 0 & 0 & & 0 \cdots & 0 \\ 0 & 0 & -\lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & -\lambda & \cdots & 0 \\ 0 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & -1 & -1 & -1 & \cdots & \lambda - 1 \end{bmatrix}.$$

Since the operation is adding a multiple of one row to another row, then such operations do not change its determinant. Thus we have $\det(B) = \det(A)$. The next operation, we only exchange among the rows of matrix B, i.e., row operations

$R_{n2}, R_{n3}, \ldots, R_{n(n-1)}$, respectively, then we get matrix C

$$C = \begin{bmatrix} \lambda - 1 & 0 & 0 & & 0 \cdots & 0 \\ 0 & -1 & -1 & -1 & \cdots & \lambda - 1 \\ 0 & 0 & -\lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -\lambda \end{bmatrix}$$

and such operations bring up two possibilities for its determinant, i.e.,

$$det(C) = \begin{cases} -\det(B), & n \text{ is an odd number} \\ \det(B), & n \text{ is an even number.} \end{cases}$$

i.: For n is an odd number
By doing elementary row operations $R_{43}(1), R_{54}(1), \ldots, R_{n(n-1)}(1)$, respectively on matrix C, we get upper triangular matrix D

$$D = \begin{bmatrix} \lambda - 1 & 0 & 0 & & 0 \cdots & 0 \\ 0 & -1 & -1 & -1 & \cdots & \lambda - 1 \\ 0 & 0 & -\lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -\lambda \end{bmatrix}$$

and such operations do not change its determinant, thus we have $\det(D) = \det(C)$. Since the matrix $D_{n\times n}$ is an upper triangular matrix, then $\det(D)$ is the product of its diagonal elements, i,e.,

$$\det(D) = (\lambda - 1)(-1)(-\lambda)^{n-2} = \lambda^{n-1} - \lambda^{n-2}.$$

Since $\det(D) = \det(C) = -\det(B) = -\det(A)$, then $\det(A) = -\lambda^{n-1} + \lambda^{n-2}$.

ii.: For n is an even number
By doing similar way as (i), we get

$$\det(D) = (\lambda - 1)(-1)(-\lambda)^{n-2} = \lambda^{n-1} - \lambda^{n-2}.$$

since $\det(D) = \det(C) = \det(B) = \det(A)$, then $\det(A) = \lambda^{n-1} - \lambda^{n-2}$.

□

In addition to the above lemma, we define a new matrix by doing a row operation R_{23} on matrix A at Lemma 3.2 to get following corollary.

Corollary 3.3. *Let E be a matrix of order $n \times n$ defined as*

$$E = \begin{bmatrix} \lambda-1 & 0 & 0 & & 0\cdots & 0 \\ 0 & -1 & \lambda-1 & -1 & \cdots & -1 \\ 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & -1 & -1 & \lambda-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & -1 & -1 & -1 & \cdots & \lambda-1 \end{bmatrix}$$

then $\det(E) = \begin{cases} \lambda^{n-1} - \lambda^{n-2}, & n \text{ is an odd number} \\ -\lambda^{n-1} + \lambda^{n-2}, & n \text{ is an even number.} \end{cases}$

Theorem 3.4. *Let B be the antiadjacency matrix of graph G_1. Then the eigenvalues of B are $0, 1,$ and $4n-1$ where the algebraic multiplicity with respect to each eigenvalue are $4n-2, 1,$ and 1, respectively.*

Proof. Suppose B is the antiadjacency matrix of graph G_1. Referring to the construction, matrix B can be written as

$$B = \begin{bmatrix} 0 & a_2 & a_3 & \cdots & a_{4n} \\ a_2 & 1 & 1 & \cdots & 1 \\ a_3 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{4n} & 1 & 1 & \cdots & 1 \end{bmatrix}$$

where $a_2, a_3, \ldots, a_{4n} = 0$, hence the polynomial characteristic of B is

$$|\lambda I - B| = \lambda^{4n} - (4n)\lambda^{4n-1} + (4n-1)\lambda^{4n-2}. \tag{7}$$

We will prove Equation 7 by mathematical induction through the second row expansion. For $n = 2$, we have

$$|\lambda I - B| = \begin{bmatrix} \lambda-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda-1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & \lambda-1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & \lambda-1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & \lambda-1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & \lambda-1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & \lambda-1 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 & \lambda-1 \end{bmatrix}$$

By the second row expansion and referring Lemma 3.2 and Corollary 3.3, we have

$$\begin{aligned} |\lambda I - B| = {} & 0 + (\lambda-1)(\lambda^7 - 7\lambda^6 + 6\lambda^5) + (-\lambda^6 + \lambda^5) + (-\lambda^6 + \lambda^5) + \\ & (-\lambda^6 + \lambda^5) + (-\lambda^6 + \lambda^5) + (-\lambda^6 + \lambda^5) + (-\lambda^6 + \lambda^5) \\ = {} & \lambda^8 - 8\lambda^7 + 7\lambda^6. \end{aligned}$$

Suppose for $n = p$ we have

$$(\lambda I - B) = \begin{bmatrix} \lambda-1 & a_2 & a_3 & a_4 & \cdots & a_{4p} \\ a_2 & \lambda-1 & -1 & -1 & \cdots & -1 \\ a_3 & -1 & \lambda-1 & -1 & \cdots & -1 \\ a_4 & -1 & -1 & \lambda-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{4p} & -1 & -1 & -1 & \cdots & \lambda-1 \end{bmatrix}$$

where $a_2, a_3, \ldots, a_{4p} = 0$ and

$$|\lambda I - B| = \lambda^{4p} - (4p)\lambda^{4p-1} + (4p-1)\lambda^{4p-2}. \tag{8}$$

Now, observe for $n = p + 1$, then we have

$$(\lambda I - B) = \begin{bmatrix} \lambda-1 & a_2 & a_3 & a_4 & \cdots & a_{4(p+1)} \\ a_2 & \lambda-1 & -1 & -1 & \cdots & -1 \\ a_3 & -1 & \lambda-1 & -1 & \cdots & -1 \\ a_4 & -1 & -1 & \lambda-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{4(p+1)} & -1 & -1 & -1 & \cdots & \lambda-1 \end{bmatrix}$$

where $a_2, a_3, \ldots, a_{4(p+1)} = 0$. Since the order of matrix $(\lambda I - B)$ is $(4p+4) \times (4p+4)$, then we use similar way as Theorem 2.4 to determine $|\lambda I - B|$.

i.: Suppose X is a matrix of order $(4p + 1) \times (4p + 1)$ defined as

$$X = \begin{bmatrix} \lambda-1 & a_2 & a_3 & a_4 & \cdots & a_{4p+1} \\ a_2 & \lambda-1 & -1 & -1 & \cdots & -1 \\ a_3 & -1 & \lambda-1 & -1 & \cdots & -1 \\ a_4 & -1 & -1 & \lambda-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{4p+1} & -1 & -1 & -1 & \cdots & \lambda-1 \end{bmatrix}$$

where $a_2, a_3, \ldots, a_{4p+1} = 0$. By the second row expansion and referring Lemma 3.2, Corollary 3.3, and Equation 8 we have

$$\begin{aligned} \det(X) = \; & 0 + (\lambda-1)(\lambda^{4p} - (4p)\lambda^{4p-1} + (4p-1)\lambda^{4p-2}) + (-\lambda^{4p-1} + \lambda^{4p-2} \\ & +(-\lambda^{4p-1} + \lambda^{4p-2}) + \ldots + (-\lambda^{4p-1} + \lambda^{4p-2}) \\ = \; & \lambda^{4p+1} - (4p+1)\lambda^{4p} + (4p)\lambda^{4p-1}. \end{aligned} \tag{9}$$

ii.: Suppose Y is a matrix of order $(4p+2) \times (4p+2)$. Using similar way and Equation 9, then we have

$$\begin{aligned} \det(Y) = \; & 0 + (\lambda-1)(\lambda^{4p+1} - (4p+1)\lambda^{4p} + (4p)\lambda^{4p-1}) + (-\lambda^{4p} + \lambda^{4p-1} \\ & +(-\lambda^{4p} + \lambda^{4p-1}) + \ldots + (-\lambda^{4p} + \lambda^{4p-1}) \\ = \; & \lambda^{4p+2} - (4p+2)\lambda^{4p+1} + (4p+1)\lambda^{4p}. \end{aligned} \tag{10}$$

iii.: Suppose Z is a matrix of order $(4p+3)\times(4p+3)$. Using similar way and Equation 10, then we have

$$\begin{aligned}\det(Z) = &\ 0+(\lambda-1)(\lambda^{4p+2}-(4p+2)\lambda^{4p+1}+(4p+1)\lambda^{4p})+(-\lambda^{4p+1}+\lambda^{4p}\\ &+(-\lambda^{4p+1}+\lambda^{4p})+...+(-\lambda^{4p+1}+\lambda^{4p})\\ = &\ \lambda^{4p+3}-(4p+3)\lambda^{4p+2}+(4p+2)\lambda^{4p+1}.\end{aligned} \tag{11}$$

Therefore, to determine the determinant of matrix $(\lambda I - B)$ of order $(4p+4)\times(4p+4)$, we use similar way and Equation 11. Therefore

$$\begin{aligned}|\lambda I - B| = &\ 0+(\lambda-1)(\lambda^{4p+3}-(4p+3)\lambda^{4p+2}+(4p+2)\lambda^{4p+1})+\\ &(-\lambda^{4p+2}+\lambda^{4p+1})+(-\lambda^{4p+2}+\lambda^{4p+1})+...+(-\lambda^{4p+2}+\lambda^{4p+1})\\ = &\ \lambda^{4(p+1)}-(4(p+1))\lambda^{4p+3}+(4p+3)\lambda^{4p+2}.\end{aligned} \tag{12}$$

It means that we proved Equation 7, hence we get the eigenvalues of B are $\lambda^{4n-2}(\lambda^2-(4n\lambda)+(4n-1))=0$ i.e., $\lambda=0$ or $\lambda=1$ or $\lambda=4n-1$. Furthermore, the algebraic multiplicity corresponding to the eigenvalue $\lambda=0, \lambda=1$, and $\lambda=4n-1$ are $4n-2, 1$, and 1, respectively. □

Corollary 3.5. *The antiadjacency matrix of graph G_1 has two non-zero eigenvalues.*

Corollary 3.6. *The energy of graph G_1 is $E(G_1)=\sum_{i=1}^{4n}|\lambda_i|=4n$.*

Hereafter, we discuss the eigenvalues of the antiadjacency matrix of graph G_2. The output of the program in MATLAB can be seen in the following table.

TABLE 4. Eigenvalues of The Adjacency Matrix of G_2

n	λ_1	λ_2	λ_3	λ_4	λ_5
3	-1.0504	1	2.9195	9.1309	0
5	-2.0301	1	5.6291	15.4010	0
7	-3	1	8.2918	21.7082	0
61	-29.0012	1	79.1868	192.8144	0
⋮	⋮	⋮	⋮	⋮	⋮
523	-251.3	1	684.9	1657.4	0
Algebraic multiplicity	1	1	1	1	$4n-4$

As we can see at Table 4, we have similar result as the previous section. There is no pattern of nonzero eigenvalues of the antiadjacency matrix of graph G_2. Therefore, in this case we only discuss on the number of zero and non-zero eigenvalues.

Theorem 3.7. *Let B be the antiadjacency matrix of graph G_2. Then B has four non-zero eigenvalues with each algebraic multiplicity is one and zero eigenvalues with algebraic multiplicity is $4n-4$.*

Proof. Using similar way to the proof of Theorem 3.4, it is easy to get that the $rank(B)$ is four. Furthermore, since the antiadjacency matrix is also a symmetry matrix, then matrix B has four non-zero eigenvalues with each algebraic multiplicity is one. Consequently, the algebraic multiplicity of zero eigenvalues of B is $4n-4$. □

4. LAPLACIAN MATRIX OF COPRIME GRAPH OF Q_{4n}

In this section, we determine the construction of Laplacian matrix of graphs G_1 and G_2 and its eigenvalues. Laplacian matrix is constructed from degree matrix and adjacency matrix. Referring the definition of Laplacian matrix and Theorem 1.3, we create the construction of the matrix as shown on the following theorems.

Theorem 4.1. *Let G_1 be the coprime graph of generalized quaternion group with the vertex set $V(G_1) = \{e, a, a^2, \dots, a^{2n-1}, b, ab, a^2b, \dots, a^{2n-1}b\}$. Then the Laplacian matrix of graph G_1 is defined as $L(G_1) = D - A$, where A is the adjacency matrix of G_1 and*

$$D = [d_{ij}], \quad d_{ij} = \begin{cases} 4n-1, & i = j = 1 \\ 1, & i = j = 2, 3, ..., 4n \\ 0, & \textit{otherwise} \end{cases}$$

is the degree matrix of G_1.

Proof. Since we have discussed matrix A in Theorem 2.1, it is sufficient to show the construction of degree matrix D to prove the theorem. Specifically, since degree matrix is a diagonal matrix, we only show the element of diagonal of the matrix. According to proof of Theorem 2.1, we get degree of vertex e is $4n-1$ implies $d_{11} = 4n-1$ and the degree of any vertex in V_2 is 1 implies $d_{ii} = 1$ for $i = 2, 3, \dots, 4n$. □

The general construction of the Laplacian matrix of graph G_1 can be written as

$$L(G_1) = \begin{bmatrix} 4n-1 & -1 & -1 & \dots & -1 & -1 & -1 & -1 & \dots & -1 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Theorem 4.2. *Let G_2 be the coprime graph of generalized quaternion group with the vertex set $V(G_2) = \{a, \dots, a^{2k+1}, a^n, b, ab, \dots, a^{2n-1}b, a^2, ..., a^{2k}, e : 1 \leq k \leq n-1, 2k+1 \neq n\}$. Then the Laplacian matrix of graph G_2 is defined as $L(G_2) =$*

$D - A$, where A is the adjacency matrix of G_2 and

$$D = [d_{ij}], \quad d_{ij} = \begin{cases} 1, & i = j = 1, 2, ..., n-1 \\ n, & i = j = n, n+1, ..., 3n \\ 2(n+1), & i = j = 3n+1, 3n+2, ..., 4n-1 \\ 4n-1, & i = j = 4n \\ 0, & otherwise \end{cases}$$

is the degree matrix of G_2.

Proof. Similar to Theorem 4.1, we only show the element of diagonal of the degree matrix D. According to proof of Theorem 2.2, we get the degree of any vertex in S_1 is 1 implies $d_{ii} = 1$ for $i = 1, 2, \ldots, n-1$. For other vertex partitions, the degree of any vertex in S_2 is n implies $d_{ii} = n$ for $i = n, n+1, \ldots, 3n$. The degree of any vertex in V_2 is $2(n+1)$ implies $d_{ii} = 2(n+1)$ for $i = 3n+1, 3n+2, \ldots, 4n-1$. For the last partition, the degree of vertex e is $4n-1$ implies $d_{(4n)(4n)} = 4n-1$. □

The general construction of the Laplacian matrix of graph G_2 can be written as

$$L(G_2) = \begin{bmatrix} 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & -1 \\ 0 & \dots & 0 & n & 0 & 0 & \dots & 0 & -1 & \dots & -1 & -1 \\ 0 & \dots & 0 & 0 & n & 0 & \dots & 0 & -1 & \dots & -1 & -1 \\ 0 & \dots & 0 & 0 & 0 & n & \dots & 0 & -1 & \dots & -1 & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots & n & -1 & \dots & -1 & -1 \\ 0 & \dots & 0 & -1 & -1 & -1 & \dots & -1 & 2(n+1) & \dots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & -1 & -1 & \dots & -1 & 0 & \dots & 2(n+1) & -1 \\ -1 & \dots & -1 & -1 & -1 & -1 & \dots & -1 & -1 & \dots & -1 & 4n-1 \end{bmatrix}$$

According to the construction in Theorem 4.1 and Theorem 4.2, we can see that the sum of all the entries of each row/column is zero. Hence, the row vectors are not linearly independent and we get the following corollary.

Corollary 4.3. *The determinant of the Laplacian matrix of graphs G_1 and G_2 are zero. Thus, zero is one of its eigenvalues.*

The next discussion, we determine the eigenvalues of the Laplacian matrix using MATLAB that refer to construction in Theorem 4.1 and Theorem 4.2. The output of the program can be seen in Table 5 and Table 6 for the matrix of graphs G_1 and G_2 respectively. We use the result to determine the pattern of the eigenvalues then prove it theoretically.

Table 5. Eigenvalues of the Laplacian Matrix of G_1

k	n	λ_1	λ_2	λ_3
1	2	1	0	8
2	4	1	0	16
6	64	1	0	256
9	512	1	0	2048
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	2^k	1	0	$4n$
Algebraic multiplicity		$4n-2$	1	1

Based on the results in Table 5, we create theorem that shows the eigenvalues of the Laplacian matrix of graph G_1 as follows.

Theorem 4.4. *Let L be the Laplacian matrix of graph G_1. Then the eigenvalues of L are $1, 0$ and $4n$ where the algebraic multiplicity with respect to each eigenvalue are $4n-2, 1$ and 1 respectively.*

Proof. Suppose L is the Laplacian matrix of graph G_1, then based on Theorem 4.1, we have matrix

$$(\lambda I - L) = \begin{bmatrix} \lambda - 4n + 1 & a_2 & a_3 & a_4 & \cdots & a_{4n} \\ a_2 & \lambda - 1 & 0 & 0 & \cdots & 0 \\ a_3 & 0 & \lambda - 1 & 0 & \cdots & 0 \\ a_4 & 0 & 0 & \lambda - 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{4n} & 0 & 0 & 0 & \cdots & \lambda - 1 \end{bmatrix}$$

where $a_2, a_3, \ldots, a_{4n} = 1$. We determine the determinant through the first row expansion to obtain triangular matrix. Hence, we have

$$\begin{aligned} |\lambda I - L| &= (\lambda - 4n + 1)(\lambda - 1)^{4n-1} - (\lambda - 1)^{4n-2} - (\lambda - 1)^{4n-2} - ... - (\lambda - 1)^{4n-2} \\ &= (\lambda - 1)^{4n-2}\lambda(\lambda - 4n). \end{aligned}$$

Consequently, the eigenvalues of L are $(\lambda-1)^{4n-2}\lambda(\lambda-4n) = 0$, i.e., $\lambda = 1$ or $\lambda = 0$ or $\lambda = 4n$. Furthermore, the algebraic multiplicity corresponding to the eigenvalue $\lambda = 1, \lambda = 0$, and $\lambda = 4n$ are $4n-2, 1$, and 1 respectively. □

Corollary 4.5. *Laplacian matrix of graph G_1 has two non-zero eigenvalues.*

Now, we discuss the eigenvalues of the Laplacian matrix of graph G_2. The output of the program in MATLAB can be seen in the following table. Based on the results in Table 6, we create Theorem 4.7 that shows the eigenvalues of the Laplacian matrix of graph G_2. However, firstly we need to introduce Lemma 4.6 that will be used for proving the theorem.

Table 6. Eigenvalues of the Laplacian Matrix of G_2

n	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
3	1	3	0	10	12	8
5	1	5	0	16	20	12
7	1	7	0	22	28	16
61	1	61	0	183	244	124
223	1	223	0	670	892	448
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	n	0	$3n+1$	$4n$	$2(n+1)$
Algebraic multiplicity	$n-1$	$2n$	1	1	1	$n-2$

Lemma 4.6. *Let n be an odd prime number and define matrix $A = [a_{ij}]$ of order $n \times n$ as*

$$a_{ij} = \begin{cases} \frac{\lambda^2-(3n+2)\lambda+(2n^2-1)}{\lambda-n}, & i=j=1,2,...,4n-1 \\ \frac{\lambda^3-5n\lambda^2+(4n^2+n-1)\lambda-(3n^2-2n-1)}{\lambda^2-(n+1)\lambda+n}, & i=j=4n \\ \frac{\lambda-(3n+1)}{\lambda-n}, & i=1,2,...,4n-1 \text{ and } j=4n \\ \frac{-(2n+2)}{\lambda-n}, & \text{otherwise} \end{cases}$$

then $\det(A) = \frac{\lambda(\lambda-(3n+1))(\lambda-4n)(\lambda-(2n+2))^{n-2}}{\lambda-n}$.

Proof. According to the definition, matrix A can be written as

$$A = \begin{bmatrix} y & \frac{-(2n+2)}{\lambda-n} & \cdots & \frac{\lambda-(3n+1)}{\lambda-n} \\ \frac{-(2n+2)}{\lambda-n} & y & \cdots & \frac{\lambda-(3n+1)}{\lambda-n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda-(3n+1)}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} & \cdots & x \end{bmatrix}$$

where $x = \frac{\lambda^3-5n\lambda^2+(4n^2+n-1)\lambda-(3n^2-2n-1)}{\lambda^2-(n+1)\lambda+n}$ and $y = \frac{\lambda^2-(3n+2)\lambda+(2n^2-1)}{\lambda-n}$.

We are going to determine the determinant of matrix A by conducting elementary row operations such that we get an upper triangular matrix. Applying elementary row operations $R_{12}, R_{23}, \ldots, R_{(n-1)n}$, respectively on matrix A, we get matrix

$$B = \begin{bmatrix} \frac{-(2n+2)}{\lambda-n} & y & \cdots & \frac{-(2n+2)}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} \\ \frac{-(2n+2)}{\lambda-n} & \frac{-(2n+2)}{\lambda-n} & \cdots & \frac{-(2n+2)}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\lambda-(3n+1)}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} & \cdots & \frac{\lambda-(3n+1)}{\lambda-n} & x \\ y & \frac{-(2n+2)}{\lambda-n} & \cdots & \frac{-(2n+2)}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} \end{bmatrix}.$$

Since the number of operation is $n-1$ for n is an odd number, then we have $\det(B) = \det(A)$. The next operation for matrix B, we conduct operations $R_{(i+1)i}(-1)$,

$R_{(i+2)i}(-1), \ldots, R_{(n-2)i}(-1)$, respectively for $i = 1, 2, \ldots, n-3$, then we get matrix

$$C = \begin{bmatrix} -\frac{2n+1}{\lambda-n} & y & \cdots & -\frac{2n+1}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} \\ 0 & -\lambda+(2n+2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\lambda-(3n+1)}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} & \cdots & \frac{\lambda-(3n+1)}{\lambda-n} & x \\ y & -\frac{2n+1}{\lambda-n} & \cdots & -\frac{2n+1}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} \end{bmatrix}.$$

Since the operation is adding a multiple of one row to another row, then such operations do not change its determinant. Thus, we have $\det(C) = \det(B)$. The next operation for matrix C, we conduct operations $R_{(n-1)i}(\frac{\lambda-(3n+1)}{2n+1})$ for $i = 1, 2, \ldots, n-2$, then we get matrix

$$D = \begin{bmatrix} -\frac{2n+1}{\lambda-n} & y & \cdots & -\frac{2n+1}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} \\ 0 & -\lambda+(2n+2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z & \vdots \\ y & -\frac{2n+1}{\lambda-n} & \cdots & -\frac{2n+1}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} \end{bmatrix}$$

and such operation do not change its determinant, thus we have $\det(D) = \det(C)$. The last row operations are $R_{ni}\left(\frac{\lambda^2-(3n+2)\lambda+(2n^2+2n+(-2n-1)i)}{2n+1}\right)$ for $i = 1, 2, \ldots, n-2$ along with $R_{n(n-1)}(-\lambda+1)$ on matrix D such that we get matrix

$$E = \begin{bmatrix} -\frac{2n+1}{\lambda-n} & y & \cdots & -\frac{2n+1}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} \\ 0 & -\lambda+(2n+2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z & \vdots \\ 0 & 0 & \cdots & 0 & -\lambda^2+4n\lambda \end{bmatrix}$$

such operation do not change its determinant as well, thus we have $\det(E) = \det(D)$. Since matrix E is an upper triangular matrix, then we have

$$\begin{aligned} \det(A) &= \det(E) = (-\tfrac{2n+1}{\lambda-n})(\lambda+(2n+2))^{n-3}(\tfrac{(\lambda-(3n+1))(\lambda-(2n+2))}{2n+1})(-\lambda^2+4n) \\ &= \frac{\lambda(\lambda-(3n+1))(\lambda-4n)(\lambda-(2n+2))^{n-2}}{\lambda-n}. \end{aligned}$$

□

Theorem 4.7. *Let L be the Laplacian matrix of graph G_2. Then the eigenvalues of L are $1, n, 0, 3n+1, 4n$, and $2(n+1)$ where the algebraic multiplicity with respect to each eigenvalue are $n-1, 2n, 1, 1, 1$, and $n-2$, respectively.*

Proof. Suppose L is the Laplacian matrix of graph G_2. According to Theorem 4.2, the matrix $(\lambda I - L)$ can be written as partition matrix as follows

$$(\lambda I - L) = \begin{bmatrix} E_{(n-1)\times(n-1)} & 0_{(n-1)\times(2n+1)} & A^t_{(n-1)\times n} \\ 0_{(2n+1)\times(n-1)} & D_{(2n+1)\times(2n+1)} & B^t_{(2n+1)\times n} \\ A_{n\times(n-1)} & B_{n\times(2n+1)} & C_{n\times n} \end{bmatrix}$$

where $E = [e_{ij}]$ and $D = [d_{ij}]$ are diagonal matrices with $e_{ii} = \lambda - 1$ and $d_{ij} = \lambda - n$ respectively, $B = [b_{ij}]$ is a matrix of ones, i.e., $b_{ij} = 1$ for all i and j. Meanwhile, for $A = [a_{ij}]$ and $C = [c_{ij}] = [c_{ji}]$ are matrices defined as

$$a_{ij} = \begin{cases} 1, & i = n,\ j = 1,2,...,n-1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and } c_{ij} = \begin{cases} 2(n+1), & i = j = 1,2,...,n-1 \\ 4n-1, & i = j = n \\ 1, & i = n,\ j = 1,2,...,n-1 \\ 0, & \text{otherwise.} \end{cases}$$

We are going to determine the determinant of matrix $(\lambda I - L)$ by conducting elementary row operations such that we get an upper triangular matrix. The first purpose of row operations is to change matrices A and B into zeroes matrices. Applying elementary row operations $R_{(4n)i}(\frac{1}{\lambda-1}), i = 1,2,...,n-1$, then $R_{ij}(\frac{1}{\lambda-1}), i = 3n+1, 3n+2, ..., 4n$ and $j = n, n+1, ..., 3n$, respectively on matrix $(\lambda I - L)$, we get matrix M as follows

$$M_{4n\times 4n} = \begin{bmatrix} E_{(n-1)\times(n-1)} & 0_{(n-1)\times(2n+1)} & A^t_{(n-1)\times n} \\ 0_{(2n+1)\times(n-1)} & D_{(2n+1)\times(2n+1)} & B^t_{(2n+1)\times n} \\ 0_{n\times(n-1)} & 0_{n\times(2n+1)} & F_{n\times n} \end{bmatrix}$$

where

$$F = \begin{bmatrix} \frac{\lambda^2-(3n+2)\lambda+(2n^2-1)}{\lambda-n} & \frac{-(2n+2)}{\lambda-n} & \cdots & \frac{\lambda-(3n+1)}{\lambda-n} \\ \frac{-(2n+2)}{\lambda-n} & \frac{\lambda^2-(3n+2)\lambda+(2n^2-1)}{\lambda-n} & \cdots & \frac{\lambda-(3n+1)}{\lambda-n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda-(3n+1)}{\lambda-n} & \frac{\lambda-(3n+1)}{\lambda-n} & \cdots & \frac{\lambda^3-5n\lambda^2+(4n^2+n-1)\lambda-(3n^2-2n-1)}{\lambda^2-(n+1)\lambda+n} \end{bmatrix}.$$

Since the operation is adding a multiple of one row to another row, then such operations do not change its determinant. Thus, we have $\det(\lambda I - L) = \det(M)$. Since matrix M is an upper triangular matrix, matrices E and D are diagonal matrices, and referring to Lemma 4.6, we have

$$\begin{aligned} \det(M) &= \det(E)\det(D)\det(F) \\ &= (\lambda-1)^{n-1}(\lambda-n)^{2n+1}\frac{\lambda(\lambda-(3n+1))(\lambda-4n)(\lambda-(2n+2))^{n-2}}{\lambda-n} \\ &= (\lambda-1)^{n-1}(\lambda-n)^{2n}\lambda(\lambda-(3n+1))(\lambda-4n)(\lambda-(2n+2))^{n-2}. \end{aligned}$$

Consequently, the eigenvalues of L are $(\lambda-1)^{n-1}(\lambda-n)^{2n}\lambda(\lambda-(3n+1))(\lambda-4n)(\lambda-(2n+2))^{n-2} = 0$, i.e., $\lambda = 1$ or $\lambda = n$ or $\lambda = 0$ or $\lambda = 3n+1$ or $\lambda = 4n$ or $\lambda = 2n+2$. Furthermore, the algebraic multiplicity corresponding to the eigenvalue $\lambda = 1, \lambda = n, \lambda = 0, \lambda = 3n+1, \lambda = 4n$, and $\lambda = 2n+2$ are $n-1, 2n, 1, 1, 1$, and $n-2$, respectively. ☐

Corollary 4.8. *The Laplacian matrix of graph G_2 has five non-zero eigenvalues.*

5. SIGNLESS LAPLACIAN MATRIX OF COPRIME GRAPH OF Q_{4n}

In this section, we determine construction of signless Laplacian matrix of graphs G_1 and G_2 and its eigenvalues. Signless Laplacian matrix is constructed from Laplacian matrix by replacing the elements -1 to 1. In another word, if the Laplacian matrix is defined as $D - A$, then the signless Laplacian matrix is defined as $D + A$. Since the construction only replacing all elements -1 to 1, we directly change the program in MATLAB from Laplacian matrix construction to determine the eigenvalues of signless Laplacian matrix. The output of the program can be seen in Table 7 and Table 8 for the matrix of graphs G_1 and G_2 respectively. We use the result to determine the pattern of the eigenvalues then prove it theoretically. Based on the results in Table 7, we create Theorem 5.1 that shows the eigenvalues of the signless Laplacian matrix of graph G_1.

TABLE 7. Eigenvalues of the Signless Laplacian Matrix of G_1

k	n	λ_1	λ_2	λ_3
1	2	1	0	8
2	4	1	0	16
6	64	1	0	256
9	512	1	0	2048
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	2^k	1	0	$4n$
Algebraic multiplicity		$4n-2$	1	1

Theorem 5.1. *Let S be the signless Laplacian matrix of graph G_1. Then the eigenvalues of S are $1, 0,$ and $4n$ where the algebraic multiplicity with respect to each eigenvalue are $4n-2, 1,$ and 1, respectively.*

Proof. Suppose S is the signless Laplacian matrix of graph G_1, then we have the following matrix

$$(\lambda I - S) = \begin{bmatrix} \lambda - 4n + 1 & b_2 & b_3 & b_4 & \cdots & ab_{4n} \\ b_2 & \lambda - 1 & 0 & 0 & \cdots & 0 \\ b_3 & 0 & \lambda - 1 & 0 & \cdots & 0 \\ b_4 & 0 & 0 & \lambda - 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{4n} & 0 & 0 & 0 & \cdots & \lambda - 1 \end{bmatrix}$$

where $a_2, a_3, \ldots, a_{4n} = -1$. Using similar way to the proof of Theorem 4.4, we get $|\lambda I - S| = (\lambda - 1)^{4n-2}\lambda(\lambda - 4n)$. Consequently, the eigenvalues of S are $\lambda = 1$ or $\lambda = 0$ or $\lambda = 4n$ with the algebraic multiplicity corresponding to the eigenvalue $\lambda = 1, \lambda = 0,$ and $\lambda = 4n$ are $4n-2, 1,$ and 1, respectively. $\square$

Corollary 5.2. *The signless Laplacian matrix of graph G_1 has two non-zero eigenvalues.*

Now, the last discussion, we are going to determine the eigenvalues of the signless Laplacian matrix of graph G_2. The output of the program in MATLAB can be seen in the following table.

TABLE 8. Eigenvalues of The Signless Laplacian Matrix of G_2

n	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
3	1	3	8	0.5140	8.2023	13.2838
5	1	5	12	0.5883	13.9204	21.4913
7	1	7	16	0.6147	19.7851	29.6002
61	1	61	124	0.6617	181.4004	245.9379
223	1	223	448	0.6653	667.3523	893.9823
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	n	$2(n+1)$	a	b	c
Algebraic multiplicity	$n-1$	$2n$	1	1	1	$n-2$

where a, b, c are non-integer eigenvalues of the matrix. Based on the results in Table 8, we create Theorem 5.3 that shows the eigenvalues of the signless Laplacian matrix of graph G_2.

Theorem 5.3. *Let S be the Laplacian matrix of graph G_2. Then the eigenvalues of S are $1, n, 2(n+1), a, b,$ and c where a, b, c are non-integer and the algebraic multiplicity with respect to each eigenvalue are $n-1, 2n, n-2, 1, 1,$ and 1, respectively.*

Proof. Let S be the signless Laplacian matrix of graph G_2. According to definition, we have matrix $(\lambda I - S)$ can be written as partition matrix as follows

$$(\lambda I - S) = \begin{bmatrix} E_{(n-1)\times(n-1)} & 0_{(n-1)\times(2n+1)} & A^t_{(n-1)\times n} \\ 0_{(2n+1)\times(n-1)} & D_{(2n+1)\times(2n+1)} & B^t_{(2n+1)\times n} \\ A_{n\times(n-1)} & B_{n\times(2n+1)} & C_{n\times n} \end{bmatrix}$$

where $E = [e_{ij}]$ and $D = [d_{ij}]$ are diagonal matrices with $e_{ii} = \lambda - 1$ and $d_{ij} = \lambda - n$, respectively, $B = [b_{ij}]$ is a matrix of ones, i.e., $b_{ij} = -1$ for all i and j. Meanwhile, for $A = [a_{ij}]$ and $C = [c_{ij}] = [c_{ji}]$ are matrices defined as

$$a_{ij} = \begin{cases} -1, & i = n, \ j = 1, 2, ..., n-1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and } c_{ij} = \begin{cases} 2(n+1), & i = j = 1, 2, ..., n-1 \\ 4n-1, & i = j = n \\ -1, & i = n, \ j = 1, 2, ..., n-1 \\ 0, & \text{otherwise.} \end{cases}$$

To determine the determinant of $(\lambda I - S)$, we use similar way to the proof of Theorem 4.7 untill we get an upper triangular matrix M as follows

$$M_{4n\times 4n} = \begin{bmatrix} E_{(n-1)\times(n-1)} & 0_{(n-1)\times(2n+1)} & A^t_{(n-1)\times n} \\ 0_{(2n+1)\times(n-1)} & D_{(2n+1)\times(2n+1)} & B^t_{(2n+1)\times n} \\ 0_{n\times(n-1)} & 0_{n\times(2n+1)} & F_{n\times n} \end{bmatrix}$$

where

$$F = \begin{bmatrix} \frac{\lambda^2-(3n+2)\lambda+(2n^2-1)}{\lambda-n} & -\frac{2n+1}{\lambda-n} & \cdots & -\frac{\lambda+n+1}{\lambda-n} \\ -\frac{2n+1}{\lambda-n} & \frac{\lambda^2-(3n+2)\lambda+(2n^2-1)}{\lambda-n} & \cdots & -\frac{\lambda+n+1}{\lambda-n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\lambda+n+1}{\lambda-n} & -\frac{\lambda+n+1}{\lambda-n} & \cdots & \frac{\lambda^3-5n\lambda^2+(4n^2+n-1)\lambda-(3n^2-2n-1)}{\lambda^2-(n+1)\lambda+n} \end{bmatrix}.$$

Next, to determine the determinant of matrix F, we use similar way to the proof of Lemma 4.6 and we get

$$\begin{aligned} \det(M) = {} & \det(E)\det(D)\det(F) \\ = {} & (\lambda-1)^{n-1}.(\lambda-n)^{2n+1}. \\ & \frac{(\lambda-2(n+1))^{n-2}(\lambda^3-(7n+1)\lambda^2+4n(3n+1)\lambda-4(2n+1)(n-1))}{\lambda-n} \\ = {} & (\lambda-1)^{n-1}.(\lambda-n)^{2n}.(\lambda-2(n+1))^{n-2}. \\ & (\lambda^3-(7n+1)\lambda^2+4n(3n+1)\lambda-4(2n+1)(n-1)). \end{aligned}$$

Consequently, the eigenvalues of S are $1, n, 2(n+1), a, b$, and c where a, b, c are non-integer and the algebraic multiplicity with respect to each eigenvalue are $n-1, 2n, n-2, 1, 1$, and 1, respectively. □

Corollary 5.4. *Signless Laplacian matrix of graph G_2 has six non-zero eigenvalues and has no zero eigenvalue.*

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