J. Indones. Math. Soc.
Vol. 31, No. 03 (2025), pp. 1–11.

On Basic Ideal Generated by Vertices in Cycles without Exits in Leavitt Path Algebras

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Abstract. Vertices located on cycles without exits have a role in constructing ideals in the Leavitt path algebras over a commutative unital ring. One key reason is that the set of such vertices is hereditary. In addition, an ideal of the commutative unital ring can be combined with these vertices to form an ideal in the Leavitt path algebra. This article focuses on creating a (basic) ideal in the Leavitt path algebras, which is generated by vertices on cycles without exits.

Key words and Phrases: vertices on cycles without exits, basic ideal, Leavitt path algebras.

1. INTRODUCTION

In addition to geometric, combinatoric, and algorithmic approaches, graphs can be viewed algebraically, commonly called graph algebra. In graph algebra, a graph consisting of vertices (or points) and edges is a directed graph, abbreviated as a digraph or a quiver. The direction of edges in a quiver or digraph (called a graph) forms two mappings that define the source and the target of the edges. A graph or quiver is defined by Assem et al. [1] as a 4-tupel $Q = \left(Q^0, Q^1, s, t\right)$ with Q^0 is a set of vertices, Q^1 is a set of edges, and $s, t : Q^1 \to Q^0$ are two functions that define the source and target of the edges, respectively.

A sequence of n edges $e_1e_2 \cdots e_n$ where the target of the i-th edge equals to the source of the $(i+1)$-th for each $i = 1, 2, \cdots, n-1$, is called a path of length n. The algebraic study of quivers leads to the development of an associative algebra known as path algebra and Leavitt path algebra. An in-depth discussion of path algebra on quiver Q over the field K denoted KQ was conducted by Assem et al. [1]. The semiprimeness of path algebras over the field has been studied by

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2020 Mathematics Subject Classification: 16S88.
Received: 26-07-2024, accepted: 08-03-2025.

Molina [2] and Pino et al. [3]. Additionally, they explored the Leavitt path algebra, whose definition is based on the path algebra with Cuntz-Krieger conditions. Meanwhile, Wardati et al. [4] investigated the properties of prime path algebra over the commutative unital ring.

The development of Leavitt path algebras started in 2005, but it is still being studied today. Key findings in the study of Leavitt path algebras include the necessary and sufficient conditions for a graph (quiver) such that the Leavitt path algebra over a field is simple [5], finite-dimensional [6], either artinian or noetherian [7], a prime algebra [8].

Tomforde [9] has studied the Leavitt path algebra over the commutative unital ring, with significant findings, including the definition of basic ideal and simple algebra. These findings are the basic for discovering the necessary and sufficient conditions of the primeness of the Leavitt path algebra over an integral domain [10]. In her dissertation, Wardati et al. [11] identified the necessary and sufficient conditions from the graph that make the basic ideal in the Leavitt path algebra over the commutative unital ring a prime basic ideal. Consequently, the necessary and sufficient conditions were also established for this algebra to be basically prime. In addition, any Leavitt path algebra over the commutative unital ring is a basically semiprime algebra. This property is similar to the findings of Pino et al. [3], which state that every Leavitt path algebra over a field is a semiprime algebra and further determines the socle of the Leavitt path algebra. However, a Leavitt path algebra over a commutative unital ring is not always semiprime unless the ring itself is semiprime. It means that the Leavitt path algebra is semiprime if and only if its algebra over the commutative unital ring is a semiprime [12]. Further research by Wardati [13] found the socle of this semiprime Leavitt path algebra, which is influenced by the ideal of its semiprime ring. This finding differs from the socle of Leavitt path algebra over the field [14] because each field has no non-trivial ideal. A notable similarity, however, is that the determination of this socle is always related to minimal left ideals of the Leavitt path algebra. These minimal left ideals are constructed by a specific vertex in its quiver, referred to as line points, which correspond to primitive idempotents. Moreover, the set of all line points is hereditary.

The topic of ideals and basic ideals of Leavitt path algebra continuous to develop and attract interest. Songul and Muge Kanuni [15] provided the necessary and sufficient conditions to assure the existence of maximal ideals in Leavitt path algebra over a field, namely the existence of a maximal saturated hereditary subset of Q^0. The Leavitt path algebra over a commutative unital ring on a finite acyclic graph (not containing cycle) is a direct sum of minimal basic ideals generated by a sink or sink point [12]. This result was strengthened by Kanwar et al., [16].

The role of sink, line points, and hereditary subsets in forming (basic) ideals in the Leavitt path algebra has been described. Notable findings of Kanwar et al., [16] revealed that if the graph satisfies the condition (L), where every cycle in the quiver contains an exit, then the Leavitt path algebra over commutative unital ring has no non-zero nilpotent basic ideal. The contraposition of this statement is that if there is a non-zero nilpotent basic ideal of the Leavitt path algebra, then the quiver does not satisfy condition (L), meaning there exists a cycle in the quiver without an exit. These results lead to investigating the role of the vertices in cycles without exits on constructing (basic) ideals in the Leavitt path algebra over the commutative unital ring. The investigation is based on the conjecture that every vertex on a cycle without an exit is a primitive idempotent element, and the set of all vertices in cycles without exits is hereditary. It is also necessary to investigate the role of ideals of the commutative unital ring in the formation of ideals of the Leavitt path algebra constructed by the combination of an ideal of this ring with a vertex on a cycle without an exit.

2. SOME PROPERTIES OF BASIC IDEALS IN LEAVITT PATH ALGEBRA OVER A COMMUTATIVE UNITAL RING

The characterization of a (basic) ideal in Leavitt path algebra over the commutative unital ring is determined by the quiver structure and the ideal of the ring. Every vertex on the quiver is an idempotent element in Leavitt path algebra. The quiver structure determines whether the vertices are primitive and whether a hereditary subset of vertices is saturated. In general, the primary reference on terminology in quivers and some previous results on Leavitt path algebra is the new book published by Abrams et al. [17], and some notions of special elements refer to Assem et al. [1].

2.1. Quiver and Leavitt Path Algebras over a Commutative Unital Ring.

A Quiver is another term for a directed graph, which is a 4-tupel $Q = \left(Q^0, Q^1, s, t\right)$ which consists of two disjoint sets Q^0, Q^1 and two mappings $s, t : Q^1 \to Q^0$. The elements in Q^0, Q^1 are called vertices and (real) edges, respectively, and for every edge that is $e \in Q^1$, then the source and the target (end) of e is $s(e), t(e) \in Q^0$. A Quiver Q is said to be finite if Q^0 is finite and Q is row-finite, i.e., $s^{-1}(u)$ a finite subset of Q^1 for every vertex u. A vertex $u \in Q^0$ is called a sink if it does not emit any edge, i.e., $u \neq s(e)$ for every $e \in Q^1$.

The sequence of edges in a quiver is called a path. Path $\mu = e_1 e_1 \cdots e_k$ with $e_i \in Q^1$ and $t(e_i) = s(e_{i+1})$ for $i = 1, 2, \cdots, k-1$ have the length of path denoted by $|\mu| = k \geq 1$. The source and target of path μ denoted by $s(\mu)$, $t(\mu)$, respectively, with $s(\mu) = s(e_1)$, $t(\mu) = t(e_k)$. A path with the same source and target is called a closed path. A closed path μ is called a cycle if no edge is repeated, meaning $s(\mu) = t(\mu)$ and for every $i \neq j$, $s(e_i) \neq s(e_j)$. A cycle with length one is called a loop. Every vertex in Q^0 is a path of length zero. The set of all paths in the quiver Q is denoted by $Path(Q)$.

Expansion of the quiver Q is denoted and defined as a new quiver $\hat{Q} = \left(Q^0, Q^1 \cup \left(Q^1\right)^*, s_{\hat{Q}}, t_{\hat{Q}}\right)$, where $\left(Q^1\right)^* = \left\{e^* : e \in Q^1\right\}$ is the set of all ghost edges, and the mapping $s_{\hat{Q}}, t_{\hat{Q}}$ is defined as:

$$s_{\hat{Q}|_{Q^1}} = s,\ t_{\hat{Q}|_{Q^1}} = t,\ s(e^*) = t(e),\ t(e^*) = s(e).$$

This extension is used to define Leavitt path algebra, which is based on path algebra. A path algebra over a commutative unital ring R, on a quiver Q, denoted RQ is an R-free algebra constructed by $Path(Q)$ and satisfies:

- (V) $uv = \delta_{u,v}u$ for every $u, v \in Q^0$.
- $(E1)$ $s(e)e = et(e) = e$ for every $e \in Q^1$.

The Leavitt path algebra denoted by $L_R(Q)$ is a path algebra on the extended quiver $\hat{Q}$ that satisfies the *Cuntz-Krieger* conditions:

- $(E2)$ $e * s(e) = t(e)e^* = e^*$ for every $e \in Q^1$.
- $(CK1)$ $e^*e' = \delta_{e,e'}t(e)$ for every $e, e' \in Q^1$.
- $(CK2)$ $v = \sum_{\{e \in Q^1 | s(e) = v\}} ee^*$ for every vertex v that emits edges.

Based on axiom (V), every vertex is idempotent because $u^2 = u$ for any $u \in Q^0$. In addition, ee^* is also an idempotent element since $(CK1)$ holds $(ee^*)^2 = e(e^*e)e^* = et(e)e^* = ee^*$. Two idempotent elements x_1, x_2 are called orthogonal if $x_1x_2 = 0$, and an idempotent element x is said to be primitive if x cannot be expressed as $x_1 + x_2$ for a non-zero orthogonal element x_1, x_2. Based on this definition and axiom $(CK2)$, the vertex v that emits more than one edge is not primitive idempotent element. Meanwhile, every sink is a primitive idempotent element.

Based on $(CK1)$ and $(CK2)$, the generator elements of $L_R(Q)$ are of the form monomial $\alpha\beta^*$ with $\alpha, \beta \in Path(Q)$, $t(\alpha) = t(\beta)$, and the assumption that for every vertex $u \in Q^0, u^* = u = t(u) = s(u)$. Given two monomials $\alpha\beta^*, \gamma\delta^* \in L_R(Q)$ then their product is defined as follows:

$$(\alpha\beta^*)(\gamma\delta^*) = \begin{cases} \alpha\gamma'\delta^* & \text{if } \gamma = \beta\gamma' \\ \alpha\delta^* & \text{if } \beta = \gamma \\ \alpha\beta'^*\delta^* & \text{if } \beta = \gamma\beta' \\ 0 & \text{if not one of the above conditions} \end{cases} \tag{1}$$

Leavitt path algebra contains all real paths, ghost paths, and vertices, which are considered zero-length paths. Every monomial in $L_R(Q)$ is of the form:

(a) ku with $k \in R, u \in Q^0$ or
(b) $ke_{i_1} \cdots e_{i_\sigma} e^*_{j_1} \cdots e^*_{j_\tau}$ with $k \in R$, $\sigma, \tau \geq 0$, $\sigma + \tau > 0$, $e_{is} \in Q^1$, $e^*_{j_t} \in (Q^1)^*$ for $0 \leq s \leq \sigma$, $0 \leq t \leq \tau$.

In general, the Leavitt path algebra over the commutative unital ring R on the quiver Q is written as:

$$L_R(Q) = Span_R \left\{ \sum_{i=1}^{n} k_i \alpha_i \beta_i^* | k_i \in R, \alpha_i, \beta_i \in Path(Q) \right\}.$$

The hereditary subset of vertices plays a crucial role in forming the (basic) ideal of the Leavitt path algebra over the commutative unital ring. A hereditary subset is defined by a preorder relation "$\leq$" on Q^0. For any two vertices v, w, we define that $v \leq w$ if and only if $v = w$ or there is a path $\mu \in Path(Q)$ satisfies $s(\mu) = v$ and $t(\mu) = w$. A subset $H \subseteq Q^0$ is said to be hereditary if for every $v, w \in Q^0$ with $v \leq w$, $v \in H$ implies $w \in H$. In addition to the hereditary property, the preorder relation is used to define the tree of a vertex. The tree of a vertex v is denoted as $T(v) = \left\{w \in E^0 : v \leq w\right\}$, which represents the set of all vertices that are preceded by v.

The subset $H \subseteq Q^0$ is called saturated if, for each vertex v with $s^{-1}(v) \neq \emptyset$, if $t\left(s^{-1}(v)\right) = \{t(e) \,|s(e) = v\} \subseteq H$ then $v \in H$. The closure of H, denoted $\bar{H}$ is the smallest saturated hereditary set that contains H. If H is hereditary, then closure $\bar{H}$ is usually called the saturation of H. It is easy to show that the intersection and union of hereditary subsets are also hereditary; the intersection of saturated subsets is also saturated, but the union is not necessarily saturated. Another property is that the set of all vertices that are in the ideal of $L_R(Q)$ forms a saturated hereditary subset, as stated in the following lemma. This lemma refers to [9], with a slightly different proof.

Lemma 2.1. [9] *Given an ideal* $\Im \subseteq L_R(Q)$ *and* $X = \left\{u \in Q^0 : u \in \Im\right\}$ $= Q^0 \cap \Im$. *Then* X *is hereditarily saturated.*

Proof. Take any vertices $u, v \in Q^0$ with $u \leq v$ and $u \in X$. Then $u \in \Im$ and there is $\mu \in Path(Q)$ such that $s(\mu) = u \in \Im$ and $v = t(\mu) = \mu^*\mu = \mu^* s(\mu)\mu = \mu^* u \mu \in \Im$. Thus, $v \in X$ and consequently X is hereditary. Next, take an arbitrary vertex v with $s^{-1}(v) \neq \emptyset$ and $t\left(s^{-1}(v)\right) \subseteq X$. Then for every edge $e \in s^{-1}(v)$, $t(e) \in X$ and $t(e) \in \Im$, so $ee^* = et(e)e^* \in \Im$. According to $(CK2)$, $v = \sum_{s(e)=v} ee^* \in \Im$ so $v \in X$. Thus, X is saturated. □

The path μ is called a closed path with base vertex v, if $v = s(\mu) = t(\mu)$. Suppose c is a cycle with base v, then it is defined that $c^0 = v$ and $c^{-n} = (c^*)^n$ for every natural number n. The edge e is called an exit for the path $\mu = e_1 e_2 \cdots e_k$ if $s(e) = s(e_i)$ and $e \neq e_i$ for some $i \in \{1, 2, \cdots, k\}$. The set of all vertices on the quiver Q that lies on a cycle without exit is denoted $P_c(Q)$. For any two vertices u, v with $u \leq v$ and u in $P_c(Q)$, then v also in $P_c(Q)$. Consequently, $P_c(Q)$ is a hereditary subset, so the vertices in $P_c(Q)$ play an important role in the construction of the (basic) ideal of the Leavitt path algebra.

2.2. On Ideal Generated by Combination of an Ideal and a Hereditary Subset.

There is a special ideal in Leavitt path algebra over commutative unital ring called the basic ideal. While every ideal is a basic ideal in Leavitt path algebra over a field, it is not over the commutative unital ring. A basic ideal is defined as follows:

Definition 2.2. [9] *Given a Leavitt path algebra, $L_R(Q)$ over a commutative unital ring R on the quiver Q. An ideal $\Im \in L_R(Q)$ is called a basic ideal if, for every non-zero element $k \in R$ and every vertex v, $kv \in \Im$ implies $v \in \Im$.*

According to Proposition 7.7 in [9], which has been thoroughly proven, saturated hereditary subsets have a significant role in forming basic ideals in Leavitt path algebra over commutative unital rings. The property is restated in the proposition below without proof.

Proposition 2.3. [9] *Given an arbitrary commutative unital ring R, quiver Q and a hereditary subset H on Q^0, then the set*

$$(H) = Span_R \left\{ \sum_{i=1}^{n} k_i \alpha_i {\beta_i}^* | k_i \in R, \alpha_{i,} \beta_i \in Path(Q), t(\alpha_i) = t(\beta_i) \in H \right\}$$

is the basic ideal of $L_R(Q)$ that is generated by H.

Other properties of hereditary subsets exist, including the role of the intersection of hereditary subsets in the construction of basic ideals in Leavitt's path of algebra. These properties are given with complete proof as follows.

Proposition 2.4. *Given any commutative unital ring R, quiver Q and hereditary subsets H_1, H_2 in Q^0. Then,*

(a) If $H_1 \subseteq H_2$ then $(H_1) \subseteq (H_2)$.
(b) $(H_1 \cap H_2) = (H_1)(H_2)$.

Proof. (a) Take an arbitrary $x \in (H_1)$, $x = \sum_{i=1}^{n} k_i \alpha_i {\beta_i}^*$ with $k_i \in R$, $\alpha_{i,} \beta_i \in Path(Q)$, $t(\alpha_i) = t(\beta_i) \in H_1$ Since $H_1 \subseteq H_2$, then $t(\alpha_i) = t(\beta_i) \in H_2$, and $x = \sum_{i=1}^{n} k_i \alpha_i {\beta_i}^*$ with $k_i \in R, \alpha_{i,} \beta_i \in Path(Q)$. This means $x \in (H_2)$ and therefore $(H_1) \subseteq (H_2)$.

(b) Take any non-zero element $y \in (H_1)(H_2)$ then $y = \sum_{a \in (H_1), b \in (H_2)} ab$. Consider a non-zero monomial ab in y with $a = \sum_{p \in R} p\alpha\beta^* \in (H_1)$, $b = \sum_{q \in R} q\gamma\delta^* \in (H_2)$ where $t(\alpha) = t(\beta) \in H_1$, $t(\gamma) = t(\delta) \in H_2$. Based on equation (1), we have $0 \neq ab = \sum_{k \in R} k\lambda\delta^*$ with $\lambda = \begin{cases} \alpha\gamma' & \text{if } \gamma = \beta\gamma' \\ \alpha & \text{if } \gamma = \beta \\ \alpha\beta'^* & \text{if } \beta = \gamma\beta' \end{cases}$

This means that $t(\lambda) = t(\delta) \in H_2$, thus:

i. If $\lambda = \alpha\gamma'$ then $t(\alpha) = s(\gamma') \in H_1$ and $t(\gamma') = t(\lambda) \in H_1$ since H_1 is hereditary. So, $t(\lambda) = t(\delta) \in H_1 \cap H_2$ or $ab = \sum_{k \in R} k\lambda\delta^* \in (H_1 \cap H_2)$.
ii. If $\lambda = \alpha$ then $t(\lambda) = t(\alpha) \in H_1$, so $t(\lambda) = t(\delta) \in H_1 \cap H_2$ or $ab = \sum_{k \in R} k\lambda\delta^* \in (H_1 \cap H_2)$.
iii. If $\lambda = \alpha\beta'^*$ then $t(\alpha) = t(\beta') \in H_1$ or $t(\alpha) = t\left(s^{-1}(s(\alpha))\right) \in H_1$. Therefore, $t(\alpha) = s(\lambda) \in H_1$, since H_1 is saturated and $t(\lambda) \in H_1$, since H_1 is hereditary. Thus, $t(\lambda) = t(\delta) \in H_1 \cap H_2$, in other words, $ab = \sum_{k \in R} k\lambda\delta^* \in (H_1 \cap H_2)$.

□

Based on Proposition 2.3, a hereditary subset H (not necessarily saturated) can construct a basic ideal (H) in $L_R(Q)$, and according to Lemma 2.1, $(H) \cap Q^0$ is hereditary saturated. Related to these two properties, we have the property $\left(\overline{H}\right) = (H)$, where $\overline{H}$ is a saturation of hereditary subset H.

Lemma 2.5. [9] *Given an arbitrary commutative unital ring R, quiver Q and a hereditary subset H in Q^0, then the basic ideal*

$$(H) = Span_R \left\{ \sum_{i=1}^{n} k_i \alpha_i \beta_i^* | k_i \in R, \alpha_i \beta_i \in Path\,(Q)\,, t\,(\beta_i) \in H \right\} = \left(\overline{H}\right)$$

Proof. Since the subset H is hereditary then $H \subseteq \overline{H}$. Based on Proposition 2.4 (a), we get $(H) \subseteq \left(\overline{H}\right)$. Conversely, according to Lemma 2.1, $X = \left\{u \in Q^0 : u \in (H)\right\}$ is a saturated hereditary subset that contains H, so that $\overline{H} \subseteq X$. Take any point $v \in \overline{H}$ then $v \in X$ so that $v \in (H)$. Consequently, for every monomial $\alpha_i \beta_i^* \in \left(\overline{H}\right)$, there is $v \in \overline{H}$ so that $v = t\,(\alpha_i) = t\,(\beta_i) \in \overline{H}$ and $\alpha_i \beta_i^* = \alpha_i v \beta_i^* \in (H)$ since $v \in (H)$. In other words, $\left(\overline{H}\right) \subseteq (H)$. Thus, $(H) = \left(\overline{H}\right)$. □

Proposition 2.3. motivates to study an ideal in a Leavitt path algebra over a commutative unital ring, which is constructed by combining the ideal of this ring with a hereditary subset. This means that it is necessary to study not only the role of the hereditary subset but also the role of the ideal of the commutative unital ring for constructing the ideal in the Leavitt path algebra $L_R(Q)$.

Proposition 2.6. *Given a Leavitt path algebra $L_R(Q)$ over a commutative unital ring R on quiver Q. If $I \subseteq R$ is an ideal and $H \subseteq E^0$ is hereditary then:*

$$(IH) = Span_I \left\{ \sum_{i=1}^{n} a_i \alpha_i \beta_i^* | a_i \in I, \alpha_i \beta_i \in Path\,(Q)\,, t\,(\alpha_i) = t\,(\beta_i) \in H \right\} \tag{2}$$

is an ideal of $L_R(Q)$. Besides, $(IH) = (I\overline{H})$ with $\overline{H}$ is the saturation of H.

Proof. Suppose $J = (IH)$ is an ideal expressed by (2). Take any element of the form $\alpha\beta^*$ with $t\,(\alpha) = t\,(\beta) = u \in H$, and every $x, y \in L_R(Q)$, $a \in I$. We will show that $ax\alpha u\beta^* y \in J$, by simply showing that $a\gamma\delta^* u\mu\sigma^* \in J$ for every $a \in I$,$u \in H$, $\gamma, \delta, \mu, \sigma \in Path\,(Q)$. First case, $a\gamma\delta^* u\mu\sigma^* = 0$, then it is clear that $a\gamma\delta^* u\mu\sigma^* \in J$. Second case, $a\gamma\delta^* u\mu\sigma^* \neq 0$ with three possibilities. Based on the definition of multiplication (1), we get $a\gamma\delta^* u\mu\sigma^* = a\gamma\mu'\sigma^*$ if $\mu = \delta\mu'$, or $a\gamma\delta^* u\mu\sigma^* = a\gamma\sigma^*$ if $\delta = \mu$, or $a\gamma\delta^* u\mu\sigma^* = a\gamma\delta'\sigma^*$ if $\delta = \mu\delta'$. Note again that $u = t(\mu) \in H$ with H a hereditary subset, then we have $t(\mu) = t(\mu') \in H$ for the first possibility, $t(\mu) = t(\delta) \in H$ for the second possibility, and $t(\delta) = t(\delta') \in H$ for the third possibility, where $s(\mu) = s(\delta) = u$. Thus, $0 \neq a\gamma\delta^* u\mu\sigma^* \in J$ for all possibilities. Therefore, $J = (IH)$ is an ideal of $L_R(Q)$. Since H is hereditary then by Lemma 2.5, we have

$$(H) = \left(\overline{H}\right) = Span_R \left\{ \sum_{i=1}^{n} k_i \alpha_i \beta_i^* | k_i \in R, \alpha_i, \beta_i \in Path\,(Q)\,, t\,(\alpha_i) = t\,(\beta_i) \in \overline{H} \right\},$$

If the coefficients in R are replaced in I, then equation (2) can also be written as

$$\begin{aligned}(IH) &= Span_I\left\{\sum_{i=1}^{n} a_i\alpha_i\beta_i^* | a_i \in R, \alpha_i, \beta_i \in Path\,(Q)\,, t\,(\alpha_i) = t\,(\beta_i) \in \overline{H}\right\} \\ &= \left(I\overline{H}\right)\end{aligned}$$

□

Proposition 2.6 has stated an important property regarding the ideal of $L_R(Q)$ constructed from an ideal $I \subseteq R$ and a hereditary subset H. As mentioned above, the union of saturated subsets is not necessarily saturated, whereas the union of hereditary subsets is hereditary. Therefore, we can develop Proposition 2.6 into more detailed properties, as stated in the following proposition.

Proposition 2.7. *Given a Leavitt path algebra $L_R(Q)$ over a commutative unital ring R on a quiver Q and an ideal $I \subseteq R$. If $\{H_i\}_{i\in\Gamma}$ is a family of hereditary disjoint subsets of Q^0, then:*

$$\left(I\overline{\bigcup_{i\in\Gamma} H_i}\right) = \left(I\bigcup_{i\in\Gamma} H_i\right) = \oplus_{i\in\Gamma}\,(IH_i) = \oplus_{i\in\Gamma}\left(I\overline{H_i}\right)$$

Proof. Since H_i is hereditary for every $i \in \Gamma$ then $H = \bigcup_{i\in\Gamma} H_i$ is hereditary. Based on Proposition 2.6, we get $\left(I\overline{H}\right) = \left(I\overline{\bigcup_{i\in\Gamma} H_i}\right) = \left(I\bigcup_{i\in\Gamma} H_i\right) = (IH)$ and $(IH_i) = \left(I\overline{H_i}\right)$ for every $i \in \Gamma$. It remains to be shown in the middle equation that $\left(I\bigcup_{i\in\Gamma} H_i\right) = \oplus_{i\in\Gamma}\,(IH_i)$. Take an arbitrary $x \in (IH)$, based on (2), $x \in \sum_{l=1}^{n} a_l\alpha_l\beta_l^*$ for some $a_l \in I$, $\alpha_l, \beta_l \in Path(Q)$, $t(\alpha_l) = t(\beta_l) \in H$. Since $H = \bigcup_{i\in\Gamma} H_i$ then for every l, $t(\alpha_l) \in H_i$ for some $i \in \Gamma$. Consequently, $x = \sum_{l=1}^{n} a_l\alpha_l\beta_l^* \in \sum_{i\in\Gamma}(IH_i)$, thus $(IH) \subseteq \sum_{i\in\Gamma}(IH_i)$. Conversely, since $(IH_i) \subseteq (IH)$ for every i then clearly that $\sum_{i\in\Gamma}(IH_i) \subseteq (IH)$. Thus, we get $\sum_{i\in\Gamma}(IH_i) = (IH)$. Finally, suppose that there exist $j \in \Gamma$ such that $\sum_{i\in\Gamma, j\neq i}(IH_i) \cap (IH_j) \neq \{0\}$. That is, there exist $0 \neq y \in \sum_{i\in\Gamma, j\neq i}(IH_i) \cap (IH_j)$, and based on (2), $y = \sum_{k=1}^{n} s_k\gamma_k\delta_k^*$ for some $s_k \in I$, $\gamma_k, \delta_k \in Path(Q)$, $t(\gamma_k) = t(\delta_k) \in H_j$ and also $t\,(\gamma_k) = t\,(\delta_k) \in \bigcup_{i\in\Gamma, j\neq i} H_i$. This means that H_j and $\bigcup_{i\in\Gamma, j\neq i} H_i$ are not disjoint and it is a contradiction. Thus, $\left(\bigcup_{i\in\Gamma, j\neq i} H_i\right) \cap H_j = \{0\}$ and therefore, $\left(I\bigcup_{i\in\Gamma} H_i\right) = \oplus_{i\in\Gamma}\,(IH_i)$ □

3. THE ROLE OF VERTICES IN CYCLE WITHOUT EXIT TO FORM BASIC IDEAL

We have studied the ideal (IH) of the Leavitt path algebra $L_R(Q)$, which is constructed by the ideal I of the commutative unital ring R and the hereditary subset H. On the other hand, we have the hereditary subset $P_c(Q)$, which is the set of vertices lying on a cycle c of the quiver Q without exit ([18]). The elements in $P_c(Q)$ play a role in constructing ideal of the Leavitt path algebra.

Theorem 3.1. *Given a commutative unital ring R and quiver Q. Let $v \in P_c(Q)$ with c is a cycle without exit such that $s(c) = v$, and Λ_v denotes the set of paths leading to v but not containing all edges in c. If I is an ideal in R then $(IP_c(Q)) = (Iv) \cong M_n\left(I\left[x, x^{-1}\right]\right)$ is an ideal in $L_R(Q)$.*

Proof. The set of all vertices on the cycle c without exit, denoted $P_c(Q)$, is a hereditary subset. This is because for every $x, y \in Q^0$ with $x \leq y$, $x \in P_c(Q)$, it must be $y \in P_c(Q)$, since the cycle c does not contain an exit. Furthermore, since $s(c) = v \in P_c(Q)$ then $\overline{P_c(Q)} = \overline{\{v\}}$. Therefore, by Proposition 2.6, $(IP_c(Q)) = (Iv) = \left(Ic^0\right)$. Let $B = \left\{\mu c^k \sigma | \mu, \sigma \in \Lambda_v, k \in \mathbb{Z}\right\}$ with $c^0 = v$, $c^k = (c^*)^{-k}$, for integer $k < 0$, then B is linearly independent over I. Take an arbitrary $k \in \mathbb{Z}$ and $\sum_{i=1}^{k} t_i \mu_i c_i^k \sigma_i^* = 0$ then $0 = \mu_j^* \left(\sum_{i=1}^{k} t_i \mu_i c_i^k \sigma_i^*\right) \sigma_j = t_j u c_j^k v = t_j c_j^k$ for every $j \in \{1, 2, \ldots, m\}$, so $t_j = 0$. According to Proposition 2.6, the form of elements in (Iv) is $a\alpha\beta^*$ with $a \in I$, and path α, β satisfying $t(\alpha) = t(\beta) \in P_c(Q) = T(v)$, so that $\alpha = \mu c^l$, $\beta = \sigma c^m$ for some $\mu, \sigma \in \Lambda_v$ and an integer $l, m \leq 0$. This implies that B generates (Iv). Thus, B is the I-basic of (Iv).
We define a function $\varphi : (Iv) \to M_n\left(I\left[x, x^{-1}\right]\right)$, $\varphi\left(a\mu c^k \sigma^*\right) = ax^k e_{\mu,\sigma}$ for any $a\mu c^k \sigma^* \in (Iv)$ with $a \in I$, monomial $\mu c^k \sigma^* \in B$, $ax^k e_{\mu,\sigma} \in M_n\left(I\left[x, x^{-1}\right]\right)$ which ax^k is an entry in (μ, σ) and the other entry is 0. It is easy to show that φ is an algebraic isomorphism, so that $(Iv) \cong M_n\left(I\left[x, x^{-1}\right]\right)$. □

It appears that the ideal (Iv) is not a basic ideal for any non-trivial ideal I in R since $v \notin (Iv)$. However, if $I = R$ then $(v) = (c^0)$ is a basic ideal because $P_c(Q)$ hereditary (according to Proposition 2.3), with v is the basic point of the cycle c without exit.

Corollary 3.2. *Given an arbitrary quiver Q and a commutative unital ring R. Take an arbitrary $v \in P_c(Q)$ and a cycle c without an exit such that $s(c) = v$. Let Λ_v is the set of paths that end at a v but do not contain all edges in c, and $n = |\Lambda_v|$, then $\left(c^0\right) = (v) \cong M_n\left(R\left[x, x^{-1}\right]\right)$ is a basic ideal in $L_R(Q)$.*

Theorem 3.3. *Let R be any commutative unital ring and Q a quiver. If I is an ideal of R, and $P_{\{c_i\}}(Q)$ is the set of all vertices on all cycles without exits of the quiver Q, then $\left(IP_{\{c_i\}}(Q)\right) \cong \oplus_{i\in\Gamma} M_{n_i}\left(I\left[x, x^{-1}\right]\right)$ where $\{c_i\}$ is the set of distinct cycle in Q without exit, $n_i = |\Lambda_{v_i}|$ with v_i is the base of the cycle c_i and Λ_{v_i} is the set of all paths that end at v_i but do not contain any edges of c_i.*

Proof. Note that $P_{\{c_i\}}(Q) = \bigcup P_{c_i}(Q)$. Since $P_{c_i}(Q)$ hereditary then $\bigcup P_{c_i}(Q)$ is also hereditary. According to Proposition 2.7 and Theorem 3.1, we have:

$$\left(I \bigcup_{i\in\Gamma} P_{c_i}(Q)\right) = \left(I\overline{\bigcup_{i\in\Gamma} P_{c_i}(Q)}\right) = \oplus_{i\in\Gamma}\left(I\overline{P_{c_i}(Q)}\right) = \oplus_{i\in\Gamma}\left(IP_{c_i}(Q)\right)$$
$$= \oplus_{i\in\Gamma}(Iv_i) \cong \oplus_{i\in\Gamma} M_{n_i}\left(I\left[x, x^{-1}\right]\right)$$

Thus, $\left(IP_{\{c_i\}}(Q)\right) \cong \oplus_{i\in\Gamma} M_{n_i}\left(I\left[x, x^{-1}\right]\right)$. □

In accordance with Corollary 3.2, if $I = R$, then we have $\left(I \bigcup_{i\in\Gamma} P_{c_i}(Q)\right) = \oplus_{i\in\Gamma}(v_i) \cong \oplus_{i\in\Gamma} M_{n_i}\left(I\left[x, x^{-1}\right]\right)$ is a basic ideal in $L_R(Q)$. This basic ideal is constructed by the set of all distinct vertices of the cycles without exits.

4. CONCLUSION

The set of all vertices in the quiver Q on a cycle c without an exit is denoted by $P_c(Q)$, and is hereditary. Given R a commutative unital ring, an ideal I of R, and a vertex $v \in P_c(Q)$ such that $s(c) = v$, then $(Iv) \cong M_n\left(I\left[x, x^{-1}\right]\right)$ is an ideal in $L_R(Q)$, where $n = |\Lambda_v|$ and Λ_v is the set of all paths that end at v but do not contain any edges from c. However, the ideal (Iv) is not a basic ideal. If $I = R$, then $(Rv) = (v) \cong M_n\left(R\left[x, x^{-1}\right]\right)$ is a basic ideal in $L_R(Q)$. This basic ideal is an ideal constructed by the set of vertices on a cycle without exit. In general, $\left(\bigcup_{i\in\Gamma} P_{c_i}(Q)\right) = \oplus_{i\in\Gamma}(v_i) \cong \oplus_{i\in\Gamma} M_{n_i}\left(R\left[x, x^{-1}\right]\right)$ is a basic ideal generated by the set of all distinct vertices of the cycles c_i without exits.

Acknowledgement. This article is part of the research results of the "research leader" cluster in 2023. We would like to thank LPPM UIN Sunan Kalijaga Yogyakarta for providing research funding for this cluster. Thanks also to the anonymous reviewers for their corrections and suggestions.

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