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Diachromatic Number of Some Acyclic Digraphs

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Abstract. A vertex coloring that ensures every pair of different colors is represented at least once is termed complete coloring. The diachromatic number of an acyclic digraph denotes the maximum number of colors required for its complete coloring. This study delves into the diachromatic numbers of lobster digraphs, fireworks digraphs, banana tree digraphs, and coconut tree digraphs under specific and arbitrary directional orientations.

Key words and Phrases: Vertex Coloring, Complete Coloring, Maximum Number of Colors

1. INTRODUCTION

A prominent concept in digraph theory is vertex coloring, which involves assigning colors to the vertices of a digraph so that no two adjacent vertices share the same color. A. F. Möbius (1790 - 1868) initially explored vertex coloring while investigating the map coloring problem, wherein vertices represent regions and adjacent regions are connected by edges. This inquiry famously led to the formulation of the four-color problem.

One advancement stemming from vertex coloring is complete coloring, which involves ensuring that each pair of different colors appears at least once in the coloring of vertices. An intriguing problem within vertex coloring concerns determining the minimum number of colors required for the coloring process, encapsulated by the chromatic number in graphs and the dichromatic number in digraphs as proposed by [1]. Specifically regarding complete coloring, [2] initially introduced the concept of the achromatic number in graphs, representing the maximum number of colors utilized in a complete coloring. Achromatic numbers are also explored in [3], with a specific focus on circulant graphs. In 2017, [4] introduced an extension of the achromatic number to digraphs, termed the diachromatic number. One of the studies on diachromatic number is conducted by [5], who specifically investigates the diachromatic number of the directed double star graph $\overrightarrow{K_{1,n,n}}$. Another study on

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diachromatic number is by [6], who explores the relationship between diachromatic number and harmonious chromatic number.

This paper considers various potential direction orientations for certain digraphs whose diachromatic numbers are under investigation. Consequently, we adopt the following direction orientations: O' for the lobster digraph $L_2(2;r)$, O'' for the firework digraph $F(2,p)$, O''' for the banana tree digraph $B_{2,t}$, and O'''' for the coconut tree digraph $CT(m,4)$.

Definition 1.1. [7] *The lobster graph $L_n(q;r)$ with $n, q, r \in \mathbb{N}$, $n \geq 2$ is a graph with vertex set*

$$V(L_n(q;r)) = \{b_i, h_{ij}, f_{ijk} | 1 \leq i \leq n, 1 \leq j \leq q, 1 \leq k \leq r, n \geq 2\},$$

and edge set

$$E(L_n(q;r)) = \{b_i b_{i+1} | 1 \leq i \leq n-1, n \geq 2\} \\ \cup \{h_{ij} b_i, f_{ijk} h_{ij} | 1 \leq i \leq n, 1 \leq j \leq q, 1 \leq k \leq r\}.$$

We define specifically for $L_2(2;r)$ orientation of direction O' is as follows; for the backbone vertices b_1 and b_2, $d_{out}(b_1) = 2$, $d_{in}(b_1) = 1$, $d_{out}(b_2) = 3$, and $d_{in}(b_2) = 0$, and while for the hand vertices h_{ij}, $d_{out}(h_{ij}) = r$ and $d_{in}(h_{ij}) = 1$ for $i = 1, 2$ and $j = 1, 2$, and for the finger vertices f_{ijk}, $d_{out}(f_{ijk}) = 0$ and $d_{in}(f_{ijk}) = 1$, for $i = 1, 2$, $j = 1, 2$ and $1 \leq k \leq r$. So we obtain a directed lobster graph $L_2(2;r)$ denoted by $\overrightarrow{L_2(2;r)}$.

Definition 1.2. [8] *The firework graph $F(n,p)$ with $n, p \geq 2$ is a graph with vertex set*

$$V(F(n,p)) = \{b_i, v_{ij} | 1 \leq i \leq n, 1 \leq j \leq p-1\}$$

and edge set

$$E(F(n,p)) = \{b_i v_{ij} | 1 \leq i \leq n, 1 \leq j \leq p-1\} \\ \cup \{v_{i1} v_{(i+1)1} | 1 \leq i \leq n-1\}.$$

Particularly, for $F(2,p)$ orientation of direction O'' is as follows; $d_{out}(b_1) = p-1$, $d_{in}(b_1) = 0$, $d_{out}(b_2) = p-1$, $d_{in}(b_2) = 0$, $d_{out}(v_{11}) = 1$, $d_{in}(v_{11}) = 1$, $d_{out}(v_{21}) = 0$, $d_{in}(v_{21}) = 2$, $d_{out}(v_{1j}) = 0$, $d_{in}(v_{1j}) = 1$, $d_{out}(v_{2j}) = 0$, and $d_{in}(v_{2j}) = 1$ for $2 \leq j \leq p-1$. Thus, we acquire a directed firework graph represented as $\overrightarrow{F(n,p)}$.

Definition 1.3. [9] *The banana tree graph $B_{n,t}$ with $n \geq 1$ and $t \geq 2$ is a graph with vertex set*

$$V(B_{n,t}) = \{v, b_i, v_{ij} | 1 \leq i \leq n, 1 \leq j \leq t-1\}$$

and edge set

$$E(B_{n,t}) = \{b_i v_{ij} | 1 \le i \le n, 1 \le j \le t-1, t \ge 2\}$$
$$\cup \{v_{i1} v | 1 \le i \le n\}.$$

Specifically, for the orientation of direction O''' in $B_{2,t}$, it is as follows; $d_{out}(b_1) = t-1$, $d_{in}(b_1) = 0$, $d_{out}(b_2) = t-1$, $d_{in}(b_2) = 0$, $d_{out}(v_{11}) = 1$, $d_{in}(v_{11}) = 1$, $d_{out}(v_{21}) = 0$, $d_{in}(v_{21}) = 2$, $d_{out}(v) = 1$, $d_{in}(v) = 1$, $d_{out}(v_{1j}) = 0$, $d_{in}(v_{1j}) = 1$, $d_{out}(v_{2j}) = 0$, and $d_{in}(v_{2j}) = 1$ for $2 \le j \le t-1$. Therefore, we obtain a directed banana tree graph denoted by $\overrightarrow{B_{2,t}}$. Subsequently, an illustration of the graph $\overrightarrow{B_{2,t}}$ with the orientation of direction O''' is presented.

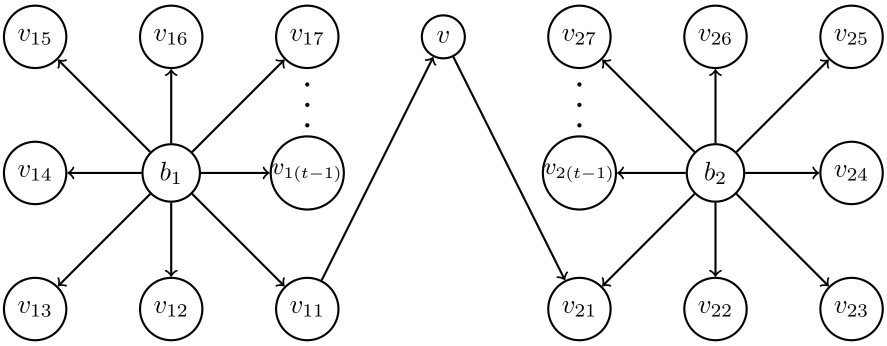

FIGURE 1. Banana Tree Graph $\overrightarrow{B_{2,t}}$

Definition 1.4. [10] *The coconut tree graph $CT(m,n)$ with $m, n \in \mathbb{N}$ is a graph with vertex set*

$$V(CT(m,n)) = \{b_i, v_j | 1 \le i \le n, 1 \le j \le m\},$$

and edge set

$$E(CT(m,n)) = \{b_i b_{i+1} | 1 \le i \le n-1\}$$
$$\cup \{b_k v_j | k = \max\{i\}, 1 \le i \le n, 1 \le j \le m\}.$$

Specifically, in the case of the orientation of direction O'''' for $CT(m,4)$, it is as follows; $d_{out}(b_1) = 1$, $d_{in}(b_1) = 0$, $d_{out}(b_2) = 1$, $d_{in}(b_2) = 1$, $d_{out}(b_3) = 1$, $d_{in}(b_3) = 1$, $d_{out}(b_4) = m$, $d_{in}(b_4) = 1$, $d_{out}(v_j) = 0$, and $d_{in}(v_j) = 1$ for $1 \le j \le m$. Hence, the result yields a directed banana tree graph represented as $\overrightarrow{CT(m,4)}$. An illustration of the graph $\overrightarrow{CT(m,4)}$ with orientation of direction O'''' is presented below.

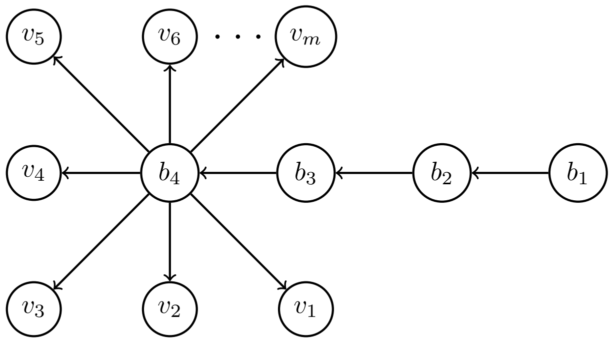

FIGURE 2. Coconut Tree Graph $\overrightarrow{CT(m,4)}$

Definition 1.5. [11] *Given a digraph $\overrightarrow{G}$. A complete coloring of the digraph $\overrightarrow{G}$ is a vertex coloring such that each different color pair appears at least once in the digraph $\overrightarrow{G}$.*

Lemma 1.6. *Given a digraph $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$. Let w be the number of colors that can be used in the complete coloring of $\overrightarrow{G}$. The value of w satisfies ${}_wP_2 \leq |A(\overrightarrow{G})|$.*

PROOF. Considering a digraph $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$ and denoting w as the number of colors that can be utilized for complete coloring of $\overrightarrow{G}$, assume ${}_wP_2 > |A(\overrightarrow{G})|$. Let $f : A(\overrightarrow{G}) \to C = \{C_1, C_2, ..., C_w\}$ represent a complete coloring function, where C is the set of color types. Without loss of generality, there must exist a color pair $\{C_i, C_j\} \subseteq C$, with $C_i \neq C_j$, such that for every arc $(x, y), (z, u) \in A(\overrightarrow{G})$, either $f(x) \neq C_i$, $f(y) \neq C_j$, $f(z) \neq C_j$, or $f(u) \neq C_i$. This contradicts the requirement of complete coloring in $\overrightarrow{G}$ that mandates every different color pair appears at least once. Hence, the assumption is incorrect. Consequently, the number of colors suitable for complete coloring satisfies ${}_wP_2 \leq |A(\overrightarrow{G})|$. □

2. MAIN RESULTS

Definition 2.1. [4] *The diachromatic number, denoted as $dac(\overrightarrow{G})$, refers to the maximum number of colors utilized in the complete coloring of an acyclic digraph $\overrightarrow{G}$.*

Theorem 2.2. *Given a lobster digraph* $\overrightarrow{L_2(2;r)}$ *with* $r \in \mathbb{N}$ *as Defined* 1.1 *and with direction orientation* O'. *It follows that* $dac(\overrightarrow{L_2(2;r)}) = 3$ *for* $r = 1, 2$ *and* $dac(\overrightarrow{L_2(2;r)}) = 4$ *for any* $r \geq 3$.

Proof. Considering a lobster digraph $\overrightarrow{L_2(2;r)}$ as Defined in Definition 1.1, where $r \in \mathbb{N}$, and with the following direction orientation: for the vertices backbone, b_1 and b_2, $d_{out}(b_1) = 2$, $d_{in}(b_1) = 1$, $d_{out}(b_2) = 3$, and $d_{in}(b_2) = 0$; for the vertices hand, h_{ij}, $d_{out}(h_{ij}) = r$ and $d_{in}(h_{ij}) = 1$ for $i = 1, 2$ and $j = 1, 2$; and for the vertices finger, f_{ijk}, $d_{out}(f_{ijk}) = 0$ and $d_{in}(f_{ijk}) = 1$ for $i = 1, 2$, $j = 1, 2$, and $1 \leq k \leq r$, let w represent the number of colors usable for digraph $\overrightarrow{L_2(2;r)}$. The lobster digraph $\overrightarrow{L_2(2;1)}$ comprises 9 arcs. Employing complete coloring, we have ${}_wP_2 \leq |A(\overrightarrow{L_2(2;1)})|$. Consequently, $w(w-1) \leq 9$, leading to the maximum $w = 3$. Let $A = \{C_1, C_2, C_3\}$ be the set of color types, and $f : V(\overrightarrow{L_2(2;1)}) \to A$ constitute a complete coloring. Without loss of generality, if $f(b_1) = C_2$ and $f(b_2) = C_1$, then $\overrightarrow{L_2(2;1)}$ can be colored with 3 colors, yielding $dac(\overrightarrow{L_2(2;1)}) = 3$. For $\overrightarrow{L_2(2;2)}$, let $B = \{C_1, C_2, C_3, C_4\}$ denote the set of color types, and $f : V(\overrightarrow{L_2(2;2)}) \to B$ be a complete coloring. Without loss of generality, if $f(b_1) = C_2$, $f(b_2) = C_1$, $f(h_{11}) = C_3$, and $f(h_{12}) = C_4$, some color pairs $\{C_2, C_t\}$ for $t \in \{1, 3, 4\}$ cannot be found in $\overrightarrow{L_2(2;2)}$ due to constraints on $d_{out}(b_1)$. Hence, $dac(\overrightarrow{L_2(2;2)}) < 4$, leading to $dac(\overrightarrow{L_2(2;2)}) = 3$. Similarly, for $\overrightarrow{L_2(2;3)}$ comprising 17 arcs, complete coloring yields ${}_wP_2 \leq |A(\overrightarrow{L_2(2;3)})|$, resulting in $w(w-1) \leq 17$ and $w = 4$. Let $C = \{C_1, C_2, C_3, C_4\}$ represent the set of color types, and $f : V(\overrightarrow{L_2(2;3)}) \to C$ constitute a complete coloring. Without loss of generality, if $f(b_1) = C_3$, $f(b_2) = C_1$, $f(h_{11}) = C_2$, $f(h_{12}) = C_1$, $f(h_{21}) = C_3$, and $f(h_{22}) = C_4$, the digraph $\overrightarrow{L_2(2;3)}$ can be colored with 4 colors, resulting in $dac(\overrightarrow{L_2(2;3)}) = 4$. For $\overrightarrow{L_2(2;r)}$ with $r \geq 4$, let $D = \{C_1, C_2, C_3, C_4, C_5\}$ denote the set of color types, and $f : V(\overrightarrow{L_2(2;r)}) \to D$ be a complete coloring. Without loss of generality, if $f(h_{21}) = C_1$, $f(h_{22}) = C_2$, $f(h_{11}) = C_4$, and $f(h_{12}) = C_3$, a pair of the form $\{C_5, C_t\}$ for $t \in \{1, 2, 3, 4\}$ cannot be found in $\overrightarrow{L_2(2;r)}$ due to constraints on $d_{out}(h_{21})$. Hence, complete coloring with 5 color types on $\overrightarrow{L_2(2;r)}$ for $r \geq 4$ is not possible, i.e., $dac(\overrightarrow{L_2(2;r)}) = 4$ for $r \geq 4$. □

Lemma 2.3. *Given a lobster digraph* $\overrightarrow{L_2(2;r)}$ *with* $r \in \mathbb{N}$ *as Definition* 1.1 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{L_2(2;r)}) \leq 5$ *for any* $r \geq 8$.

Proof. Consider a lobster digraph $\overrightarrow{L_2(2;r)}$ Defined in Definition 1.1 with $r \in \mathbb{N}$ and arbitrary orientation. When $r \geq 8$, $\overrightarrow{L_2(2;r)}$ comprises four star graphs, each with at least 8 leaves (vertices representing fingers). Let $A = \{C_1, C_2, C_3, C_4, C_5\}$ denote the set of color types, and the function $f : V(\overrightarrow{L_2(2;r)}) \to A$ ensures a complete coloring for any $r \geq 8$. From A, color pairs $\{C_i, C_k\}$ with $C_i \neq C_k$ are derived where $1 \le i \le 4$ and $i+1 \le k \le 5$. Note that the maximum number of such pairs with $i = 1$ is 4. Since each pair requires 2 arcs, there must be at least 8 arcs in $\overrightarrow{L_2(2;r)}$ to achieve complete coloring with 5 colors. Without loss of generality, if $f(h_{21}) = C_1$, $f(h_{22}) = C_2$, $f(h_{11}) = C_4$, and $f(h_{12}) = C_3$, and each vertex h_{21}, h_{22}, h_{11}, and h_{12} has 8 arcs, then complete coloring of $\overrightarrow{L_2(2;r)}$ with 5 colors is feasible. This yields $dac(\overrightarrow{L_2(2;r)}) \ge 5$. Now, let $B = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ represent the set of color types, and $f : V(\overrightarrow{L_2(2;r)}) \to B$ achieves complete coloring for any $r \ge 8$. From B, color pairs $\{C_i, C_k\}$ with $C_i \ne C_k$ are obtained where $1 \le i \le 5$ and $i+1 \le k \le 6$ to color $\overrightarrow{L_2(2;r)}$. Without loss of generality, if $f(h_{22}) = C_1, f(h_{21}) = C_2, f(h_{11}) = C_3$ and $f(h_{12}) = C_4$, then there exist pairs $\{C_5, C_6\}$ such that for all $(x,y),(z,u) \in A(\overrightarrow{L_2(2;r)})$, it holds $f(x) \ne C_5$ or $f(y) \ne C_6$ or $f(z) \ne C_6$ or $f(u) \ne C_5$. Thus, it is demonstrated that $\overrightarrow{L_2(2;r)}$ cannot be colored with complete coloring using 6 color types. It is observed that $dac(\overrightarrow{L_2(2;r)}) = 5$ for any $r \ge 8$. Moreover, due to the arbitrary orientation allowing the utilization of as many color types as possible, we conclude $dac(\overrightarrow{L_2(2;r)}) \le 5$, for any $r \ge 8$. □

Lemma 2.4. *Given a lobster digraph* $\overrightarrow{L_2(2;r)}$ *with* $r = 6$ *or* $r = 7$ *as Definition* 1.1 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{L_2(2;r)}) \le 5$ *for* $r = 6$ *or* $r = 7$.

Proof. Considering the lobster digraph $\overrightarrow{L_2(2;r)}$ Defined in Definition 1.1 with arbitrary orientation, let $A = \{C_1, C_2, C_3, C_4, C_5\}$ denote the set of color types, and the function $f : V(\overrightarrow{L_2(2;r)}) \to A$ represents the complete coloring for $r = 6$. Moreover, without loss of generality, suppose $f(h_{11}) = C_1$, $f(h_{12}) = C_2$, $f(h_{21}) = C_3$, $f(b_2) = C_5$, $f(b_1) = C_3$, and $f(h_{22}) = C_4$. It is evident that the digraph $\overrightarrow{L_2(2;6)}$ can be entirely colored using 5 colors, yielding $dac(\overrightarrow{L_2(2;6)}) = 5$. Furthermore, due to the arbitrary orientation allowing the utilization of as many color types as possible, we conclude $dac(\overrightarrow{L_2(2;6)}) \le 5$. The scenario for $r = 7$ mirrors that of $r = 6$. □

Lemma 2.5. *Given a lobster digraph* $\overrightarrow{L_2(2;5)}$ *as Definition* 1.1 *and with arbitrary digraph direction orientation. It follows that* $dac(\overrightarrow{L_2(2;5)}) \le 5$.

Proof.Given the lobster digraph $\overrightarrow{L_2(2;5)}$ defined in Definition 1.1 with arbitrary orientation, let $A = \{C_1, C_2, C_3, C_4, C_5\}$ represent the set of color types, and the function $f : V(\overrightarrow{L_2(2;5)}) \to A$ denotes a complete coloring. Furthermore, without loss of generality, if $f(h_{11}) = C_1$, $f(h_{12}) = C_2$, $f(h_{21}) = C_3$, $f(h_{22}) = C_5$, $f(b_1) = C_4$, and $f(f_{211}) = C_1$, then for every pair $\{C_i, C_j\} \subseteq A$ with $C_i \ne C_j$, there exist $(x,y),(z,u) \in A(\overrightarrow{L_2(2;5)})$ such that $f(x) = C_i$, $f(y) = C_j$, $f(z) = C_j$, and $f(u) = C_i$. This demonstrates that the lobster digraph $\overrightarrow{L_2(2;5)}$ can be colored using 5 color types with complete coloring. Hence, $dac(\overrightarrow{L_2(2;5)}) = 5$ is established. Moreover, due to the arbitrary orientation allowing the usage of as many colors as possible, we conclude $dac(\overrightarrow{L_2(2;5)}) \leq 5$. □

Lemma 2.6. *Given a lobster digraph $\overrightarrow{L_2(2;4)}$ as Definition* 1.1 *and with arbitrary direction orientation. It follows that $dac(\overrightarrow{L_2(2;4)}) \leq 4$.*

Proof. Given the lobster digraph $\overrightarrow{L_2(2;4)}$ defined in Definition 1.1 with arbitrary orientation, let $A = \{C_1, C_2, C_3, C_4, C_5\}$ denote the set of color types, and the function $f : V(\overrightarrow{L_2(2;4)}) \to A$ represent a complete coloring. Furthermore, without loss of generality, if $f(h_{11}) = C_1$, $f(h_{12}) = C_2$, $f(h_{21}) = C_3$, $f(h_{22}) = C_5$, $f(b_1) = C_5$, and $f(b_2) = C_1$, then there exists a pair $\{C_1, C_4\}$ such that for all $(x, y), (z, u) \in A(\overrightarrow{L_2(2;4)})$ it holds $f(x) \neq C_1$ or $f(y) \neq C_4$ or $f(z) \neq C_4$ or $f(u) \neq C_1$. Hence, complete coloring with 5 color types on the digraph $\overrightarrow{L_2(2;4)}$ is not feasible. Furthermore, let $B = \{C_1, C_2, C_3, C_4\}$ be the set of color types, and the function $f : V(\overrightarrow{L_2(2;4)}) \to B$ represents a complete coloring. Without loss of generality, if $f(h_{11}) = C_1$, $f(h_{12}) = C_2$, $f(h_{21}) = C_3$, and $f(h_{22}) = C_1$, then it evidently satisfies complete coloring with 4 color types. Thus, $dac(\overrightarrow{L_2(2;4)}) = 4$. Additionally, since the orientation is arbitrary allowing the usage of as many color types as possible, we conclude $dac(\overrightarrow{L_2(2;4)}) \leq 4$. □

Lemma 2.7. *Given a lobster digraph $\overrightarrow{L_2(2;3)}$ as Definition* 1.1 *and with arbitrary direction orientation. It follows that $dac(\overrightarrow{L_2(2;3)}) \leq 4$.*

Proof. Given the lobster digraph $\overrightarrow{L_2(2;3)}$ defined in Definition 1.1 with arbitrary orientation, let $A = \{C_1, C_2, C_3, C_4\}$ represent the set of color types, and the function $f : V(\overrightarrow{L_2(2;3)}) \to A$ denote a complete coloring. Without loss of generality, if $f(h_{11}) = C_1$, $f(h_{12}) = C_4$, $f(h_{21}) = C_2$, and $f(h_{22}) = C_3$, then it evidently satisfies complete coloring with 4 color types. Therefore, $dac(\overrightarrow{L_2(2;3)}) = 4$. Furthermore, since the orientation is arbitrary allowing the usage of as many color types as possible, we conclude $dac(\overrightarrow{L_2(2;3)}) \leq 4$. □

Lemma 2.8. *Given a lobster digraph $\overrightarrow{L_2(2;2)}$ as Definition* 1.1 *and with arbitrary direction orientation. It follows that $dac(\overrightarrow{L_2(2;2)}) \leq 3$.*

Proof. Given the lobster digraph $\overrightarrow{L_2(2;2)}$ defined in Definition 1.1 with arbitrary orientation, let $A = \{C_1, C_2, C_3, C_4\}$ represent the set of color types, and the function $f : V(\overrightarrow{L_2(2;2)}) \to A$ denote a complete coloring. Without loss of generality, if $f(h_{11}) = C_1$, $f(h_{12}) = C_2$, $f(h_{21}) = C_3$, and $f(h_{22}) = C_4$, then there exist pairs $\{C_1, C_3\}$ such that for all $(x, y), (z, u) \in A(\overrightarrow{L_2(2;2)})$ it holds $f(x) \neq C_1$ or $f(y) \neq C_3$ or $f(z) \neq C_3$ or $f(u) \neq C_1$. Hence, complete coloring with 4 color types on the digraph $\overrightarrow{L_2(2;2)}$ is not possible. Furthermore, suppose $B = \{C_1, C_2, C_3\}$ is the set of color types, and the function $f : V(\overrightarrow{L_2(2;2)}) \to B$ is a complete coloring. Without loss of generality, if $f(h_{11}) = C_1$, $f(h_{12}) = C_2$, and $f(h_{21}) = C_3$, then it evidently satisfies complete coloring with 3 color types. Thus, $dac(\overrightarrow{L_2(2;2)}) = 3$. Additionally, since the orientation is arbitrary allowing the usage of as many color types as possible, we conclude $dac(\overrightarrow{L_2(2;2)}) \leq 3$. □

Lemma 2.9. *Given a lobster digraph* $\overrightarrow{L_2(2;1)}$ *as Definition* 1.1 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{L_2(2;1)}) \leq 3$.

Proof. Given the lobster digraph $\overrightarrow{L_2(2;1)}$ defined in Definition 1.1 with arbitrary orientation, let $A = \{C_1, C_2, C_3\}$ represent the set of color types, and the function $f : V(\overrightarrow{L_2(2;1)}) \to A$ denote a complete coloring. Without loss of generality, if $f(h_{11}) = C_1$, $f(h_{12}) = C_3$, and $f(h_{21}) = C_2$, then it's evident that 3 color types can be utilized to fully color the lobster digraph $\overrightarrow{L_2(2;1)}$. Therefore, we have $dac(\overrightarrow{L_2(2;1)}) = 3$. Furthermore, since the orientation is arbitrary allowing the usage of as many colors as possible, we conclude $dac(\overrightarrow{L_2(2;1)}) \leq 3$. □

Theorem 2.10. *Given a lobster digraph* $\overrightarrow{L_2(2;r)}$ *defined in Definition* 1.1*, where* $r \in \mathbb{N}$*, and featuring arbitrary orientation, the following relationship is observed:*

$$dac(\overrightarrow{L_2(2;r)}) \leq \begin{cases} 3 & ; r = 1, 2, \\ 4 & ; r = 3, 4, \\ 5 & ; r \geq 5. \end{cases}$$

Proof. Given a lobster digraph $\overrightarrow{L_2(2;r)}$ defined in Definition 1.1, where $r \in \mathbb{N}$, and featuring arbitrary orientation. According to Lemma 2.3, 2.4, and 2.5, $dac(\overrightarrow{L_2(2;r)}) \leq 5$ holds for every $r \geq 5$. Subsequently, from Lemma 2.6 and 2.7, $dac(\overrightarrow{L_2(2;r)}) \leq 4$ is derived for $r = 3$ and $r = 4$. Moreover, according to Lemma 2.8 and 2.9, $dac(\overrightarrow{L_2(2;r)}) \leq 3$ is established for $r = 1$ and $r = 2$. Hence, we conclude:

$$dac(\overrightarrow{L_2(2;r)}) \leq \begin{cases} 3 & ; r = 1, 2, \\ 4 & ; r = 3, 4, \\ 5 & ; r \geq 5. \end{cases}$$

. □

Theorem 2.11. *Given a firework digraph* $\overrightarrow{F(2,p)}$ *with* $p \in \mathbb{N}$ *and* $p \geq 2$ *as defined by Definition 1.2 and with direction orientation* O''*. It follows that* $dac(\overrightarrow{F(2,p)}) = 2$ *for arbitrary* $p \geq 2$.

Proof. Given a firework digraph $\overrightarrow{F(2,p)}$ defined in Definition 1.2, where $p \in \mathbb{N}$ and $p \geq 2$, the orientation of the digraph follows as: $d_{out}(b_1) = p-1$, $d_{in}(b_1) = 0$, $d_{out}(b_2) = p-1$, $d_{in}(b_2) = 0$, $d_{out}(v_{11}) = 1$, $d_{in}(v_{11}) = 1$, $d_{out}(v_{21}) = 0$, $d_{in}(v_{21}) = 2$, $d_{out}(v_{1j}) = 0$, $d_{in}(v_{1j}) = 1$, $d_{out}(v_{2j}) = 0$, and $d_{in}(v_{2j}) = 1$ for $2 \leq j \leq p-1$. The firework digraph $\overrightarrow{F(2,2)}$ comprises 3 arcs. Let w denote the number of colors that can be utilized to color the digraph $\overrightarrow{F(2,p)}$. The coloring of the digraph adopts a complete coloring approach where each different color pair appears at least once. Hence, ${}_wP_2 \leq |A(\overrightarrow{F(2,2)})|$. Consequently, $w(w-1) \leq 3$. Thus, the largest value of w satisfying the inequality is $w = 2$. Subsequently, let $A = \{C_1, C_2\}$ be the set of color types, and the function $f : V(\overrightarrow{F(2,2)}) \to A$ represent a complete coloring. Without loss of generality, if $f(b_1) = C_1$ and $f(v_{11}) = C_2$, then evidently 2 color types of the digraph $\overrightarrow{F(2,2)}$ can be completely colored. Therefore, we have $dac(\overrightarrow{F(2,2)}) = 2$. Furthermore, the diachromatic number of the digraph $\overrightarrow{F(2,p)}$ can be at least completely colored with 2 colors. Thus, $dac(\overrightarrow{F(2,2)}) \geq 2$ is obtained. Next, let $B = \{C_1, C_2, C_3\}$ denote the set of color types, and the function $f : V(\overrightarrow{F(2,p)}) \to B$ represent a complete coloring. Without loss of generality, if $f(b_1) = C_1$, $f(v_{11}) = C_2$, $f(v_{21}) = C_1$, and $f(b_2) = C_3$, then there are pairs of the form $\{C_2, C_3\}$ such that for all $(x,y),(z,u) \in A(\overrightarrow{F(2,p)})$ it holds $f(x) \neq C_2$ or $f(y) \neq C_3$ or $f(z) \neq C_3$ or $f(u) \neq C_2$. Therefore, complete coloring with 3 color types on the digraph $\overrightarrow{F(2,p)}$ is not possible. Hence, $dac(\overrightarrow{F(2,p)}) < 3$. Consequently, $dac(\overrightarrow{F(2,p)}) = 2$. □

Lemma 2.12. *Given a firework digraph $\overrightarrow{F(2,p)}$ with $p \in \mathbb{N}$ and $p \geq 5$ as Definition 1.2 and with arbitrary direction orientation. It follows that $dac(\overrightarrow{F(2,p)}) \leq 3$ for any $p \geq 5$.*

Proof. Given a firework digraph $\overrightarrow{F(2,p)}$ defined in Definition 1.2, where $p \in \mathbb{N}$ and $p \geq 5$, with arbitrary orientation. For $p \geq 5$, the firework digraph $\overrightarrow{F(2,p)}$ consists of 2 star graphs, each with at least 4 leaves. Let $A = \{C_1, C_2, C_3\}$ represent the set of color types, and the function $f : V(\overrightarrow{F(2,p)}) \to A$ denote a complete coloring. Without loss of generality, if $f(b_1) = C_1$, $f(b_2) = C_3$, and $f(v_{11}) = C_3$, then it evidently achieves complete coloring with 3 color types. Thus, for $p \geq 5$, we have $dac(\overrightarrow{F(2,p)}) \geq 3$. Furthermore, let $B = \{C_1, C_2, C_3, C_4\}$ denote the set of color types, and the function $f : V(\overrightarrow{F(2,p)}) \to B$ represent a complete coloring. Without loss of generality, if $f(b_1) = C_1$ and $f(b_2) = C_2$, then there exists a pair $\{C_3, C_4\}$ such that for all $(x,y),(z,u) \in A(\overrightarrow{F(2,p)})$ it holds $f(x) \neq C_3$ or $f(y) \neq C_4$ or $f(z) \neq C_4$ or $f(u) \neq C_3$. Hence, complete coloring with 4 color types on the digraph $\overrightarrow{F(2,p)}$ is not feasible. We conclude $dac(\overrightarrow{F(2,p)})) = 3$ for any $p \geq 5$. Additionally, since the orientation is arbitrary, allowing the utilization of as many colors as possible, $dac(\overrightarrow{F(2,p)}) \leq 3$ for any $p \geq 5$. □

Lemma 2.13. *Given a firework digraph* $\overrightarrow{F(2,4)}$ *as Definition* 1.2 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{F(2,4)}) \leq 3$.

Proof. Considering a firework digraph $\overrightarrow{F(2,4)}$ defined in Definition 1.2, with arbitrary orientation. Let w represent the number of colors that can be employed to color the digraph $\overrightarrow{F(2,4)}$. Since $|A(\overrightarrow{F(2,4)})| = 7$ and utilizing complete coloring for the digraph $\overrightarrow{F(2,4)}$, we have ${}_wP_2 \leq 7$. Consequently, there exists no possibility of using 4 types of colors to completely color the digraph $\overrightarrow{F(2,4)}$. Furthermore, let $A = \{C_1, C_2, C_3\}$ denote the set of color types, and the function $f : V(\overrightarrow{F(2,4)}) \to A$ represent a complete coloring. Without loss of generality, if $f(b_1) = C_1$, $f(b_2) = C_2$, $f(v_{11}) = C_3$, and $f(v_{21}) = C_1$, then evidently the firework digraph $\overrightarrow{F(2,4)}$ can be completely colored with 3 color types. Hence, we conclude $dac(\overrightarrow{F(2,4)}) = 3$. Moreover, since the orientation is arbitrary, allowing the utilization of as many colors as possible, $dac(\overrightarrow{F(2,4)}) \leq 3$. □

Lemma 2.14. *Given a firework digraph* $\overrightarrow{F(2,3)}$ *as Definition* 1.2 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{F(2,3)}) \leq 2$.

Proof. Given a firework digraph $\overrightarrow{F(2,3)}$ defined in Definition 1.2, with arbitrary orientation. Let w denote the number of colors available to color the digraph $\overrightarrow{F(2,3)}$. Since $|A(\overrightarrow{F(2,3)})| = 5$, and employing complete coloring for the digraph $\overrightarrow{F(2,3)}$, we have ${}_wP_2 \leq 5$. Consequently, there is no possibility of using 3 color types to fully color the digraph $\overrightarrow{F(2,3)}$. Next, let $A = \{C_1, C_2\}$ represent the set of color types, and function $f : V(\overrightarrow{F(2,3)}) \to A$ denote a complete coloring. Without loss of generality, if $f(b_1) = C_1$ and $f(b_2) = C_2$, then clearly the firework digraph $\overrightarrow{F(2,3)}$ can be completely colored with 2 color types. Thus, we conclude $dac(\overrightarrow{F(2,3)}) = 2$. Furthermore, since the orientation is arbitrary, allowing the utilization of as many color types as possible, $dac(\overrightarrow{F(2,3)}) \leq 2$. □

Lemma 2.15. *Given a firework digraph* $\overrightarrow{F(2,2)}$ *as Definition* 1.2 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{F(2,2)}) \leq 2$.

Proof. Given a firework digraph $\overrightarrow{F(2,2)}$ defined in Definition 1.2, with arbitrary orientation. Let w represent the number of colors available to color the digraph $\overrightarrow{F(2,2)}$. Since $|A(\overrightarrow{F(2,2)})| = 3$, and utilizing complete coloring for the digraph $\overrightarrow{F(2,2)}$, we have ${}_wP_2 \leq 3$. Consequently, it is not possible to use 3 types of colors to fully color the digraph $\overrightarrow{F(2,2)}$. Next, suppose $A = \{C_1, C_2\}$ denotes the set of color types, and function $f : V(\overrightarrow{F(2,2)}) \to A$ represents a complete coloring. Without loss of generality, if $f(b_1) = C_1$ and $f(b_2) = C_2$, then evidently the firework digraph $\overrightarrow{F(2,2)}$ can be completely colored with 2 color types. Thus, we obtain $dac(\overrightarrow{F(2,2)}) = 2$. Furthermore, since the orientation is arbitrary, allowing the utilization of as many color types as possible, $dac(\overrightarrow{F(2,2)}) \leq 2$. □

Theorem 2.16. *Considering a firework digraph $\overrightarrow{F(2,p)}$ defined in Definition 1.2, with arbitrary orientation. We deduce from the following inequality:*

$$dac(\overrightarrow{F(2,p)}) \leq \begin{cases} 2 & ; p = 2, 3, \\ 3 & ; p \geq 4. \end{cases}$$

that for $p \in \mathbb{N}$ and $p \geq 2$, the diachromatic number of $\overrightarrow{F(2,p)}$ satisfies the given conditions.

Proof. Given a firework digraph $\overrightarrow{F(2,p)}$ defined in Definition 1.2, with arbitrary orientation. It is established through Lemma 2.12 and 2.13 that $dac(\overrightarrow{F(2,p)}) \leq 3$ holds true for every $p \geq 4$. Furthermore, according to Lemma 2.14 and 2.15, $dac(\overrightarrow{F(2,p)}) \leq 2$ is attained for $p = 3$ and $p = 4$. Consequently, we conclude that:

$$dac(\overrightarrow{F(2,p)}) \leq \begin{cases} 2 & ; p = 2, 3, \\ 3 & ; p \geq 4. \end{cases}$$

□

Theorem 2.17. *Given a banana tree digraph $\overrightarrow{B_{2,t}}$ with $t \in \mathbb{N}$ and $t \geq 2$ as Definition 1.3 and with direction orientation O'''. It follows that $dac(\overrightarrow{B_{2,t}}) = 2$ for any $t \geq 2$.*

Proof. Given a banana tree digraph $\overrightarrow{B_{2,t}}$ defined in Definition 1.3, where $t \in \mathbb{N}$ and $t \geq 2$, the orientation of the digraph is as follows: $d_{out}(b_1) = t-1$, $d_{in}(b_1) = 0$, $d_{out}(b_2) = t-1$, $d_{in}(b_2) = 0$, $d_{out}(v_{11}) = 1$, $d_{in}(v_{11}) = 1$, $d_{out}(v_{21}) = 0$, $d_{in}(v_{21}) = 2$, $d_{out}(v) = 1$, $d_{in}(v) = 1$, $d_{out}(v_{1j}) = 0$, $d_{in}(v_{1j}) = 1$, $d_{out}(v_{2j}) = 0$, and $d_{in}(v_{2j}) = 1$ for $2 \leq j \leq t-1$. The banana tree digraph $\overrightarrow{B_{2,2}}$ has 4 arcs. Let w denote the number of colors that can be used to color the digraph $\overrightarrow{B_{2,2}}$. The coloring of the digraph is complete coloring, ensuring that each different color pair appears at least once. Thus, ${}_wP_2 \leq |A(\overrightarrow{B_{2,2}})|$, leading to $w(w-1) \leq 4$. Consequently, the largest value of w satisfying the inequality is $w = 2$. Next, let $A = \{C_1, C_2\}$ represent the set of color types, and the function $f : V(\overrightarrow{B_{2,2}}) \to A$ represents a complete coloring. Without loss of generality, if $f(b_1) = C_1$ and $f(v_{11}) = C_2$, then obviously 2 color types of digraph $\overrightarrow{B_{2,2}}$ can be utilized for complete coloring. Hence, $dac(\overrightarrow{B_{2,2}}) = 2$. Furthermore, the digraph $\overrightarrow{B_{2,t}}$ can be at least completely colored with 2 colors. Therefore, $dac(\overrightarrow{B_{2,t}}) \geq 2$. Additionally, consider $A = \{C_1, C_2, C_3\}$ as the set of color types, and let $f : V(\overrightarrow{B_{2,t}}) \to A$ represent a complete coloring. Without loss of generality, if $f(b_1) = C_1, f(b_2) = C_2, f(v_{11}) = C_3, f(v) = C_2$, and $f(v_{21}) = C_3$, then there exists a pair $\{C_3, C_1\}$ such that for all $(x,y), (z,u) \in A(\overrightarrow{B_{2,t}})$, $f(x) \neq C_3$

or $f(y) \neq C_1$ or $f(z) \neq C_1$ or $f(u) \neq C_3$. Therefore, complete coloring with 3 color types on the digraph $\overrightarrow{B_{2,t}}$ is not feasible. Consequently, $dac(\overrightarrow{B_{2,t}}) < 3$, yielding $dac(\overrightarrow{B_{2,t}}) = 2$. □

Lemma 2.18. *Given a banana tree digraph $\overrightarrow{B_{2,t}}$ with $t \in \mathbb{N}$ and $t \geq 2$ as Definition* 1.3 *and with arbitrary direction orientation. It follows that $dac(\overrightarrow{B_{2,t}}) \leq 4$ for any $t \geq 7$.*

Proof. Consider a banana tree digraph $\overrightarrow{B_{2,t}}$ defined in Definition 1.3, where $t \in \mathbb{N}$ and $t \geq 2$, with an arbitrary direction orientation. For $t \geq 7$, $\overrightarrow{B_{2,t}}$ constitutes the union of 2 star graphs, each containing 6 or more leaves. Let $A = \{C_1, C_2, C_3, C_4\}$ denote the set of color types, and the function $f : V(\overrightarrow{B_{2,t}}) \to A$ represents a complete coloring for any $t \geq 7$. From A, color pairs $\{C_i, C_k\}$ with $C_i \neq C_k$ are derived, where $1 \leq i \leq 3$ and $i + 1 \leq k \leq 4$. It is observed that the maximum number of color pairs $\{C_i, C_k\}$ for $i = 1$ is 3 pairs. To ensure complete coloring with 4 color types, there must be at least 6 arcs in $\overrightarrow{B_{2,t}}$, considering each pair requires 2 arcs. Without loss of generality, if $f(b_1) = C_1, f(b_2) = C_2, f(v_{11}) = C_3, f(v) = C_4$, and $f(v_{21}) = C_3$, and each of the vertices b_1 and b_2 possesses 6 arcs, it is evident that complete coloring of $\overrightarrow{B_{2,t}}$ with 4 color types is feasible. Thus, $dac(\overrightarrow{B_{2,t}}) \geq 4$ for $t \geq 7$. Next, consider $B = \{C_1, C_2, C_3, C_4, C_5\}$ as the set of color types, and let $f : V(\overrightarrow{B_{2,t}}) \to B$ represent a complete coloring for any $t \geq 7$. Without loss of generality, if $f(b_1) = C_1, f(b_2) = C_2, f(v_{11}) = C_4, f(v) = C_5$, and $f(v_{21}) = C_4$, then there exist pairs $\{C_3, C_4\}$ such that for all $(x, y), (z, u) \in A(\overrightarrow{B_{2,t}})$, $f(x) \neq C_3$ or $f(y) \neq C_4$ or $f(z) \neq C_4$ or $f(u) \neq C_3$. Thus, complete coloring with 5 color types on $\overrightarrow{B_{2,t}}$ is infeasible, implying $dac(\overrightarrow{B_{2,t}}) < 5$. Consequently, $dac(\overrightarrow{B_{2,t}}) = 4$ for $t \geq 7$. Furthermore, since the orientation is arbitrary to maximize the utilization of color types, we obtain $dac(\overrightarrow{B_{2,t}}) \leq 4$ for $t \geq 7$. □

Lemma 2.19. *Given a banana tree digraph $\overrightarrow{B_{2,6}}$ as Definition* 1.3 *and with arbitrary direction orientation. It follows that $dac(\overrightarrow{B_{2,6}}) \leq 4$.*

Proof. Consider a banana tree digraph $\overrightarrow{B_{2,6}}$ as defined in Definition 1.3, with an arbitrary direction orientation. Let $A = \{C_1, C_2, C_3, C_4\}$ denote the set of color types, and the function $f : V(\overrightarrow{B_{2,6}}) \to A$ represents a complete coloring. Furthermore, without loss of generality, if $f(b_1) = C_1, f(b_2) = C_2, f(v_{22}) = C_1, f(v_{11}) = C_4, f(v) = C_3$, and $f(v_{21}) = C_4$, then it is evident that complete coloring of the digraph $\overrightarrow{B_{2,6}}$ with 4 color types is feasible. Hence, $dac(\overrightarrow{B_{2,6}}) = 4$ is achieved. Furthermore, since the orientation is arbitrary to maximize the utilization of color types, we deduce $dac(\overrightarrow{B_{2,6}}) \leq 4$. □

Lemma 2.20. *Given a banana tree digraph $\overrightarrow{B_{2,5}}$ as Definition* 1.3 *and with arbitrary direction orientation. It follows that $dac(\overrightarrow{B_{2,5}}) \leq 3$.*

Proof. Consider a banana tree digraph $\overrightarrow{B_{2,5}}$ as defined in Definition 1.3, with an arbitrary direction orientation. The banana tree digraph $\overrightarrow{B_{2,5}}$ comprises 10 arcs. Let w denote the number of colors that can be utilized to color the digraph $\overrightarrow{B_{2,5}}$. The coloring of the digraph is a complete coloring where each different color pair appears at least once. Therefore, ${}_wP_2 \leq |A(\overrightarrow{B_{2,5}})|$. This implies $w(w-1) \leq 10$. Consequently, the largest value of w satisfying the inequality is $w = 3$. Next, let $A = \{C_1, C_2, C_3\}$ represent the set of color types, and the function $f : V(\overrightarrow{B_{2,5}}) \to A$ signifies a complete coloring. Without loss of generality, if $f(b_1) = C_1$ and $f(b_2) = C_2$, then it is evident that 3 color types of the digraph $\overrightarrow{B_{2,5}}$ can be colored using complete coloring. Thus, $dac(\overrightarrow{B_{2,5}}) = 3$ is established. Furthermore, since the orientation is arbitrary to maximize the utilization of colors, we infer $dac(\overrightarrow{B_{2,5}}) \leq 3$. □

Lemma 2.21. *Given a banana tree digraph* $\overrightarrow{B_{2,4}}$ *as Definition* 1.3 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{B_{2,4}}) \leq 3$.

Proof. Given a banana tree digraph $\overrightarrow{B_{2,4}}$ as defined in Definition 1.3, with arbitrary direction orientation. Let $A = \{C_1, C_2, C_3\}$ represent the set of color types, and the function $f : V(\overrightarrow{B_{2,4}}) \to A$ denotes a complete coloring. Furthermore, without loss of generality, if $f(b_1) = C_1, f(b_2) = C_2, f(v_{11}) = C_2$, and $f(v_{21}) = C_1$, then it is evident that complete coloring of the digraph $\overrightarrow{B_{2,4}}$ with 3 color types is feasible. Thus, we conclude $dac(\overrightarrow{B_{2,4}}) = 3$. Moreover, since the orientation is arbitrary to maximize the utilization of color types, we have $dac(\overrightarrow{B_{2,4}}) \leq 3$. □

Lemma 2.22. *Given a banana tree digraph* $\overrightarrow{B_{2,3}}$ *as Definition* 1.3 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{B_{2,3}}) \leq 3$.

Proof. Given a banana tree digraph $\overrightarrow{B_{2,3}}$ as described in Definition 1.3, with arbitrary direction orientation. Let $A = \{C_1, C_2, C_3\}$ denote the set of color types, and the function $f : V(\overrightarrow{B_{2,3}}) \to A$ represent a complete coloring. Furthermore, without loss of generality, if $f(b_1) = C_1, f(b_2) = C_2, f(v) = C_1, f(v_{11}) = C_2$, and $f(v_{21}) = C_3$, then it is evident that complete coloring of the digraph $\overrightarrow{B_{2,3}}$ with 3 color types is possible. Therefore, we conclude $dac(\overrightarrow{B_{2,3}}) = 3$. Moreover, since the orientation is arbitrary to maximize the utilization of color types, we have $dac(\overrightarrow{B_{2,3}}) \leq 3$. □

Lemma 2.23. *Given a banana tree digraph* $\overrightarrow{B_{2,2}}$ *as Definition* 1.3 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{B_{2,2}}) \leq 2$.

Proof. Given a banana tree digraph $\overrightarrow{B_{2,2}}$ as defined in Definition 1.3, with arbitrary direction orientation. The digraph $\overrightarrow{B_{2,2}}$ consists of 4 arcs. Let w represent the number of colors that can be utilized to color the digraph $\overrightarrow{B_{2,2}}$. The coloring of the digraph is conducted as a complete coloring where each different color pair appears at least once. Consequently, ${}_wP_2 \leq |A(\overrightarrow{B_{2,2}})|$. This implies that $w(w-1) \leq 4$, leading to the largest feasible value of w being $w = 2$. Additionally, let $A = \{C_1, C_2\}$ denote the set of color types, and the function $f : V(\overrightarrow{B_{2,2}}) \to A$ represent a complete coloring. Without loss of generality, if $f(b_1) = C_1$ and $f(b_2) = C_1$, then it's evident that 2 color types can completely color the digraph $\overrightarrow{B_{2,2}}$. Hence, $dac(\overrightarrow{B_{2,2}}) = 2$. Furthermore, since the orientation is arbitrary and aims to utilize as many colors as possible, we conclude that $dac(\overrightarrow{B_{2,2}}) \leq 2$. □

Theorem 2.24. *Given a banana tree digraph $\overrightarrow{B_{2,t}}$ defined in Definition 1.3, with arbitrary direction orientation. It follows that*

$$dac(\overrightarrow{B_{2,t}}) \leq \begin{cases} 2 & ; t = 2, \\ 3 & ; t = 3, 4, 5, \\ 4 & ; t \geq 6. \end{cases}$$

Proof. Given a banana tree digraph $\overrightarrow{B_{2,t}}$ as defined in Definition 1.3, with arbitrary direction orientation. According to Lemma 2.18 and 2.19, $dac(\overrightarrow{B_{2,t}}) \leq 4$ holds for every $t \geq 6$. Subsequently, Lemma 2.20, Lemma 2.21, and Lemma 2.22 yield $dac(\overrightarrow{B_{2,t}}) \leq 3$ for $t = 3$, $t = 4$, and $t = 5$. Moreover, Lemma 2.23 establishes $dac(\overrightarrow{B_{2,t}}) \leq 2$ for $t = 2$. Consequently, we conclude

$$dac(\overrightarrow{B_{2,t}}) \leq \begin{cases} 2 & ; t = 2, \\ 3 & ; t = 3, 4, 5, \\ 4 & ; t \geq 6. \end{cases}$$

□

Theorem 2.25. *Given a coconut tree digraph $\overrightarrow{CT(m,4)}$ with $m \in \mathbb{N}$ as Definition* 1.4 *and with direction orientation O''''. Then we have $dac(\overrightarrow{CT(m,4)}) = 2$.*

Proof. Given a coconut tree digraph $\overrightarrow{CT(m,4)}$ as defined in Definition 1.4, with $m \in \mathbb{N}$. The direction orientation of the digraph is as follows: $d_{out}(b_1) = 1$, $d_{in}(b_1) = 0$, $d_{out}(b_2) = 1$, $d_{in}(b_2) = 1$, $d_{out}(b_3) = 1$, $d_{in}(b_3) = 1$, $d_{out}(b_4) = m$, $d_{in}(b_4) = 1$, $d_{out}(v_j) = 0$, and $d_{in}(v_j) = 1$ for $1 \leq j \leq m$. The coconut tree digraph $\overrightarrow{CT(1,4)}$ has 4 arcs. Let w represent the number of colors that can be used to color the digraph $\overrightarrow{CT(m,4)}$. The coloring of the digraph utilizes complete coloring, ensuring that each different color pair appears at least once. Thus, ${}_wP_2 \leq |A(\overrightarrow{CT(1,4)})|$. Consequently, $w(w-1) \leq 4$. Therefore, the largest value of w satisfying this inequality is $w = 2$. Next, let $A = \{C_1, C_2\}$ denote the set of color types, and the function $f : V(\overrightarrow{CT(1,4)}) \to A$ represents a complete coloring. Without loss of generality, if $f(b_1) = C_1$, $f(b_2) = C_2$, and $f(b_3) = C_1$, then it's evident that 2 color types of the digraph $\overrightarrow{CT(1,4)}$ can be completely colored. Thus, we have $dac(\overrightarrow{CT(1,4)}) = 2$. Moreover, the diachromatic number of the digraph $\overrightarrow{CT(m,4)}$ can be at least completely colored with 2 colors. Hence, $dac(\overrightarrow{CT(m,4)}) \geq 2$. Furthermore, let $B = \{C_1, C_2, C_3\}$ represent the set of color types, and the function $f : V(\overrightarrow{CT(m,4)}) \to B$ is a complete coloring. Without loss of generality, if $f(b_4) = C_1$, $f(b_1) = C_2$, $f(b_2) = C_3$, and $f(b_3) = C_2$, then there exists a pair of the form $\{C_3, C_1\}$ such that for all $(x,y),(z,u) \in A(\overrightarrow{CT(m,4)})$ holds $f(x) \neq C_3$ or $f(y) \neq C_1$ or $f(z) \neq C_1$ or $f(u) \neq C_3$. Hence, complete coloring with 3 color types on the digraph $\overrightarrow{CT(m,4)}$ is not possible. Therefore, $dac(\overrightarrow{CT(m,4)}) < 3$. Consequently, $dac(\overrightarrow{CT(m,4)}) = 2$. □

Lemma 2.26. *Given a coconut tree digraph* $\overrightarrow{CT(m,4)}$ *with* $m \in \mathbb{N}$ *as Definition* 1.4 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{CT(m,4)}) \leq 3$ *for arbitrary* $m \geq 3$.

Proof. Given a coconut tree digraph $\overrightarrow{CT(m,4)}$ with $m \in \mathbb{N}$ and $m \geq 3$ as specified in Definition 1.4, with arbitrary directional orientation. For $m \geq 3$, the coconut tree digraph $\overrightarrow{CT(m,4)}$ comprises a combination of a star digraph and a path digraph, where the star digraph has at least 3 leaves. Let $A = \{C_1, C_2, C_3\}$ represent the set of color types, and the function $f : V(\overrightarrow{CT(m,4)}) \to A$ denotes a complete coloring. Without loss of generality, if $f(b_4) = C_1$, $f(b_1) = C_2$, $f(b_2) = C_3$, and $f(b_3) = C_2$, then it evidently achieves complete coloring with 3 color types. Hence, for $m \geq 3$, $dac(\overrightarrow{CT(m,4)}) \geq 3$ is established. Furthermore, let $B = \{C_1, C_2, C_3, C_4\}$ denote the set of color types, and the function $f : V(\overrightarrow{CT(m,4)}) \to B$ signifies a complete coloring. Without loss of generality, if $f(b_4) = C_1$, $f(b_1) = C_2$, $f(b_2) = C_3$, and $f(b_3) = C_2$, then pairs of the form $\{C_3, C_4\}$ are present such that for all $(x,y),(z,u) \in A(\overrightarrow{CT(m,4)})$ it holds that $f(x) \neq C_3$ or $f(y) \neq C_4$ or $f(z) \neq C_4$ or $f(u) \neq C_3$. Therefore, complete coloring with 4 color types on the digraph $\overrightarrow{CT(m,4)}$ is not achievable. Thus, $dac(\overrightarrow{CT(m,4)}) = 3$ for any $m \geq 3$. Furthermore, considering the arbitrary orientation to utilize as many colors as possible, $dac(\overrightarrow{CT(m,4)}) \leq 3$ for any $m \geq 3$. □

Lemma 2.27. *Given a coconut tree digraph* $\overrightarrow{CT(2,4)}$ *as Definition* 1.4 *and with arbitrary direction orientation. It follows that* $dac(\overrightarrow{CT(2,4)}) \leq 2$.

Proof. Given a coconut tree digraph $\overrightarrow{CT(2,4)}$ as defined in Definition 1.4, with arbitrary directional orientation. Let w denote the number of colors that can be utilized to color the digraph $\overrightarrow{CT(2,4)}$. Considering that $|A(\overrightarrow{CT(2,4)})| = 5$ and employing complete coloring for the digraph, we have ${}_wP_2 \leq 5$. Consequently, it's impossible to employ 3 types of colors for complete coloring of the digraph $\overrightarrow{CT(2,4)}$.

Next, let's consider $A = \{C_1, C_2\}$ as the set of color types, and the function $f : V(\overrightarrow{CT(2,4)}) \to A$ represents a complete coloring. Without loss of generality, if $f(b_1) = C_1$, $f(b_2) = C_2$, and $f(b_3) = C_1$, it's evident that the coconut tree digraph $\overrightarrow{CT(2,4)}$ can be entirely colored with 2 color types. Hence, $dac(\overrightarrow{CT(2,4)}) = 2$. Furthermore, considering the arbitrary orientation to employ as many color types as possible, we obtain $dac(\overrightarrow{CT(2,4)}) \leq 2$. □

Lemma 2.28. *Given a coconut tree digraph $\overrightarrow{CT(1,4)}$ as Definition* 1.4 *and with arbitrary direction orientation. It follows that $dac(\overrightarrow{CT(1,4)}) \leq 2$.*

Proof. Given a coconut tree digraph $\overrightarrow{CT(1,4)}$ as defined in Definition 1.4, with arbitrary directional orientation. Let $A = \{C_1, C_2\}$ represent the set of color types, and the function $f : V(\overrightarrow{CT(1,4)}) \to A$ denotes a complete coloring. Without loss of generality, if $f(b_1) = C_1$, $f(b_2) = C_2$, and $f(b_3) = C_1$, it's evident that the coconut tree digraph $\overrightarrow{CT(1,4)}$ can be completely colored with 2 color types. Therefore, $dac(\overrightarrow{CT(1,4)}) = 2$. Furthermore, considering the arbitrary orientation to utilize as many color types as possible, we obtain $dac(\overrightarrow{CT(1,4)}) \leq 2$. □

Theorem 2.29. *Given a coconut tree digraph $\overrightarrow{CT(m,4)}$ as defined in Definition* 1.4*, with arbitrary directional orientation. It follows that*

$$dac(\overrightarrow{CT(m,4)}) \leq \begin{cases} 2 & ; m = 1, 2, \\ 3 & ; m \geq 3. \end{cases}$$

Proof. Given a coconut tree digraph $\overrightarrow{CT(m,4)}$ with $m \in \mathbb{N}$ as defined in Definition 1.4, and with arbitrary directional orientation. Lemma 2.26 establishes that $dac(\overrightarrow{CT(m,4)}) \leq 3$ holds for every $m \geq 3$. Furthermore, according to Lemma 2.27 and 2.28, $dac(\overrightarrow{CT(m,4)}) \leq 2$ is valid for $m = 1$ and $m = 2$. Thus, we conclude that

$$dac(\overrightarrow{CT(m,4)}) \leq \begin{cases} 2 & ; m = 1, 2, \\ 3 & ; m \geq 3. \end{cases}$$

□

3. CONCLUSION

In this paper, we investigate the diachromatic numbers of several acyclic directed graphs, leading to the following conclusions:

(1) For the lobster digraph $\overrightarrow{L_2(2;r)}$ with the direction orientation O', we find that $dac(\overrightarrow{L_2(2;r)}) = 3$ holds when $r = 1, 2$, and $dac(\overrightarrow{L_2(2;r)}) = 4$ for any $r \geq 3$.

(2) Regarding the lobster digraph $\overrightarrow{L_2(2;r)}$ with an arbitrary direction orientation, the diachromatic number satisfies

$$dac(\overrightarrow{L_2(2;r)}) \leq \begin{cases} 3 & ; r = 1, 2, \\ 4 & ; r = 3, 4, \\ 5 & ; r \geq 5. \end{cases}.$$

(3) For the fireworks digraph $\overrightarrow{F(2,p)}$ with the direction orientation O'', $dac(\overrightarrow{F(2,p)}) = 2$ is observed for any $p \geq 2$.

(4) The diachromatic number of the fireworks digraph $\overrightarrow{F(2,p)}$ with an arbitrary direction orientation is given by

$$dac(\overrightarrow{F(2,p)}) \leq \begin{cases} 2 & ; p = 2, 3, \\ 3 & ; p \geq 4. \end{cases}.$$

(5) The diachromatic number of the banana tree digraph $\overrightarrow{B_{2,t}}$ with the direction orientation O''' is $dac(\overrightarrow{B_{2,t}}) = 2$ for any $t \geq 2$.

(6) When considering the banana tree digraph $\overrightarrow{B_{2,t}}$ with an arbitrary direction orientation, we find

$$dac(\overrightarrow{B_{2,t}}) \leq \begin{cases} 2 & ; t = 2, \\ 3 & ; t = 3, 4, 5, \\ 4 & ; t \geq 6. \end{cases}$$

(7) The diachromatic number of the coconut tree digraph $\overrightarrow{CT(m,4)}$ with the direction orientation O'''' is $dac(\overrightarrow{CT(m,4)}) = 2$ for any $m \in \mathbb{N}$.

(8) For the coconut tree digraph with an arbitrary direction orientation, the diachromatic number is bounded by

$$dac(\overrightarrow{CT(m,4)}) \leq \begin{cases} 2 & ; m = 1, 2, \\ 3 & ; m \geq 3. \end{cases}$$

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