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Analysis of Primary Resonance in Damped Duffing-Helmholtz Oscillator Excited by an Amplitude-Modulated Signal

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Abstract. The effect of amplitude modulated signal on the primary resonance on a Duffing-Helmholtz oscillator has been investigated using (i) a multiple time scale perturbation method involving two different time scales i.e., regular time (T_0) and a slow time, (T_1), to analyze the system response and (ii) its modified version which is valid for larger values of expansion parameter viz., asymptotic ordering parameter, incorporating Lindstedt-Poincare transformation method that allows the interaction of the amplitude and frequency of the response. The modification of the amplitude- frequency plots, exhibiting bi-stable behavior, caused by different values of modulation amplitude and damping has been obtained both analytically and numerically for various parameters of the system near the primary resonance. The peak amplitude of the response is shown to get lowered in strongly nonlinear Duffing-type system. Phase plane trajectories of stable and unstable manifolds are presented in different instances.

Key words and Phrases: Duffing-Helmholtz oscillator, bi-stability, amplitude modulation, primary and secondary resonance, stable and unstable manifolds.

1. INTRODUCTION

Oscillators comprising of both cubic and quadratic nonlinear terms are termed in Kovacic and Gatti [1] as Duffing-Helmholtz oscillators. In case of a linear damped system with displacement, $Q(t)$, we may represent Duffing-Helmholtz system as:

$$\ddot{Q} + 2\mu_0\dot{Q} + \omega_0^2 Q + I_1 Q^2 + I_2 Q^3 \quad = \quad 0, \tag{1}$$

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where μ_0, ω_0, I_1 and I_2 refer the linear damping coefficient, the natural frequency, strength of quadratic and cubic non-linearities of the system, respectively. It may be noted that, for the presence of only quadratic non-linearity, i.e. $I_2 = 0$, equation (1) represents a Helmholtz asymmetric oscillator as in Kovacic and Brennan [2] while in the absence of the quadratic nonlinear term i.e., $I_1 = 0$, equation (1) refers to the Duffing type oscillator used earlier to study pendulum dynamics with cubic nonlinear restoring term (Kovacic and Gatti [1]). The application of Cardano's transformation i.e., $q(t) = Q(t) + \frac{I_1}{3I_2}$ allows one to write the Duffing-Helmholtz system (1.1) as a Duffing type oscillator,

$$\ddot{q} + 2\mu_0\dot{q} + \Omega_0^2 q + I_2 q^3 \quad = \quad f - k_0, \tag{2}$$

with $\Omega_0^2 = \omega_0^2 - \frac{I_1^2}{3I_2}$ defining the transformed natural frequency of the system, $k_0 = \frac{2I_1^3}{27I_2^2} - \omega_0^2\frac{I_1}{3I_2}$ is a constant and may be considered as a steady bias for the oscillator as in Das [3] and f is the time dependent excitation that controls the dynamics externally (Mallik [4]). Various instances of the Duffing type oscillator and synchronization of coupled oscillators have been investigated earlier, both analytically and numerically in Bhardwaj and Das [5], Mickens [6] and Nayfeh and Mook [7] and, in recent years in Brennan and Kovacic [8], Chen, et.al. [9], Das and Bhardwaj [10], Das [3], Kalmár-Nagy and Balachandran [11], Lie, et.al. [12], Mickens [13], Ramlan, et.al. [14], WaWrzynski [15], WaWrzynski [16]. Among the commonly used tools of nonlinear system analysis (Chaudhary, et.al. [17] and Kovacic and Brennan [18]) viz., frequency response curve, backbone curve, time histories, phase portraits, bifurcation analysis, Poincare section, basin of attraction, etc., the frequency response curve is commonly used in engineering mechanics to estimate the resonance frequency, jump-up/down frequencies etc. In particular, recently in WaWrzynski [16], the idea of origin point is introduced, based on bistability region and unstable solution domain of the periodically excited Duffing system, to understand its dynamics. Earlier, in Brennan, et.al. [19], Nayfeh and Mook [7], using Harmonic balance method (Mickens [6]) wherein a periodic solution is expressed in terms of superposition of sinusoidal functions and multi-scale perturbation analysis involving two time scales, a linearly damped harmonically excited Duffing type system is known to exhibit jump-up frequency as dependent mainly on the nonlinearity parameter, I_2, whereas the jump-down frequency is observed to be dependent on both I_2 and the damping parameter, μ, respectively. The primary aim of the present work is to recast the usual small perturbation method (Mickens [13], Nayfeh and Mook [7]) of multiple time scales perturbation method [MTPM] (applicable to only weakly nonlinear forced oscillatory system) to be amenable to investigate the *strongly nonlinear* oscillator, *i.e.*, larger values of the nonlinear parameter I_2, as well. Such a procedure involves (i) defining a new expansion parameter, α as a function of weak nonlinear parameter, ϵ and (ii) the introduction of a frequency de-tuning parameter, σ, in the squared frequency, Ω^2, of the forcing term as in Burton and Rahman [20]. Based on the improved/modified method, the effect of more general form of excitation, *i.e.*, amplitude modulated excitation (AME) on the primary and secondary resonance of a linearly damped

Duffing type system, *i.e.*, equation (2), has been investigated. Using the improved procedure, the amplitude-frequency response curves of Duffing type system when excited by AME, for different parameter settings are obtained in case of primary resonance.

In section II, we briefly outline the method of MTPM applied to weakly nonlinear Duffing type oscillator and its extension for the strongly nonlinear case. The numerical results regarding the effect of AME on the primary resonance both for (*i*) weakly and (*ii*) strongly nonlinear oscillator, for different values of parameters of the system are discussed. In section III, we present the results of numerical simulation of the response of the system. Appendix further provides some insight into the stability of the system.

2. AMPLITUDE MODULATION EXCITATION EFFECT ON PRIMARY RESONANCE OF DUFFING-TYPE OSCILLATOR

In this section, we consider the forcing function, $f(t)$, of the externally excited Duffing system, equation (2), to be amplitude modulated. Therefore, we write, $f(t)$, as (WaWrzynski [16]):

$$f(t) \quad = \quad [f_{10} + 2g_{10}\cos(\Omega_m t)]\sin(\Omega t), \tag{3}$$

where f_{10} and g_{10} are amplitudes of the external harmonic excitation and modulation with main excitation frequency or carrier frequency, Ω, and modulating frequency, Ω_m, respectively. Since, $2\cos(\Omega_m t)\sin(\Omega t) = \sin(\Omega + \Omega_m)t + \sin(\Omega - \Omega_m)t$, the AME excitation of the Duffing type system therefore comprises of two additional frequencies, namely, (1) $\Omega + \Omega_m$ and (2) $\Omega - \Omega_m$ besides the main excitation frequency , Ω. Therefore, with increased complexity of excitation, we may write the reduced form of the Duffing- type equation as,

$$\ddot{q} + 2\mu_0\dot{q} + \Omega_0^2 q + I_2 q^3 \quad = \quad -k_0 + [f_{10} + 2g_{10}\cos(\Omega_m t)]\sin(\Omega t). \tag{4}$$

The dynamical response of the AME Duffing type system, *i.e.*, equation (4), is investigated, in the following, using the multi time scale expansion method (Mickens [13], Mickens [6], Nayfeh and Mook [7]) for both instances of weak and strong nonlinearity.

2.1. Response of the weakly nonlinear AME excited Duffing- type system using Multi Time Scale Perturbation Method.

In this section, we focus our attention to the comparison of the dynamic response of a forced dynamic system with weak nonlinearity and the corresponding forced linear case. In order to facilitate such a comparison, a small asymptotic ordering parameter, $\epsilon << 1$, is introduced and rewrite $k_0 = \epsilon k$, $f_{10} = \epsilon f_1$, $g_{10} = \epsilon g_1$ and both damping parameter, $\mu_0 = \epsilon\mu$ and the nonlinearity parameter as $I_2 = \epsilon \mathrm{I}$ in Duffing type equation (4), as

$$\ddot{q} + 2\epsilon\mu\dot{q} + \Omega_0^2 q + \epsilon \mathrm{I} q^3 \quad = \quad \epsilon\left\{[f_1 + 2g_1\cos(\Omega_m t)]\sin(\Omega t) - k\right\}, \tag{5}$$

where besides weak damping, weak non-linearity, AME term, f is also considered to be weak. The foregoing system, in the absence of quadratic nonlinearity, i.e.,

$I_1 = 0$ appearing in equation (1) and only harmonically excited Duffing system has been investigated earlier in Chen, et.al. [9].

In order to obtain the solution for the case of weak forcing, weak damping and weakly nonlinear Duffing type equation (5) we define (Mickens [13], Nayfeh and Mook [7]),

$$\mathrm{T}_n = \epsilon^n t \quad \text{with } n = 0, 1, 2, \cdots . \tag{6}$$

Therefore, the derivative with respect to t is now written as expansion in terms of partial derivative with respect to T_n and hence, we have

$$\frac{d}{dt} = \frac{\partial}{\partial \mathrm{T}_0} + \epsilon \frac{\partial}{\partial \mathrm{T}_1} + \cdots \quad = D_0 + \epsilon D_1 + \cdots \quad \text{and} \quad \frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \cdots \tag{7}$$

where expansions are taken up to $\mathrm{O}(\epsilon^1)$.

Let the solution of equation (5) be represented in terms of perturbation parameter ϵ as

$$q(\mathrm{T};\epsilon) = q_0(\mathrm{T}_0, \mathrm{T}_1) + \epsilon q_1(\mathrm{T}_0, \mathrm{T}_1, \cdots) + \cdots . \tag{8}$$

Substituting for $q(\mathrm{T};\epsilon)$ in equation (5) and equating the terms with same powers of ϵ, we get the following equations as,

$$\epsilon^0 : \quad D_0^2 q_0 + \Omega_0^2 q_0 = 0, \tag{9}$$

$$\begin{aligned} \epsilon^1 : \quad D_0^2 q_1 + \Omega_0^2 q_1 &= -2D_1 D_0 q_0 - 2\mu D_0 q_0 - \mathrm{I} q_0^3 \\ &\quad + (f_1 + 2g_1 \cos \Omega_m t) \sin \Omega t - k. \end{aligned} \tag{10}$$

The general solution of equation (9) may be written as,

$$q_0(\mathrm{T}_0, \mathrm{T}_1) = A(\mathrm{T}_1) e^{i\Omega_0 \mathrm{T}_0} + A^*(\mathrm{T}_1) e^{-i\Omega_0 \mathrm{T}_0}, \tag{11}$$

where $A(\mathrm{T}_1)$ and its complex conjugate $A^*(\mathrm{T}_1)$ are functions of T_1. It may be noted that $A(\mathrm{T}_1)$ and its conjugate need to be determined. Following the method described in Nayfeh and Mook [7], the solution for $A(\mathrm{T}_1)$ can be represented in polar form as

$$A(\mathrm{T}_1) = \frac{1}{2}\, a(\mathrm{T}_1)\, \exp\{i\phi(\mathrm{T}_1)\} \quad \text{and} \quad A^*(\mathrm{T}_1) = \frac{1}{2}\, a(\mathrm{T}_1)\, \exp\{-i\phi(\mathrm{T}_1)\}, \tag{12}$$

where $a(T_1)$ and $\phi(T_1)$ refers to the amplitude and phase of the response.

As a result of substitution for $q_0(\mathrm{T}_0, \mathrm{T}_1)$, from equation (11), in equation (10), we get

$$\begin{aligned} D_0^2 q_1 + \Omega_0^2 q_1 &= -\left\{2i\Omega_0 A' + 2i\mu\Omega_0 A + 3A^2 A^* I\right\} e^{i\Omega_0 \mathrm{T}_0} - \mathrm{I}_2 A^3 e^{3i\Omega_0 \mathrm{T}_0} \\ &\quad - \frac{i}{2}\left[f_1 e^{i\Omega\ t} + g_1 \left\{e^{i(\Omega+\Omega_m)\ t} + e^{i(\Omega-\Omega_m)\ t}\right\}\right] - k + \mathrm{cc}, \end{aligned} \tag{13}$$

where $'$ denotes the derivative with respect to T_1 and cc represents complex conjugate terms. Considering the setting where the main external forcing frequency is close to the system's natural frequency, Ω_0, we may write Ω in terms of detuning parameter σ as,

$$\Omega \equiv \Omega_0 + \epsilon\sigma. \tag{14}$$

Depending on the values of excitation and modulating frequencies, we consider the following different scenarios,

$$\text{(i)}\ \ \Omega_m << \Omega, \quad \text{(ii)}\ \ \Omega_m = \Omega.$$

In general, as in the case of communication engineering (Kennedy, Davis and Prasanna [21]), the modulating frequency Ω_m is much less than that of the excitation or carrier frequency, Ω. Therefore, using definitions viz., equations (12) and (14), equation (13) becomes

$$\begin{aligned} D_0^2 q_1 + \Omega_0^2 q_1 &= -\left\{ i\Omega_0(a' + ia\phi') + i\Omega_0\mu a + \frac{3}{8}a^3\mathrm{I} + \frac{i}{2}(f_1 + 2g_1)e^{i(\sigma\mathrm{T}_1 - \phi)} \right\} \times \\ &\times \quad e^{i(\Omega_0\mathrm{T}_0 + \phi)} - \frac{\mathrm{I}}{8}a^3 e^{3i(\Omega_0\mathrm{T}_0 + \phi)} - k + \mathrm{cc}. \end{aligned} \tag{15}$$

Putting $\sigma\mathrm{T}_1 - \phi = \gamma$ and then equating the secular term to zero, we get the following equations for the amplitude and phase response of the present Duffing type system as,

$$\Omega_0 a' = -\Omega_0\mu a - \frac{f_1 + 2g_1}{2}\cos\gamma, \tag{16}$$

$$\Omega_0 a\gamma\,' = \Omega_0\sigma a - \frac{3}{8}\mathrm{I}a^3 + \frac{f_1 + 2g_1}{2}\sin\gamma. \tag{17}$$

The foregoing expression for the amplitude and phase response results in the relation for the detuning parameter σ, in the steady state as,

$$\sigma = \frac{3\mathrm{I}}{8\Omega_0}a^2 \pm \left[\frac{(f_1 + 2g_1)^2}{4\Omega_0^2 a^2} - \mu^2\right]^{1/2}. \tag{18}$$

and hence the excitation frequency Ω near the resonant frequency, Ω_0, would be

$$\Omega = \left(\Omega_0 + \epsilon\ \frac{3\mathrm{I}}{8\Omega_0}a^2\right) \pm\ \epsilon\ \left[\frac{(f_1 + 2g_1)^2}{4\Omega_0^2 a^2} - \mu^2\right]^{1/2}, \tag{19}$$

where the term $\left(\Omega_0 + \epsilon\ \frac{3I}{8\Omega_0}a^2\right)$ defines the back-bone curve, *i.e.*, frequency-amplitude relation of the undamped free oscillation of system.

From equation (16), the maximum value, that the amplitude of the oscillation can achieve, is given by

$$a_{max} = \frac{f_1 + 2g_1}{2\Omega_0\mu} \tag{20}$$

The equation for amplitude response, in this case, is given by

$$\frac{a}{|f_1 + 2g_1|} = \frac{1}{2\sqrt{\Omega_0^2\mu^2 + (\Omega_0\sigma - \frac{3}{8}Ia^2)^2}}. \tag{21}$$

Therefore, for the case (i), *i.e.*, $\Omega_m << \Omega$, the AME significantly affects the amplitude and phase response of the Duffing type system with weak damping, weak nonlinearity and weak AME forcing term. Fig. 2.1 and Fig. 2.2 show the pattern of variation of amplitude, a, with detuning parameter, σ, for specific values of modulation amplitude, g_1, and for different values of damping coefficient, μ.

Direct simulation of equations (16) and (17) result in exhibiting the possibilities of coexistence of three equilibrium points, of which two are stable and one unstable saddle point (Fig. 2.3). It may be further noted that for case (ii) $i.e.$, $\Omega_m = \Omega$, the right hand side of the amplitude and phase responses, $i.e.$, equations (16) and (17), remain independent of the amplitude modulation term, g_1.

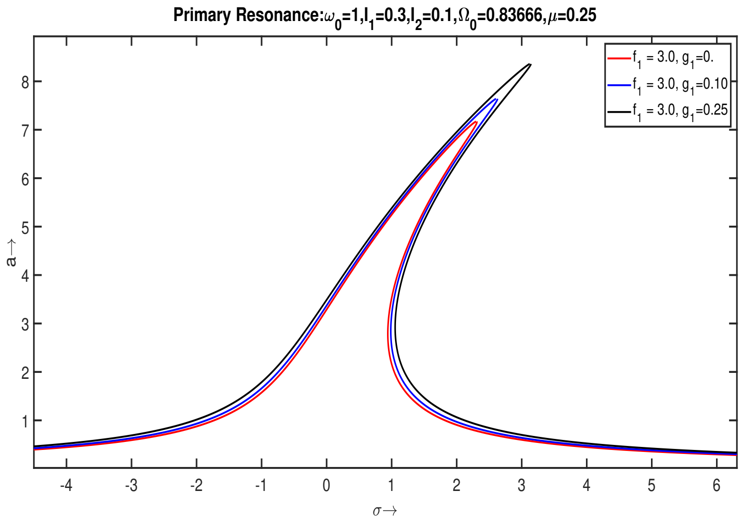

FIGURE 2.1. Frequency Response curve for different values of modulation parameter, g_1.

2.2. **Response of the strongly nonlinear AME excited Duffing- type system using Modified Multi Time Scale Perturbation Method.**

In this section, we consider the strongly damped and nonlinear AME excited Duffing type system. In such a case, we need to incorporate modifications in the foregoing analysis. Therefore, with the following details, the present analysis in this section is termed as modified multi time scale perturbation method (hereafter MMTPM). Following Nayfeh and Mook [7], we need to replace the expansion parameter, ϵ, in the MTPM by a new parameter, $\alpha = \alpha(\epsilon)$. Such a change is facilitated by first redefining the squared frequency, Ω^2, of the forced Duffing system in terms of detuning parameter, σ, and the parameter, α, as

$$\Omega^2 \quad = \quad p\left(\Omega_0^2 + \frac{3}{4}\epsilon a_0^2\right)[1 + \alpha\sigma]\,, \tag{22}$$

where p is a constant while a_0 refers to the amplitude of the fundamental harmonic. Subsequently the parameter α may be defined as,

$$\alpha = \frac{\epsilon a_0^2}{4\left(\Omega_0^2 + \frac{3}{4}\epsilon a_0^2\right)} \tag{23}$$

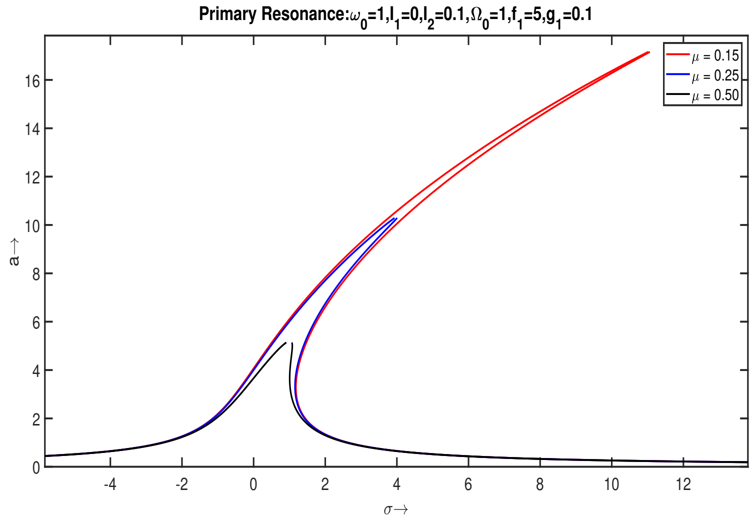

FIGURE 2.2. Frequency Response curve for different values of damping parameter, (μ).

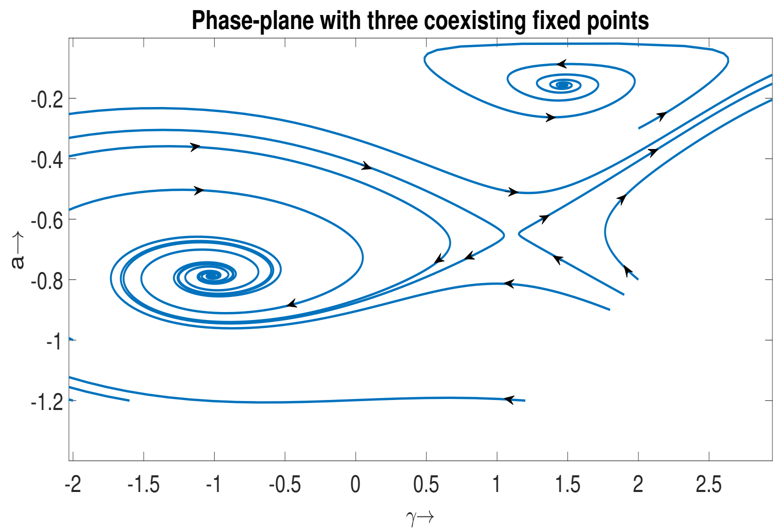

FIGURE 2.3. Three coexisting equilibrium solution of eqs.(2.14-2.15) with $f_1 = 0.25,\ \mu = 0.10,\ \Omega = 1.2, \epsilon = 0.2$ and $I = 5.0$.

As a result, for arbitrarily large values of the term ϵa_0^2, note that the parameter, α get restricted to the value of $\leq \frac{1}{3}$. The relation between the parameter, ϵ and α

may therefore be written as,

$$\epsilon = \frac{4\alpha}{a_0^2(1-3\alpha)}\Omega_0^2. \tag{24}$$

Therefore, using equations (23) and (24) into equation (22), it is readily observed that

$$\Omega \quad \simeq \quad \sqrt{p}\Omega_0 + \alpha\sigma_M, \tag{25}$$

where $\sigma_M = \sqrt{p}\frac{\Omega_0(3+\sigma)}{2}$ and may be termed as modified detuning parameter.

Having introduced the parameter, α, and Ω^2, we may recast, equation (5), using the Lindstedt-Poincáre transformation, as $\mathrm{T} = \Omega t$ (Mickens [13]), to get

$$\Omega^2 q'' + 2\epsilon\mu\Omega q' + \Omega_0^2 q + \epsilon \mathrm{I} q^3 = \epsilon\left[\left\{f_1 + 2g_1\cos\frac{\Omega_m}{\Omega}\mathrm{T}\right\}\sin\mathrm{T} - k\right]. \tag{26}$$

Here, $'$ represents derivative with respect to T. Further substitution of ϵ and Ω^2, from equations (22) and (24), last equation results in

$$\begin{aligned}(1+\alpha\sigma)q'' + \frac{(1-3\alpha)}{p}q \quad &+ \quad \frac{4\alpha}{a_0^2 p}\, 2\mu\Omega q' + \frac{4\alpha}{a_0^2 p}\mathrm{I}q^3 \\ &= \quad \frac{4\alpha}{a_0^2 p}\left[\left\{f_1 + 2g_1\cos\frac{\Omega_m}{\Omega}\mathrm{T}\right\}\sin\mathrm{T} - k\right]. \end{aligned}\tag{27}$$

Non-dimensionalizing q by defining $u = \frac{q}{a_0}$, equation (27) takes the following form

$$(1+\alpha\sigma)u'' + 2\alpha\eta u' + \frac{(1-3\alpha)}{p}u + 4\frac{\alpha}{p}\mathrm{I}u^3 = \alpha\left[\left\{F + 2G\cos\frac{\Omega_m}{\Omega}\mathrm{T}\right\}\sin\mathrm{T} - K\right], \tag{28}$$

where various factors have been redefined as follows

$$\eta = \frac{4\Omega}{a_0^2 p}\,\mu, \quad F = \frac{4f_1}{a_0^3 p}, \quad G = \frac{4g_1}{a_0^3 p} \quad \text{and} \quad K = \frac{4}{a_0^3 p}\,k. \tag{29}$$

With $\Omega_m << \Omega$, equation (28) may be rewritten as,

$$(1+\alpha\sigma)\,u'' + 2\alpha\eta\,u' + \frac{(1-3\alpha)}{p}\,u + 4\frac{\alpha\mathrm{I}}{p}\,u^3 = \alpha\left\{F + 2G\right\}\sin\mathrm{T} - \alpha K. \tag{30}$$

The improved multiple scales method, *i.e.*, MMTPM, with the new expansion parameter α, time scales are defined as follows

$$\mathrm{T}_n = \alpha^n\mathrm{T}, \quad \Longrightarrow \quad \frac{d}{d\mathrm{T}} = D_0 + \alpha D_1 + \cdots \quad \text{and} \quad \frac{d^2}{d\mathrm{T}^2} = D_0^2 + 2\alpha D_1 D_0 + \cdots, \tag{31}$$

and is used again to analyze primary resonances for system (30). Let the solution of equation (30), as earlier, be assumed as,

$$u(\alpha;\mathrm{T}) = u_0(\mathrm{T}_0, \mathrm{T}_1, \cdots) + \alpha u_1(\mathrm{T}_0, \mathrm{T}_1\cdots) + \cdots. \tag{32}$$

Using equations (31) and (32), up to first order in α, equation (30) becomes

$$(1+\alpha\sigma)D_0^2 u_0 + \alpha D_0^2 u_1 \quad + \quad 2\alpha D_1 D_0 u_0 + 2\alpha\eta D_0 u_0 + \frac{1-3\alpha}{p}\,u_0 + \frac{1}{p}\alpha u_1 +$$

$$+ \quad \frac{4}{p}\alpha \mathrm{I} u_0^3 = \alpha \left\{F + 2G\right\} \sin \mathrm{T} - \alpha K. \qquad (33)$$

Further equating coefficient of powers of α, equation (33) results in following two equations

$$\begin{aligned} \alpha^0: \quad D_0^2 u_0 + \frac{1}{p} u_0 &= 0, & (34)\\ \alpha^1: \quad D_0^2 u_1 + \frac{1}{p} u_1 &= -\sigma D_0^2 u_0 - 2D_1 D_0 u_0 - 2\eta D_0 u_0 + \frac{3}{p} u_0 - \frac{4}{p} \mathrm{I} u_0^3 + \\ &+ \quad (F + 2G) \sin \mathrm{T} - K & (35) \end{aligned}$$

Based on equations (11) and (12), it readily follows that

$$u_0(\mathrm{T}_0, \mathrm{T}_1) = \frac{a}{2} \exp\left\{ i \left(\frac{\mathrm{T}_0}{\sqrt{p}} + \phi \right) \right\} + cc, \qquad (36)$$

where a and ϕ have the same meaning as in section (2.1).

Substitution of equation (36) into (35) results in

$$\begin{aligned} D_0^2 u_1 + \frac{1}{p} u_1 \quad &= \quad \left[\frac{1}{2p} \sigma a - i \frac{a'}{\sqrt{p}} + a \frac{\phi'}{\sqrt{p}} - i\eta \frac{a}{\sqrt{p}} + \frac{3}{2p} a \left(1 - \mathrm{I}\ a^2\right) \right] e^{\left\{ i \left(\frac{\mathrm{T}_0}{\sqrt{p}} + \phi \right) \right\}} \\ &- \quad \frac{\mathrm{I}}{2p} a^3 e^{3i\left(\frac{\mathrm{T}_0}{\sqrt{p}} + \phi \right)} - K - \frac{i}{2} (F + 2G)\, e^{i\mathrm{T}_0} + \mathrm{cc}. \qquad (37) \end{aligned}$$

It may be noted that oscillator in equation (33) corresponds to the normalized natural frequency $\omega = \frac{1}{\sqrt{p}}$ whereas the external frequency Ω is normalized to 1. Therefore, for primary resonance to occur, $p = 1$. In that event, the AME excitation term contributes to the secular term in the solution to equation (37). Therefore, on eliminating the resulting secular terms, we obtain the following set of equations for the amplitude and phase response as,

$$\begin{aligned} a' \quad &= \quad -\eta a - \frac{(F + 2G)}{2} \cos \phi, & (38)\\ a\phi' \quad &= \quad -\frac{\sigma a}{2} - \frac{3}{2} a(1 - I a^2) + \frac{(F + 2G)}{2} \sin \phi. & (39) \end{aligned}$$

In the steady state, for $I = 1$, the amplitude $a \to 1$. Therefore detuning frequency, σ, at the steady state may be written as,

$$\sigma \quad = \quad -Z_1 \pm \sqrt{Z_1^2 + Z_2}, \qquad (40)$$

where $Z_1 = -\frac{32\mu^2\alpha}{I^2 a_0^4 (1-3\alpha)}$ and $Z_2 = \frac{64}{I^2 a_0^4} \left[\left(\frac{F+2G}{a_0} \right)^2 - \frac{\mu^2 \Omega_0^2}{1-3\alpha} \right]$. The effect of AME excited steady state amplitude response curve, for $\Omega_m << \Omega$, is shown in Fig. 2.4 for different values of the parameter, ϵ. It is to be noted that the maximum amplitude, $a_{max} = \left(\frac{F+2G}{\mu \Omega_0} \right) \sqrt{1 - 3\alpha}$ and therefore tends to reduce with increase in ϵ while retaining other parameter values. The MMTPM, therefore, enables us to obtain the amplitude- frequency response curve even for larger values of the parameter, ϵ.

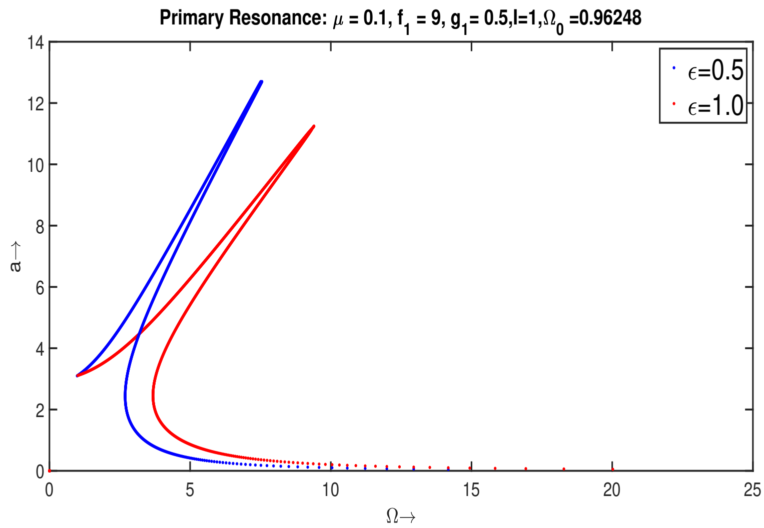

FIGURE 2.4. Amplitude-frequency Response curve for varying ϵ in AME excited Duffing type oscillator.

3. MAIN RESULTS

In this paper, the problem of a Duffing- Helmholtz oscillator excited by an amplitude modulated signal is considered. Using Cardano's transformation, the problem is shown to get reduced to a Duffing -type system excited by an amplitude modulated (AME) signal and a constant bias signal that depends on the parameters of the original system. Assuming the excitation amplitudes (f_1), modulation amplitude(g_1), damping (μ) and nonlinearity parameter, I, to be a perturbation on the system, a multi time scale perturbation method (MTPM) involving an expansion parameter, ϵ, has been used to understand the dynamical complexities involved in getting the amplitude- frequency relationship, exhibiting bi-stability on varying the modulation parameter, g_1, and the damping parameter, μ, of the Duffing-type system (Figs.2.1-2.2) near primary-resonance. Phase plane trajectories exhibiting the stable and unstable manifolds for specific values of the parameters involved in the system is further illustrated in Fig.2.3.

A modified perturbation scheme (MMTPM) extending the MTPM scheme is further developed, incorporating the Lindstedt-Poincare transformation and a transformation $\epsilon \to \alpha$, so as to obtain the response of the AME excited Duffing -type system even for very high values of the parameter, ϵ i.e, strongly nonlinear system. Simulation results showing the amplitude-frequency response around the back bone curve, for $\epsilon = 0.5$ and $\epsilon = 1.0$, as obtained in Fig.2.4 further reveal the bi-stable behavior of the system. For the parameter of the Duffing-type system as given in Fig.2.4, the peak amplitude, a_{max} for $\epsilon = 0.5$ is observed to be larger than that when $\epsilon = 1.0$.

Data Availability Statement. No data was used for the research described in the article.

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Appendix A.

The linear stability analysis of the amplitude and phase response for the Duffing-type system can be analyzed by considering equations (16)-(17) and writing its Jacobian matrix as,

$$J = \begin{vmatrix} -\xi & \theta\sin\gamma \\ \frac{\left(\sigma-\frac{9}{8}Ia^2\right)}{a} & \theta\cos\gamma \end{vmatrix} = \begin{vmatrix} -\xi & -a\left[\sigma-\frac{3Ia^2}{8\Omega_0}\right] \\ \frac{\left(\sigma-\frac{9Ia^2}{8\Omega_0}\right)}{a} & -\xi \end{vmatrix}, \tag{A.1}$$

where $\theta = \frac{f_1+2g_1}{2\Omega_0}$.

The characteristic roots, $\lambda_{1,2}$, may therefore be written as,

$$\lambda_{1,2} = -\xi \pm \sqrt{\left(\sigma-\frac{3Ia^2}{8\Omega_0}\right)\left(\frac{9Ia^2}{8\Omega_0}-\sigma\right)} \tag{A.2}$$

Alternatively, it is observed that, the determinant, Δ, and the trace of J, are given by,

$$\Delta = det(J) = \xi^2 + \left(\sigma-\frac{3Ia^2}{8\Omega_0}\right)\left(\frac{9Ia^2}{8\Omega_0}-\sigma\right);\ Tr(J) = -2\xi. \tag{A.3}$$

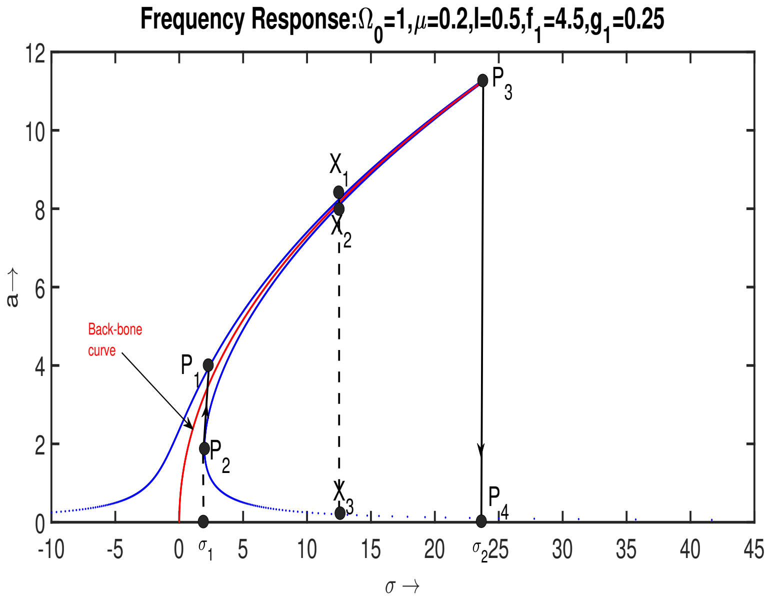

FIGURE 3.5. Frequency response diagram in case of AME excited Duffing-type oscillator based on MTPM.

$$\Delta = \xi^2 + \left(\sigma-\frac{3}{8}Ia^2\right)\left(\frac{9}{8}Ia^2-\sigma\right) = 0 \tag{A.4}$$

It is to be noted that, sum of the eigenvalues, *i.e.*, $Tr(J)$, is negative. Therefore, for the stability of the fixed point of the amplitude and phase response, both the eigenvalues viz., $\lambda_{1,2}$ should have negative real part. However, if one of the eigenvalue say λ_1 is having negative real part, then the fixed point of the system would be saddle only if the real part of the other eigenvalue (λ_2), is positive. In case, one of the eigenvalue , say λ_2, is zero, then the system undergoes a bifurcation, *i.e.*, saddle node/ pitch-fork bifurcation and the following condition holds,

A typical frequency-response curve is shown in Fig.3.5 where the point P_2 and P_3 refers to the jump-up and jump-down points respectively.The curve between the point P_2 and P_3 corresponds to the unstable domain, whereas the curve joining the points P_1, X_1 and P_3 along with the curve joining P_2, X_3 and P_4 refers to the stable region (Kalmár-Nagy and Balachandran [11]).