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Some Results on Inclusive Distance Antimagic Labeling of Graphs

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Abstract. Let $G = (V, E)$ be a graph of order n. A bijection $f : V(G) \to \{1, 2, \cdots, n\}$ is called inclusive distance antimagic labeling if $w(u) \neq w(v)$ for any two distinct vertices $u, v \in V(G)$, where $w(v) = \sum\limits_{x \in N[v]} f(x)$. We start our discussion with the connection between distance magic labeling and inclusive distance antimagic labeling. Then, we investigate the existence of an inclusive distance antimagic labeling for circulant graphs, disjoint union graphs, and join graphs.

Key words and Phrases: Inclusive Distance Antimagic, Distance Magic, Circulant Graph, Disjoint Union Graph, Join Graph.

1. INTRODUCTION

All graphs in this paper are considered to be finite, simple, and undirected. The concept of magic labeling based on distance was separately introduced by Vilfred [1] in his doctoral thesis in 1994 and Miller et al. [2] in 2003 with the following definition.

Definition 1.1. *Let $G = (V, E)$ be a graph of order n. A bijection $f : V(G) \to \{1, 2, \cdots, n\}$ is called a distance magic labeling if there exists a positive integer k such that $\sum_{x \in N(v)} f(x) = k$, for every $v \in V(G)$. The constant k is referred to as the magic constant of the labeling f. The sum $\sum_{x \in N(v)} f(x)$ is called the weight of vertex v under f, and is denoted as $w(v)$. If a graph G admits a distance magic labeling, then G is called a distance magic graph, or G is distance magic.*

Research on distance magic labeling has been extensively conducted for several families of graphs. Furthermore, studies on distance-based labeling has expanded into several variants, one example of which is graph labeling considering that all weights must be distinct, with the vertex weight defined as the sum of labels

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of all its closed neighbors. This notion was termed as inclusive distance antimagic labeling, introduced by Dafik et al. [3] with the definition as follows:

Definition 1.2. *Let $G = (V, E)$ be a graph of order n. A bijection $f : V(G) \to \{1, 2, \cdots, n\}$ is called an inclusive distance antimagic labeling if $w(u) \neq w(v)$ for any two distinct vertices $u, v \in V(G)$, where $w(v) = \sum_{x \in N[v]} f(x)$. The set $N[v]$ is the closed neighborhod of vertex v, and is defined as $N(v) \cup v$. If the graph G admits such a labeling, then G is said to be an inclusive distance antimagic graph, or G is inclusive distance antimagic.*

Observation 1.3. *If a graph G is inclusive distance antimagic, then for any two distinct vertices u and v, we have $N[u] \neq N[v]$.*

Conjecture 1.4. *A graph G is inclusive distance antimagic if and only if G does not have two vertices with the same closed neighborhood.*

From the notion of inclusive distance antimagic labeling, they investigate the existence of such a labeling for various simple graphs.

Theorem 1.5. [3] *The Complete graph K_n is not inclusive distance antimagic.*

Theorem 1.6. [3] *The Path graph P_n is inclusive distance antimagic, for $n \neq 2$.*

Theorem 1.7. [3] *The Cycle graph C_n is inclusive distance antimagic, for $n \neq 2, 3$.*

The term distance-based labeling was generalized by O'Neal and Slater [4, 5] for distance magic labeling, and by Simanjuntak and Wijaya [6] for distance antimagic labeling, by defining the notation for the labeling weight as $w(v) = \sum_{x \in N_D(v)} f(x)$, where $N_D(v) = \{y \in V(G) | d(v, y) \in D\}$ and $D \subseteq \{0, 1, \cdots, diam(G)\}$, with $diam(G)$ denotes the diameter of graph G. If all vertices have the same weight, we call the labeling a $D-$distance magic labeling, whereas if all vertices have distinct weights, we call the labeling a $D-$distance antimagic labeling.

If $D = \{1\}$, a D-distance magic labeling is a distance magic labeling and D-distance antimagic labeling is distance antimagic labeling. For $D = \{0, 1\}$, a D-distance antimagic labeling is an inclusive distance antimagic labeling. Ngurah [7] proved that for $D \subseteq \{0, 1, \cdots diam(G)\}$, if a graph G has a $D-$distance magic labeling, then G also has a $(D \cup \{0\})-$distance antimagic labeling. For the case $D = \{1\}$, then this statement can be rewritten in the theorem as follows:

Theorem 1.8. [7] *If G is a distance magic graph, then G is inclusive distance antimagic.*

Extensive research has been conducted on the existence of distance magic labeling in certain graphs. Miller et al. [2] provided several simple observations for certain graphs that have distance magic labeling, such as path graphs P_1 and P_3, cycle graph C_4, complete graph K_1, and wheel graph W_4 (Wheel graph W_4 isomorphic with $C_4 + K_1$). Additionally, Cichacz and Froncek [8] proved that the circulant graph $C_{2p+2}(1, p)$ is distance magic, followed by the circulant graph

$C_{2(p^2-1)}(1,p)$, for p even, is also distance magic. According to Theorem 1.8, these graphs are also inclusive distance antimagic.

The following corollary represents the contrapositive of Theorem 1.8:

Corollary 1.9. *If graph G is not inclusive distance antimagic, then G is not distance magic.*

Corollary 1.10. *If graph G has pairs of vertices with the same closed neighborhood, then G is not distance magic.*

In this paper, we will present the existence of inclusive distance antimagic labeling for circulant graphs, disjoint union graphs, and join graphs. This paper also discusses some examples of graph that have pairs of vertices with the same closed neighborhood. According to Corollary 1.10, these graphs are not distance magic.

2. CIRCULANT GRAPHS

As stated in Theorem 1.5, complete graph K_n is not inclusive distance antimagic, mainly because all vertices in K_n has the same closed neighborhood set, which is $V(K_n)$ itself. Since K_n is a regular graph, then Dafik et al. [3] proposed a conjecture regarding the inclusive distance antimagic property in regular graphs.

Conjecture 2.1. *Every r-regular graph except complete graph K_n is inclusive distance antimagic.*

In this section, several examples of regular graphs will be given that have at least two vertices with the same closed neighborhood set. According to Observation 1.3, these graphs are not inclusive distance antimagic. This also serves as a counterexample to Conjecture 2.1. Circulant graphs are regular graphs where some of them have at least one pair of vertices with the same closed neighborhood set.

Definition 2.2. *Let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq \left\lfloor \frac{n}{2} \right\rfloor$, where n and a_i, $i = 1, 2, \cdots, k$ are positive integers. The Circulant Graph $C_n(a_1, a_2, \cdots, a_k)$ is a regular graph of order n with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ and edge set $E = \{v_i v_{(i+a_j) \bmod n} \mid i = 0, 1, \cdots, n-1 \text{ and } j = 1, 2, \cdots, k\}$. The numbers $a_1, a_2, \cdots, a_k$ are called the generators of the circulant graph.*

For example, circulant graph $C_6(1,3)$ is a graph with vertex set $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ and edge set $E = \{v_0v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_0, v_0v_3, v_1v_4, v_2v_5\}$, as shown in Figure 1.

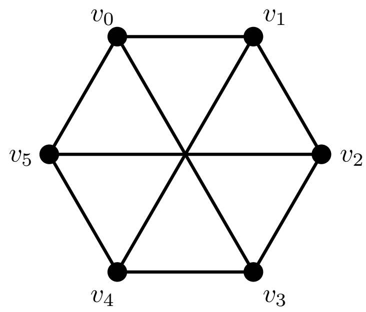

FIGURE 1. A Circulant Graph $C_6(1,3)$

Theorem 2.3. *Let $1 \leq a_1 < a_2 < n$ are positive integers such that $a_1 + a_2 = n$. Then circulant graph $C_{2n}(a_1, a_2, n)$ has pairs of vertices with the same closed neighborhood set, which means the graph is not inclusive distance antimagic.*

Proof. Suppose graph $C_{2n}(a_1, a_2, n)$ with vertex set $V = \{v_0, v_1, v_2, \cdots v_{2n-1}\}$. Consider two vertices v_0 and v_n. Since $a_1 < a_2 < n$, the closed neighborhood of v_0 is $N[v_0] = \{v_0, v_{a_1}, v_{a_2}, v_n, v_{2n-a_2}, v_{2n-a_1}\}$, and the closed neighborhood of v_n is $N[v_n] = \{v_0, v_{n-a_2}, v_{n-a_1}, v_n, v_{n+a_1}, v_{n+a_2}\}$. However, since $a_1 + a_2 = n$, it follows that $v_{a_1} = v_{n-a_2}, v_{a_2} = v_{n-a_1}, v_{2n-a_2} = v_{n+a_1}$, and $v_{2n-a_1} = v_{n+a_2}$. Therefore, v_0 and v_n have the same closed neighborhood set. □

Figure 2 below shows the circulant graph $C_8(1,3,4)$, which, according to Theorem 2.3 is not inclusive distance antimagic since v_0 and v_4 have the same closed neighborhood set $N[v_0] = N[v_4] = \{v_0, v_1, v_3, v_4, v_5, v_7\}$.

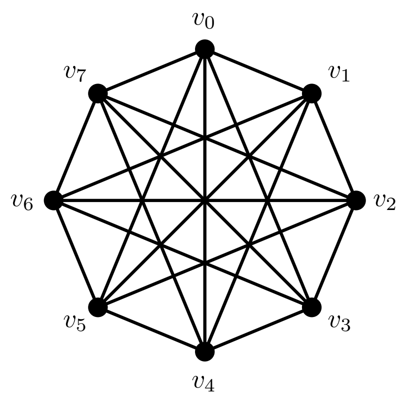

FIGURE 2. A Circulant Graph $C_8(1,3,4)$

Here we present several corollaries from Theorem 2.3.

Corollary 2.4. *Let k is an even number, and $a_1, a_2, ..., a_k, n$ are natural numbers with $a_1 < a_2 < \cdots < a_k < n$. If $a_i + a_{k-i+1} = n$ for every $i \leq \frac{k}{2}$, then circulant graph $C_{2n}(a_1, a_2, \cdots, a_k, n)$ is not inclusive distance antimagic.*

Corollary 2.5. *Let k and n are even number, and $a_1, a_2, ..., a_k$ are natural numbers with $a_1 < a_2 < \cdots < a_k < n$. If $a_i + a_{k-i+1} = n$ for every $i \leq \frac{k}{2}$, then circulant graph $C_{2n}(a_1, a_2, \cdots, a_{\frac{k}{2}}, \frac{n}{2}, a_{\frac{k}{2}+1}, \cdots, a_k, n)$ is not inclusive distance antimagic.*

Theorem 2.6. *Suppose the set $S \subset V(G)$ forms a complete subgraph in graph G. If for every pair of vertices $u, v \in S$ it holds that $N(u) - S = N(v) - S$, then graph G is not inclusive distance antimagic.*

Proof. For two vertices $u, v \in S$, $N[u] = S \cup (N(u) - S)$ and $N[v] = S \cup (N(v) - S)$. Since $N(u) - S = N(v) - S$, it follows that $N[u] = N[v]$. □

Let $p, q, n > 1$ be positive integers such that $pq = n$. The circulant graph $C_n(p, 2p, 3p, \cdots)$ with vertex set $V = \{v_0, v_2, \cdots, v_{n-1}\}$ and all of its generators less than or equal to $\lfloor \frac{n}{2} \rfloor$, is isomorphic to pK_q, that is p copies of the complete graph K_q. For a positive integer $k \leq \lfloor \frac{p}{2} \rfloor$, by adding generators $k, p-k, p+k, 2p-k, 2p+k, \cdots$, circulant graph $C_n(k, p-k, p, p+k, 2p-k, 2p, 2p+k, \cdots)$ is an example of the graph that satisfies Theorem 2.6, with one of its sets $S = \{v_0, v_p, \cdots v_{(q-1)p}\}$.

Corollary 2.7. *Let $p, q, n > 1$ be positive integers such that $pq = n$. For a positive integer $k \leq \lfloor \frac{p}{2} \rfloor$, circulant graph $C_n(k, p-k, p, p+k, 2p-k, 2p, 2p+k, \cdots)$ is not inclusive distance antimagic.*

3. DISJOINT UNION OF GRAPHS

In this section, we provide the definition of disjoint union of two or more graphs, then we explore the properties of inclusive distance antimagic labeling of the disjoint union of graphs.

Definition 3.1. *The disjoint union between two graphs G and H, denoted as $G \cup H$, is a disconnected graph with its components are G and H. It means that $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The disjoint union of more than two graphs, $G_1, G_2, \cdots, G_n$, is denoted as $\bigcup_{i=1}^{n} G_i$. If each graph G_i is isomorphic to graph G, then the disjoint union is denoted as nG.*

Theorem 3.2. *Let G be an inclusive distance antimagic graph, and H be an $h-$regular inclusive distance antimagic graph. If $\Delta(G) \leq h$, then $G \cup H$ is also an inclusive distance antimagic graph.*

Proof. Let f_G and f_H be the inclusive distance antimagic labeling for G and H, respectively. Let $|V(G)| = m$, and define the labeling $f_{G \cup H}$ for $G \cup H$ as follows:

$$f_{G \cup H}(v) = \begin{cases} f_G(v), & v \in V(G) \\ f_H(v) + m, & v \in V(H). \end{cases}$$

Let w_G, w_H, and $w_{G\cup H}$ be the weight of vertices based on the labeling f_G, f_H, and $f_{G\cup H}$, respectively. Since H is an $h-$regular graph, then the weight of vertices in graph $G \cup H$ are:

$$w_{G\cup H}(v) = \begin{cases} w_G(v), & v \in V(G) \\ w_H(v) + (h+1)m, & v \in V(H). \end{cases}$$

For each vertex v in graph G, it holds that $w_G(v) < m(\Delta(G)+1)$. Since $\Delta G \leq h$, therefore for every vertex v in G and y in H

$$w_{G\cup H}(v) = w_G(v) < m(\Delta(G)+1) \leq m(h+1) < w_H(y) + m(h+1) = w_{G\cup H}(y).$$

Hence, under the labeling $f_{G\cup H}$, every vertex in graph $G \cup H$ has distinct weight. Thus, $G \cup H$ is an inclusive distance antimagic graph. □

Theorem 3.3. *Let $R_1, R_2, \cdots, R_n$ be regular inclusive distance antimagic graphs. Then $\bigcup_{i=1}^{n} R_i$ is an inclusive distance antimagic graph.*

Proof. Let r_i be the degree of regular graph R_i, for $1 \leq i \leq n$. Without loss of generality, assume $r_1 \leq r_2 \leq \cdots \leq r_n$. We will use mathematical induction to prove the statement that $\bigcup_{i=1}^{k} R_i$ is inclusive distance antimagic for all k. For $k = 1$, the statement is true since R_1 is inclusive distance antimagic. Now assume that the statement holds true for $k = n-1$, i.e., $\bigcup_{i=1}^{n-1} R_i$ is inclusive distance antimagic. We want to prove that $(\bigcup_{i=1}^{n-1} R_i) \cup R_n$ is also inclusive distance antimagic. Note that since $r_1 \leq r_2 \leq \cdots \leq r_n$, then $\Delta(\bigcup_{i=1}^{n-1} R_i) = r_{n-1} \leq r_n$. By Theorem 3.2, $(\bigcup_{i=1}^{n-1} R_i) \cup R_n$ is also inclusive distance antimagic. □

Since $\Delta(P_n) = \Delta(C_n) = 2$, then based on the results by Dafik et al. [3] in Theorems 1.6 and 1.7, and using Theorems 3.2 and 3.3, we can derive the following corollaries:

Corollary 3.4. *Graph $P_m \cup C_n$ is inclusive distance antimagic, for $m \neq 2$ and $n \neq 2, 3$.*

Corollary 3.5. *Graph $\bigcup_{i=1}^{n} C_{k_i}$ is inclusive distance antimagic, for $k_i \neq 2, 3$.*

4. JOIN OF GRAPHS

In this section, we provide a definition of the join of graphs, followed by presenting several theorems related to the existence of inclusive distance antimagic property of join of graphs.

Definition 4.1. *The join between two graphs G and H, denoted by $G + H$, is a graph with vertex set and edge set as follows:*

$$V(G+H) = V(G) \cup V(H)$$
$$E(G+H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H).\}$$

Dafik et al. [3] provide an example of join graphs that are not inclusive distance antimagic, such as $P_2 + H, (P_2 \cup mK_1) + H$, and $K_n + H$. The following theorem summarizes those results more generally.

Theorem 4.2. *Let H be any graph, and G be a graph that has pairs of vertices with the same closed neighborhood set (meaning graph G is not inclusive distance antimagic). Then $G + H$ is not an inclusive distance antimagic graph.*

Proof. If there are two vertices in G, say u and v, have same closed neighborhood set in G, then u and v also have same closed neighborhood set in $G + H$. This is because in $G + H$, every vertex in graph G is connected to every vertex in graph H, including u and v. □

Corollary 4.3. *If any of the graphs $G_1, G_2, \cdots G_n$ has pairs of vertices with the same closed neighborhood set, then $G_1 + G_2 + \cdots + G_n$ is not an inclusive distance antimagic graph.*

Theorem 4.4. *Let graph G of order n be an inclusive distance antimagic graph. Then the graph $G + K_1$ is an inclusive distance antimagic graph if and only if $\Delta(G) \neq n - 1$.*

Proof. If $\Delta(G) = n - 1$, then in graph $G + K_1$ there exist at least two vertices of degree n. Since graph $G + K_1$ has order $n + 1$, then these vertices of degree n will have same close neighborhood set. For $\Delta(G) < n - 1$, let $V(G) = \{v_1, v_2, \cdots v_n\}$ and $f : V(G) \to \{1, 2, \cdots n\}$ be the inclusive distance antimagic labeling of G. Let $w(v_i)$ denote the weight of vertex v_i under the labeling f. Now, let u be the vertex of K_1. Define a bijection f' for $G+K_1$ such that $f'(v_i) = f(v_i)$ for every $1 \leq i \leq n$ and $f'(u) = n+1$. Then, the weights based on bijection f' are $w'(v_i) = w(v_i)+n+1$, for $1 \leq i \leq n$, and $w'(u) = 1 + 2 + \cdots + (n + 1) = \frac{(n+1)(n+2)}{2}$. Since there are no vertices in graph G of degree $n - 1$, then in $G + K_1$ only vertex u has degree n. Consequently, $\max_i w'(v_i) < w'(u)$, ensuring that each vertex in graph $G + K_1$ has distinct weight under the bijection f'. □

Aside from paths and cycles, Dafik et al. [3] also provide examples of graphs that are inclusive distance antimagic, such as star S_n, star DS_n, broom $Br_{n,m}$, and wheel W_n. Among all those graphs, only the wheel graph W_n that satisfies $\Delta = |V| - 1$, while the others do not. Hence based on Theorem 4.4, $S_n + K_1, DS_n + K_1$, and $Br_{n,m} + K_1$ are an inclusive distance antimagic graphs, whereas $W_n + K_1$ is not.

Theorem 4.5. *Let graph H of order m be an inclusive distance antimagic graph. For $2 \leq m \leq n$, graph $H + \overline{K_n}$ is also an inclusive distance antimagic graph.*

Proof. Let f_H be the inclusive distance antimagic labeling for H, with w_H be the weight of vertices under f_H. Define a bijection $f_{H+\overline{K_n}}$ such that $f_{H+\overline{K_n}}(v) = f_H(v)$, for every $v \in V(H)$, and each vertex in $\overline{K_n}$ can be labeled arbitrarily within the range $\{m + 1, m + 2, \cdots , m + n\}$. Then, the weight of vertices in graph $H + \overline{K_n}$ are:

$$w_{H+\overline{K_n}}(v) = \begin{cases} w_H(v) + (mn + 1 + 2 + \cdots + n), & v \in V(H) \\ f_{H+\overline{K_n}}(v) + (1 + 2 + \cdots + m), & v \in V(\overline{K_n}). \end{cases}$$

Obviously, $f_{H+\overline{K_n}}(v) \leq m+n \leq mn$, for every $v \in V(\overline{K_n})$, and $w_H(v) > 0$, for every $v \in V(H)$. Since $m \leq n$, then $1+2+\cdots+n \geq 1+2+\cdots+m$. Thus, for every $u \in V(H), v \in V(\overline{K_n})$, it holds that $w_{H+\overline{K_n}}(u) > w_{H+\overline{K_n}}(v)$. Moreover, it can be observed that for every vertex in H, and every vertex in $\overline{K_n}$, has distinct weights. Therefore, $H+\overline{K_n}$ is inclusive distance antimagic. □

There are several examples of graph $H+\overline{K_n}$, such as the fan graph $F_{n,m}$ which is the join graph $P_m+\overline{K_n}$, and the cone graph $C_{m,n}$ which is the join graph $C_m+\overline{K_n}$. According to Theorem 4.5, the fan graph $F_{n,m}$ for $2 < m \leq n$, and the cone graph $C_{m,n}$ for $3 < m \leq n$, are inclusive distance antimagic.

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