J. Indones. Math. Soc.
Vol. 31, No. 01 (2025), pp. 1–15.

Monophonic Polynomial of the Join of Graphs

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Abstract. The monophonic polynomial of a graph G, denoted by $M(G,x)$, is the polynomial $M(G,x) = \sum_{k=m(G)}^{|G|} M(G,k)x^k$, where $|G|$ is the order of G and $M(G,k)$ is the number of monophonic sets in G with cardinality k. In this paper, we delve into some characterizations of monophonic sets in the join of two graphs and use it to determine its corresponding monophonic polynomial. Moreover, we also present the monophonic polynomials of the complete graph K_n $(n \geq 1)$, the path P_n $(n \geq 3)$, the cycle C_n $(n \geq 4)$, the fan F_n $(n \geq 3)$, the wheel W_n $(n \geq 4)$, the complete bipartite $K_{m,n}$ $(m, n \geq 1)$, $P_m + P_n$ $(m, n \geq 3)$, $C_m + C_n$ $(m, n \geq 4)$, $P_m + C_n$ $(m \geq 3$ and $n \geq 4)$, $P_m + \overline{K_n}$ $(m \geq 3$ and $n \geq 2)$, and $C_m + \overline{K_n}$ $(m \geq 4$ and $n \geq 2)$. In general, we obtain the monophonic polynomial of the join of two graphs.

Key words and Phrases: graph, monophonic path, monophonic set, monophonic number, monophonic polynomial.

1. INTRODUCTION

The study of polynomials in graph theory has been fundamental in understanding the properties of graphs. Since the introduction of the chromatic polynomial by George Birkhoff in [1], various polynomial functions have been developed to quantify different characteristics of graphs. In [2], Gutman and Harary defined the independence polynomial. Hoffman discussed various properties of graphs related to eigenvalues and colorings, including the concept of domination polynomials in [3]. Henning in [4], discussed the concept of total domination in graphs and introduced the total domination polynomial as a tool to analyze this aspect of graph theory. In [5], Hedetniemi and Slater introduced the monophonic polynomials of graphs, which is a relatively recent addition to the compendium of graph polynomials.

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2020 Mathematics Subject Classification: 05C12.
Received: 14-04-2024, accepted: 23-02-2025.

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite set of elements called vertices and $E(G)$ is a set of unordered pairs in $V(G)$ called edges. The set $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. Let G be a graph and $S \subseteq V(G)$. The graph induced by S, denoted by $\langle S\rangle_G$, is the graph with vertex set S and edge set $\{uv : uv \in E(G) \text{ and } u, v \in S\}$. A path $P = [a_1, a_2, ..., a_n]$ in G is said to contain a chord if there exist i and j, with $1 \le i \le n-2$ and $i+2 \le j$, such that $a_i a_j \in E(G)$. A monophonic path is a path that contains no chord. A subset S of $V(G)$ is a monophonic set in G if for every $x \in V(G)\setminus S$, there exist $u, v \in S$ such that x is in a monophonic path with endpoints u and v. The minimum cardinality of a monophonic set in G is the monophonic number of G and is denoted by $m(G)$. The monophonic polynomial of G is defined by $M(G, x) = \sum_{k=m(G)}^{|G|} M(G, k)x^k$, where $|G|$ is the order of G and $M(G, k)$ is the number of monophonic sets in G with cardinality k.

2. MONOPHONIC POLYNOMIALS OF SOME GRAPHS

In this section, we determine the monophonic polynomials of the complete graph K_n $(n \ge 1)$, the path P_n $(n \ge 3)$, the cycle C_n $(n \ge 4)$, and the complete bipartite $K_{m,n}$ $(m, n \ge 1)$.

Theorem 2.1. Let $n \ge 1$. Then, the monophonic polynomial of K_n is

$$M(K_n, x) = x^n.$$

Proof. Suppose S is a monphonic subset in K_n and $S \ne V(K_n)$. Let $a \in V(K_n)\setminus S$. Since S is monophonic in K_n, there exist $u, v \in S$ such that a is in a monophonic path with endpoints u and v. Now, $\langle\{a, u, v\}\rangle$ is complete, which contradicts the definition of a monophonic path. Hence, $S = V(K_n)$. Accordingly, $M(K_n, x) = x^n$. □

Theorem 2.2. Let $n \ge 3$. Then, the monophonic polynomial of P_n is

$$M(P_n, x) = x^2(x + 1)^{n-2}.$$

Proof. Let $V(P_n) = \{a_i : 1 \le i \le n\}$ and $E(P_n) = \{a_i a_{i+1} : 1 \le i \le n-1\}$. Let S be a monophonic set in P_n. Note that S is monphonic in P_n if and only if $a_1, a_n \in S$. Thus, we have the following:

(i) $M(P_n, 1) = 0$;
(ii) $M(P_n, 2) = 1$; and
(iii) $M(P_n, k) = \binom{n-2}{k-2}$, if $3 \le k \le n$.

Consequently,

$$\begin{aligned} M(P_n, x) &= x^2 + \sum_{k=3}^{n} \binom{n-2}{k-2} x^k \\ &= x^2 + \binom{n-2}{1} x^3 + \binom{n-2}{2} x^3 + \ldots + \binom{n-2}{n-2} x^k \end{aligned}$$

$$= x^2\left(\sum_{k=0}^{n-2}\binom{n-2}{k}x^k\right)$$
$$= x^2(x+1)^{n-2}.$$

□

Theorem 2.3. Let $n \geq 4$. Then, the monophonic polynomial of C_n is

$$M(C_n, x) = (x+1)^n - (nx^2 + nx + 1).$$

Proof. Let $V(C_n) = \{a_i : 1 \leq i \leq n\}$ and $E(C_n) = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{a_1 a_n\}$. Clearly, $M(C_n, 1) = 0$, since we need at least two elements for a monophonic set. Note that the number of subsets of $V(C_n)$ with cardinality two is $\binom{n}{2}$. Moreover, adjacent vertices cannot form a monophonic set while any two non-adjacent vertices forms a monophonic set. Thus, $M(C_n, 2) = \binom{n}{2} - n$. Suppose $3 \leq k \leq n$. Note that any set of vertices with at least three vertices will have a pair of non-adjacent vertices, and so it should be monophonic. Thus, $M(C_n, k) = \binom{n}{k}$, if $3 \leq k \leq n$. Hence,

$$\begin{aligned} M(C_n, x) &= \left[\binom{n}{2} - n\right]x^2 + \sum_{k=3}^{n}\binom{n}{k}x^k \\ &= \binom{n}{2}x^2 + \sum_{k=3}^{n}\binom{n}{k}x^k - nx^2 \\ &= \sum_{k=0}^{n}\binom{n}{k}x^k - nx^2 - nx - 1 \\ &= (x+1)^n - (nx^2 + nx + 1). \end{aligned}$$

□

In the succeeding theorems, we obtain the monophonic polynomial of $K_{m,n}$, when $m, n \geq 1$.

Theorem 2.4. Let $n \geq 3$. Then, the monophonic polynomial of $K_{1,n}$ is

$$M(K_{1,n}, x) = x^n(x+1).$$

Proof. Let $V(K_{1,n}) = \{a_i : 0 \leq i \leq n\}$ and $E(K_{1,n}) = \{a_0 a_i : 1 \leq i \leq n\}$. Clearly, $M(K_{1,n}, 1) = 0$. Note that $M(K_{1,n}, n+1) = 1$. Let S be a monophonic set in $K_{1,n}$. If $a_i \notin S$ for some $i = 1, 2, \ldots, n$, then we get a contradiction since it is impossible to have a monophonic path containing a_i with endpoints in S . In effect, $\{a_i : 1 \leq i \leq n\} \subseteq S$. Consequently, $M(K_{1,n}, n) = 1$ and $M(K_{1,n}, j) = 0$, if $1 \leq j \leq n-1$. Hence, $M(K_{1,n}, x) = x^n + x^{n+1} = x^n(x+1)$. □

Theorem 2.5. Let $2 \leq m \leq n$. Then, the monophonic polynomial of $K_{m,n}$ is

$$\begin{aligned} M(K_{m,n}, x) \quad = \quad & (x+1)^{m+n} + (mx+1)\left[x^n - (x+1)^n\right] \\ & + (nx+1)\left[x^m - (x+1)^m\right] + (mx+1)(nx+1) \end{aligned}$$

Proof. Let $V(K_{m,n}) = A \cup B$, where $A = \{a_i : 1 \leq i \leq m\}$, $B = \{b_i : 1 \leq i \leq n\}$, and $E(K_{m,n}) = \{a_i b_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. Let S be a monophonic set in $K_{m,n}$. Consider the following cases:

Case 1. When $m = 2$.
In this case, we have the following:

(i) If $|A \cap S| \leq 1$, then $|B \cap S| = n$.
(ii) If $|A \cap S| = 2$, then $0 \leq |B \cap S| \leq n$.

Sub-case (i) gives the terms $x^n + 2x^{n+1}$ while Sub-case (ii) provides the terms $\binom{n}{0}x^2 + \binom{n}{1}x^3 + \ldots + \binom{n}{n}x^{n+2}$. Thus,

$$\begin{aligned} M(K_{2,n}, x) &= \binom{n}{0}x^2 + \binom{n}{1}x^3 + \ldots + \binom{n}{n}x^{n+2} + x^n + 2x^{n+1} \\ &= x^2\left[\binom{n}{0} + \binom{n}{1}x + \ldots + \binom{n}{n}x^n\right] + x^n(2x+1) \\ &= x^2(x+1)^n + x^n(2x+1). \end{aligned}$$

Now,

$$\begin{aligned} &(x+1)^{m+n} + (mx+1)\left[x^n - (x+1)^n\right] + (nx+1)\left[x^m - (x+1)^m\right] + (mx+1)(nx+1) \\ &= (x+1)^{2+n} + (2x+1)\left[x^n - (x+1)^n\right] + (nx+1)\left[x^2 - (x+1)^2\right] + (2x+1)(nx+1) \\ &= (x+1)^{2+n} - (2x+1)(x+1)^n + (2x+1)x^n - (nx+1)(2x+1) + (2x+1)(nx+1) \\ &= (x+1)^n\left[(x+1)^2 - (2x+1)\right] + (2x+1)x^n \\ &= (x+1)^n x^2 + (2x+1)x^n \\ &= x^2(x+1)^n + x^n(2x+1). \end{aligned}$$

Hence, the theorem holds when $m = 2$.

Case 2. When $m = 3$.
In this instance, we observe the following:

(i) If $|A \cap S| \leq 1$, then $|B \cap S| = n$.
(ii) If $|A \cap S| = 2$, then $|B \cap S| \geq 2$.
(iii) If $|A \cap S| = 3$, then $0 \leq |B \cap S| \leq n$.

Sub-case (i) contributes the terms $x^n + 3x^{n+1}$. Sub-case (ii) gives us the terms

$$\binom{3}{2}\binom{n}{2}x^4 + \binom{3}{2}\binom{n}{3}x^5 + \ldots + \binom{3}{2}\binom{n}{n}x^{n+2}.$$

Sub-case (iii) provides the terms

$$\binom{n}{0}x^3 + \binom{n}{1}x^4 + \ldots + \binom{n}{n}x^{n+3}.$$

Thus,

$$\begin{aligned}M(K_{3,n},x) &= x^3\left[\sum_{k=0}^{n}\binom{n}{k}x^k\right] + 3x^2\left[\binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n\right] + x^n + 3x^{n+1}\\ &= x^3(x+1)^n + 3x^2(x+1)^n - 3x^2(nx+1) + x^n(3x+1)\\ &= (x+1)^n(x^3+3x^2) - 3x^2(nx+1) + x^n(3x+1).\end{aligned}$$

Now,

$$\begin{aligned}&(x+1)^{m+n} + (mx+1)\left[x^n - (x+1)^n\right] + (nx+1)\left[x^m - (x+1)^m\right]\\ &\qquad + (mx+1)(nx+1)\\ &= (x+1)^{3+n} + (3x+1)\left[x^n - (x+1)^n\right] + (nx+1)\left[x^3 - (x+1)^3\right]\\ &\qquad + (3x+1)(nx+1)\\ &= (x+1)^{3+n} - (3x+1)(x+1)^n + (3x+1)x^n - (nx+1)(3x^2+3x+1)\\ &\qquad + (3x+1)(nx+1)\\ &= (x+1)^n\left[(x+1)^3 - (3x+1)\right] + (3x+1)x^n - 3x^2(nx+1)\\ &= (x+1)^n(x^3+3x^2) + x^n(3x+1) - 3x^2(nx+1).\end{aligned}$$

Hence, the theorem holds when $m = 3$.

Case 3. When $m \geq 4$.
Under this condition, we have the following:

(i) If $|A \cap S| \leq 1$, then $|B \cap S| = n$.
(ii) If $2 \leq |A \cap S| \leq m-1$, then $|B \cap S| \geq 2$.
(iii) If $|A \cap S| = m$, then $0 \leq |B \cap S| \leq n$.

Sub-case (i) contributes the terms $x^n + mx^{n+1}$. Sub-case (ii) generates the following terms:

$$\begin{aligned}&\binom{m}{2}\binom{n}{2}x^4 + \binom{m}{2}\binom{n}{3}x^5 + \ldots + \binom{m}{2}\binom{n}{n}x^{n+2}\\ &+ \binom{m}{3}\binom{n}{2}x^5 + \binom{m}{3}\binom{n}{3}x^6 + \ldots + \binom{m}{3}\binom{n}{n}x^{n+3}\\ &\quad\vdots\\ &+ \binom{m}{m-1}\binom{n}{2}x^{m+1} + \binom{m}{m-1}\binom{n}{3}x^{m+2} + \ldots + \binom{m}{m-1}\binom{n}{n}x^{m+n-1}.\end{aligned}$$

Sub-case (iii) provides the terms $\binom{n}{0}x^m + \binom{n}{1}x^{m+1} + \ldots + \binom{n}{n}x^{m+n}$. Thus,

$$\begin{aligned}M(K_{m,n},x) = \sum_{i=0}^{m}\binom{n}{i}x^{m+i} + \sum_{i=2}^{n}\binom{m}{2}\binom{n}{i}x^{2+i} + \sum_{i=2}^{n}\binom{m}{3}\binom{n}{i}x^{3+i} + \ldots\\ + \sum_{i=2}^{n}\binom{m}{m-1}\binom{n}{i}x^{m+i-1} + x^n + mx^{n+1}\end{aligned}$$

$$
\begin{aligned}
&= x^m \sum_{i=0}^{m} \binom{n}{i} x^i + \binom{m}{2} x^2 \left[\sum_{i=2}^{n} \binom{n}{i} x^i \right] + \binom{m}{3} x^3 \left[\sum_{i=2}^{n} \binom{n}{i} x^i \right] + \dots \\
&\quad + \binom{m}{m-1} x^{m-1} \left[\sum_{i=2}^{n} \binom{n}{i} x^i \right] + x^n(mx+1) \\
&= x^m \sum_{i=0}^{m} \binom{n}{i} x^i + \binom{m}{2} x^2 \left[\sum_{i=0}^{n} \binom{n}{i} x^i \right] + \binom{m}{3} x^3 \left[\sum_{i=0}^{n} \binom{n}{i} x^i \right] + \dots \\
&\quad + \binom{m}{m-1} x^{m-1} \left[\sum_{i=0}^{n} \binom{n}{i} x^i \right] - (nx+1) \left[\binom{m}{2} x^2 \right] \\
&\quad - (nx+1) \left[\binom{m}{3} x^3 \right] - \dots - (nx+1) \left[\binom{m}{m-1} x^{m-1} \right] + x^n(mx+1) \\
&= x^m (x+1)^n + \binom{m}{2} x^2 (x+1)^n + \binom{m}{3} x^3 (x+1)^n + \dots \\
&\quad + \binom{m}{m-1} x^{m-1} (x+1)^n - (nx+1) \left[\sum_{i=2}^{m-1} \binom{m}{i} x^i \right] + x^n(mx+1) \\
&= (x+1)^n \left[\sum_{i=2}^{m} \binom{m}{i} x^i \right] - (nx+1) \left[\sum_{i=2}^{m} \binom{m}{i} x^i \right] + x^n(mx+1) \\
&= (x+1)^n \left[\sum_{i=0}^{m} \binom{m}{i} x^i \right] - (nx+1) \left[\sum_{i=0}^{m} \binom{m}{i} x^i \right] - (mx+1)(x+1)^n \\
&\quad + (nx+1)(x^m + mx + 1) + x^n(mx+1) \\
&= (x+1)^n (x+1)^m - (nx+1)(x+1)^m - (mx+1)(x+1)^n \\
&\quad + (nx+1)(x^m + mx + 1) + x^n(mx+1) \\
&= (x+1)^{m+n} - (nx+1)(x+1)^m - (mx+1)(x+1)^n \\
&\quad + x^m(nx+1) + (mx+1)(nx+1) + x^n(mx+1) \\
&= (x+1)^{m+n} + (mx+1) \left[x^n - (x+1)^n \right] + (nx+1) \left[x^m - (x+1)^m \right] \\
&+ (mx+1)(nx+1).
\end{aligned}
$$

□

3. MONOPHONIC POLYNOMIAL OF THE JOIN OF GRAPHS

In [6], E.M. Paluga and S. R. Canoy, Jr. provided characterizations of monophonic sets in the join of two graphs, which are stated below without proofs. In this section, we use the characterization for the case $G + K_n$, to obtain its monophonic polynomial. For the join of non-complete graphs, we use the characterization given in [6] to obtain another characterization, with mutually exclusive cases, and correspondingly obtain the monophonic polynomial of the join of two non-complete graphs.

Theorem 3.1. [6] Let G be a non-complete graph and $n \geq 1$. Then $S \subseteq V(G+K_n)$ is monophonic in $G+K_n$ if and only if $V(G) \cap S$ is monophonic in G.

Theorem 3.2. [6] Let G and H be non-complete graphs. Then $S \subseteq V(G+H)$ is monophonic in $G+H$ if and only if one of the following holds:

(i) $V(G) \cap S$ is monophonic in G;
(ii) $V(H) \cap S$ is monophonic in H; and
(iii) There exist $x, y \in V(G) \cap S$ and $u, v \in V(H) \cap S$ such that $xy \notin E(G)$ and $uv \notin E(H)$.

The next theorem provides the monophonic polynomial of the join $G+K_n$, where $n \geq 1$.

Theorem 3.3. Let G be a non-complete graph and $n \geq 1$. Then, the monophonic polynomial of $G+K_n$ is

$$M(G+K_n, x) = M(G, x)(x+1)^n.$$

Proof. Let G be a non-complete graph and $n \geq 1$. By Theorem 3.1, we have

$$\begin{aligned}
M(G+K_n, x) &= \sum_{r=0}^{n} M(G, m(G)) \binom{n}{r} x^{m(G)+r} + \sum_{r=0}^{n} M(G, m(G)+1) \binom{n}{r} x^{m(G)+r+1} \\
&\quad + \ldots + \sum_{r=0}^{n} M(G, |G|-1) \binom{n}{r} x^{|G|+r-1} + \sum_{r=0}^{n} M(G, |G|) \binom{n}{r} x^{|G|+r} \\
&= M(G, m(G)) x^{m(G)} \sum_{r=0}^{n} \binom{n}{r} x^r + M(G, m(G)+1) x^{m(G)+1} \sum_{r=0}^{n} \binom{n}{r} x^r \\
&\quad + \ldots + M(G, |G|-1) x^{|G|-1} \sum_{r=0}^{n} \binom{n}{r} x^r + M(G, |G|) x^{|G|} \sum_{r=0}^{n} \binom{n}{r} x^r \\
&= \left[\binom{n}{r} x^r \right] \left[\sum_{r=m(G)}^{|G|} M(G, r) x^r \right] \\
&= (x+1)^n M(G, x).
\end{aligned}$$

□

The following corollaries show the direct consequences of Theorem 3.3.

Corollary 3.4. Let $n \geq 3$. Then

$$M(F_n, x) = x^2 (x+1)^{n-1}.$$

Proof. Let $n \geq 3$. Note that $F_n = P_n + K_1$. Then by Theorem 3.3 and Theorem 2.2,

$$M(F_n, x) = M(P_n, x)(x+1) = \left[x^2(x+1)^{n-2}\right](x+1) = x^2(x+1)^{n-1}.$$

□

Corollary 3.5. Let $n \geq 4$. Then

$$M(W_n, x) = (x+1)^{n+1} - (x+1)(nx^2 + nx + 1).$$

Proof. Let $n \geq 4$. Note that $W_n = C_n + K_1$. Then by Theorem 3.3 and Theorem 2.3, $M(C_n, x) = (x+1)^n - (nx^2 + nx + 1)$.

$$\begin{aligned} M(W_n, x) &= M(W_n, x)(x+1) \\ &= \left[(x+1)^n - (nx^2 + nx + 1)\right](x+1) \\ &= (x+1)^{n+1} - (x+1)(nx^2 + nx + 1). \end{aligned}$$

□

Let us consider the concepts of clique polynomials and incomplete polynomials in graphs. We will use these ideas to obtain monophonic polynomial of $G + H$, where G and H are non-complete graphs.

Let G be a graph and $r \geq 1$. A subset S of $V(G)$ is called a clique if $\langle S \rangle_G$ is complete. The r-clique index of G , denoted by $\kappa(G, r)$, is the number of cliques in G with cardinality r. That is, $\kappa(G, r) = |\{S : S \subseteq V(G) \text{ and } \langle S \rangle_G \cong K_r\}|$. The clique polynomial of G, denoted by $\kappa(G, x)$, is defined by $\kappa(G, x) = \sum_{r=1}^{\omega(G)} \kappa(G, r)x^r$, where $\omega(G)$ is the maximum clique. In [7], Stevanovic provided a comprehensive characterization of clique polynomials specifically for threshold graphs, demonstrating that these polynomials uniquely determine threshold graphs within their class.

The following examples present the clique polynomial of P_n, C_n, and $\overline{K_n}$.

Example 3.6. Let $n \geq 1$. Then

$$\begin{aligned} \kappa(K_n, x) &= \binom{n}{1}x + \binom{n}{2}x^2 + \dots \binom{n}{n}x^n \\ &= (x+1)^n - 1. \end{aligned}$$

Example 3.7. Let $n \geq 3$. Then $\kappa(P_n, x) = (n-1)x^2 + nx$.

Example 3.8. Let $n \geq 4$. Then $\kappa(C_n, x) = nx^2 + nx$.

Example 3.9. Let $n \geq 1$. Then $\kappa(\overline{K_n}, x) = nx$.

For a graph G and $r \geq 1$, we define the concept of the r-incomplete index of G, denoted by $\pi(G, r)$, which is the number of subsets in G with cardinality r that generates a non-complete graph. That is,

$$\pi(G, r) = |\{S : S \subseteq V(G), |S| = r, \text{ and } \langle S \rangle_G \ncong K_r\}|.$$

Correspondingly, we define the incomplete polynomial of G, denoted by $\pi(G,x)$, is defined by $\pi(G,x) = \sum_{r=2}^{|G|} \pi(G,r)x^r$.

Note that $\pi(G,r) = \binom{|G|}{r} - \kappa(G,r)$, where $2 \leq r \leq |G|$.

The next examples provide the incomplete polynomial of K_n, P_n, C_n, and $\overline{K_n}$.

Example 3.10. Let $n \geq 1$. Then $\pi(K_n, x) = 0$.

Example 3.11. Let $n \geq 3$. Then

$$\begin{aligned}\pi(P_n,x) &= \left[\binom{n}{2} - (n-1)\right] x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n \\ &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n - (n-1)x^2 - \binom{n}{0} - \binom{n}{1}x \\ &= (x+1)^n - \left[(n-1)x^2 + nx + 1\right].\end{aligned}$$

Example 3.12. Let $n \geq 4$. Then

$$\begin{aligned}\pi(C_n,x) &= \left[\binom{n}{2} - n\right] x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n \\ &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n - nx^2 - \binom{n}{0} - \binom{n}{1}x \\ &= (x+1)^n - (nx^2 + nx + 1).\end{aligned}$$

Example 3.13. Let $n \geq 4$. Then

$$\begin{aligned}\pi(\overline{K_n},x) &= \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots \binom{n}{n}x^n \\ &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n - \binom{n}{0} - \binom{n}{1}x \\ &= (x+1)^n - (nx+1).\end{aligned}$$

The next theorem provides a characterization of monophonic sets in the join of two non-complete graphs, which we use to generate the monophonic polynomial of the join of two non-complete graphs.

Theorem 3.14. Let G and H be non-complete graphs. Then $S \subseteq V(G+H)$ is monophonic in $G+H$ if and only if one of the following holds:

(i) $\langle V(G) \cap S\rangle_G$ and $\langle V(H) \cap S\rangle_H$ are non-empty and non-complete graphs;
(ii) $V(G) \cap S = \emptyset$ and $V(H) \cap S$ is monophonic in H;
(iii) $V(H) \cap S = \emptyset$ and $V(G) \cap S$ is monophonic in G;
(iv) $V(G) \cap S$ is a clique in G and $V(H) \cap S$ is monophonic in H; and
(v) $V(H) \cap S$ is a clique in H and $V(G) \cap S$ is monophonic in G.

Proof. Suppose S is monophonic in $G+H$. Then by 3.2 (iii), $\langle V(G) \cap S\rangle_G$ and $\langle V(H) \cap S\rangle_H$ are non-empty and non-complete graphs. Thus, (i) of Theorem 3.14 holds. By Theorem 3.2 (i), (iii) and (v) of Theorem 3.14 hold. By Theorem 3.2 (ii), (ii) and (iv) of Theorem 3.14 hold. □

Suppose conditions (i)-(iii) of Theorem 3.14 hold. Then by Theorem 3.2, S is monophonic in $G+H$. Note that the conditions in Theorem 3.14 are mutually exclusive.

Theorem 3.15. Let G and H be non-complete graphs. Then

$$M(G+H,x)=M(G,x)\left[\kappa(H,x)+1\right]+M(H,x)\left[\kappa(G,x)+1\right]+\pi(G,x)\pi(H,x).$$

Proof. By Theorem 3.14(i), we obtain the following terms:

$$\begin{aligned}
&\sum_{r=2}^{|H|}\pi(G,2)\pi(H,r)x^{r+2}+\sum_{r=2}^{|H|}\pi(G,3)\pi(H,r)x^{r+3}+\ldots\\
&\quad+\sum_{r=2}^{|H|}\pi(G,|G|)\pi(H,r)x^{r+|G|}\\
&=\pi(G,2)x^2\sum_{r=2}^{|H|}\pi(H,r)x^r+\pi(G,3)x^3\sum_{r=2}^{|H|}\pi(H,r)x^r+\ldots\\
&\quad+\pi(G,|G|)x^{|G|}\sum_{r=2}^{|H|}\pi(H,r)x^r\\
&=\left(\sum_{r=2}^{|G|}\pi(G,r)x^r\right)\left(\sum_{r=2)}^{|H|}M(H,r)x^r\right)\\
&=\pi(G,x)\pi(H,x)
\end{aligned}$$

By Theorem 3.14(ii), we have the following terms:

$$M(H,m(H))x^{m(H)}+M(H,m(H))x^{m(H)+1}+\ldots+M(H,|H|)x^{|H|}=M(H,x).$$

By Theorem 3.14(iii), we have the following terms:

$$M(G,m(G))x^{m(G)}+M(G,m(G)+1)x^{m(G)+1}+\ldots+M(G,|G|)x^{|G|}=M(G,x).$$

By Theorem 3.14(iv), we obtain the following terms:

$$\begin{aligned}
&\sum_{r=m(H)}^{|H|}\kappa(G,1)M(H,r)x^{r+1}+\sum_{r=m(H)}^{|H|}\kappa(G,2)M(H,r)x^{r+2}+\ldots\\
&\quad+\sum_{r=m(H)}^{|H|}\kappa(G,\omega(G))M(H,r)x^{r+\omega(G)}\\
&=\kappa(G,1)x\left(\sum_{r=m(H)}^{|H|}M(H,r)x^r\right)+\kappa(G,2)x^2\left(\sum_{r=m(H)}^{|H|}M(H,r)x^r\right)+\ldots\\
&\quad+\kappa(G,\omega(G))x^{\omega(G)}\left(\sum_{r=m(H)}^{|H|}M(H,r)x^r\right)
\end{aligned}$$

$$= \left(\sum_{r=1}^{\omega(G)} \kappa(G,r)x^r\right)\left(\sum_{r=m(H)}^{|H|} M(H,r)x^r\right)$$
$$= \kappa(G,x)M(H,x)$$

By Theorem 3.14(v), we generate the following terms:

$$\sum_{r=m(G)}^{|G|} \kappa(H,1)M(G,r)x^{r+1} + \sum_{r=m(G)}^{|G|} \kappa(H,2)M(G,r)x^{r+2} + \dots$$
$$+ \sum_{r=m(G)}^{|G|} \kappa(H,\omega(H))M(G,r)x^{r+\omega(H)}$$
$$= \kappa(H,1)x \sum_{r=m(G)}^{|G|} M(G,r)x^r + \kappa(H,2)x^2 \sum_{r=m(G)}^{|G|} M(G,r)x^r + \dots$$
$$+ \kappa(H,\omega(H))x^{\omega(H)} \sum_{r=m(G)}^{|G|} M(G,r)x^r$$
$$= \left(\sum_{r=1}^{\omega(H)} \kappa(H,r)x^r\right)\left(\sum_{r=m(G)}^{|G|} M(G,r)x^r\right)$$
$$= \kappa(H,x)M(G,x)$$

Now, we note that the conditions specified in Theorem 3.14 are mutually exclusive. Consequently, the coefficient of x^r, which represents the number of monophonic sets with cardinality r, can be determined by adding the coefficients of x^r derived from each case outlined in Theorem 3.14. Hence,

$$\begin{aligned} M(G+H,x) &= M(G,x) + M(H,x) + \kappa(G,x)M(H,x) + \kappa(H,x)M(G,x) \\ &\quad + \pi(G,x)\pi(H,x) \\ &= M(G,x)\left[\kappa(H,x)+1\right] + M(H,x)\left[\kappa(G,x)+1\right] + \pi(G,x)\pi(H,x) \end{aligned}$$

□

In the subsequent results, we derive the monophonic polynomials of $P_m + P_n$ $(m,n \geq 3)$, $C_m + C_n$ $(m,n \geq 4)$, $P_m + C_n$ $(m \geq 3$ and $n \geq 4)$, $P_m + \overline{K_n}$ $(m \geq 3$ and $n \geq 2)$, and $C_m + \overline{K_n}$ $(m \geq 4$ and $n \geq 2)$.

Corollary 3.16. Let $m,n \geq 3$. Then, the monophonic polynomial of $P_m + P_n$ is

$$\begin{aligned} M(P_m + P_n, x) &= (x+1)^{m+n} - (2x+1)(x+1)^{m-2}\left[(n-1)x^2 + nx + 1\right] \\ &\quad - (2x+1)(x+1)^{n-2}\left[(m-1)x^2 + mx + 1\right] \\ &\quad + \left[(m-1)x^2 + mx + 1\right]\left[(n-1)x^2 + nx + 1\right] \end{aligned}$$

Proof. By Theorem 2.2, $M(P_n, x) = x^2(x+1)^{n-2}$. By Example 3.7, $\kappa(P_n, x) = (n-1)x^2+nx$. By Example 3.11, $\pi(P_n, x) = (x+1)^n - \left[(n-1)x^2+nx+1\right]$. Thus by Theorem 3.15, we have the following:

$$\begin{aligned}
M(P_m+P_n,x) &= M(P_m,x)\left[\kappa(P_n,x)+1\right]+M(P_n,x)\left[\kappa(P_m,x)+1\right]+\pi(P_m,x)\pi(P_n,x)\\
&= x^2(x+1)^{m-2}\left[(n-1)x^2+nx+1\right]+x^2(x+1)^{n-2}\left[(m-1)x^2+mx+1\right]\\
&\quad+\left[(x+1)^m-[(m-1)x^2+mx+1]\right]\left[(x+1)^n-[(n-1)x^2+nx+1]\right]\\
&= x^2(x+1)^{m-2}\left[(n-1)x^2+nx+1\right]+x^2(x+1)^{n-2}\left[(m-1)x^2+mx+1\right]\\
&\quad+(x+1)^{m+n}-(x+1)^m\left[(n-1)x^2+nx+1\right]\\
&\quad-(x+1)^n\left[(m-1)x^2+mx+1\right]\\
&\quad+\left[(m-1)x^2+mx+1\right]\left[(n-1)x^2+nx+1\right]\\
&= (x+1)^{m+n}+(x+1)^{m-2}\left[(n-1)x^2+nx+1\right]\left[x^2-(x+1)^2\right]\\
&\quad+(x+1)^{n-2}\left[(m-1)x^2+mx+1\right]\left[x^2-(x+1)^2\right]\\
&\quad+\left[(m-1)x^2+mx+1\right]\left[(n-1)x^2+nx+1\right]\\
&= (x+1)^{m+n}-(2x+1)(x+1)^{m-2}\left[(n-1)x^2+nx+1\right]\\
&\quad-(2x+1)(x+1)^{n-2}\left[(m-1)x^2+mx+1\right]\\
&\quad+\left[(m-1)x^2+mx+1\right]\left[(n-1)x^2+nx+1\right]
\end{aligned}$$

□

Corollary 3.17. Let $m, n \geq 4$. Then, the monophonic polynomial of $C_m + C_n$ is

$$M(C_m+C_n,x) = (x+1)^{m+n}-(mx^2+mx+1)(nx^2+nx+1).$$

Proof. By Theorem 2.3, $M(C_n, x) = (x+1)^n - (nx^2+nx+1)$. By Example 3.8, $\kappa(C_n, x) = nx^2+nx$. By Example 3.12, $\pi(C_n, x) = (x+1)^n - (nx^2+nx+1)$. Thus by Theorem 3.15, we have the following:

$$\begin{aligned}
M(C_m+C_n,x) &= M(C_m,x)\left[\kappa(C_n,x)+1\right]+M(C_n,x)\left[\kappa(C_m,x)+1\right]+\pi(C_m,x)\pi(C_n,x)\\
&= \left[(x+1)^m-(mx^2+mx+1)\right]\left[nx^2+nx+1\right]\\
&\quad+\left[(x+1)^n-(nx^2+nx+1)\right]\left[mx^2+mx+1\right]\\
&\quad+\left[(x+1)^m-(mx^2+mx+1)\right]\left[(x+1)^n-(nx^2+nx+1)\right]\\
&= (x+1)^m(nx^2+nx+1)-(mx^2+mx+1)(nx^2+nx+1)\\
&\quad+(x+1)^n(mx^2+mx+1)-(mx^2+mx+1)(nx^2+nx+1)\\
&\quad+(x+1)^{m+n}-(x+1)^m(nx^2+nx+1)-(x+1)^n(mx^2+mx+1)\\
&\quad+(mx^2+mx+1)(nx^2+nx+1)\\
&= (x+1)^{m+n}-(mx^2+mx+1)(nx^2+nx+1).
\end{aligned}$$

□

Corollary 3.18. Let $m \geq 3$ and $n \geq 4$. Then, the monophonic polynomial of $P_m + C_n$ is

$$M(P_m + C_n, x) = (x+1)^{m+n} - (2x+1)(nx^2+nx+1)(x+1)^{m-2}.$$

Proof. By Theorem 2.2, $M(P_n, x) = x^2(x+1)^{n-2}$. By Theorem 2.3, $M(C_n, x) = (x+1)^n - (nx^2+nx+1)$. By Example 3.7, $\kappa(P_n, x) = (n-1)x^2+nx$. By Example 3.8, $\kappa(C_n, x) = nx^2+nx$. By Example 3.11, $\pi(P_n, x) = (x+1)^n - \big[(n-1)x^2+nx+1\big]$. By Example 3.12, $\pi(C_n, x) = (x+1)^n - (nx^2+nx+1)$. Thus by Theorem 3.15, we have the following:

$$\begin{aligned}
M(P_m + C_n, x) &= M(P_m, x)\left[\kappa(C_n, x)+1\right] + M(C_n, x)\left[\kappa(P_m, x)+1\right] + \pi(P_m, x)\pi(C_n, x)\\
&= x^2(x+1)^{m-2}(nx^2+nx+1)\\
&\quad + \big[(x+1)^n - (nx^2+nx+1)\big]\big[(m-1)x^2+mx+1\big]\\
&\quad + \big[(x+1)^m - [(m-1)x^2+mx+1]\big]\big[(x+1)^n - (nx^2+nx+1)\big]\\
&= x^2(x+1)^{m-2}(nx^2+nx+1) + (x+1)^n\big[(m-1)x^2+mx+1\big]\\
&\quad - (nx^2+nx+1)\big[(m-1)x^2+mx+1\big] + (x+1)^{m+n}\\
&\quad - (x+1)^m(nx^2+nx+1) - (x+1)^n\big[(m-1)x^2+mx+1]\big]\\
&\quad + (nx^2+nx+1)\big[(m-1)x^2+mx+1\big]\\
&= (x+1)^{m+n} + (nx^2+nx+1)(x+1)^{m-2}\big[x^2-(x+1)^2\big]\\
&= (x+1)^{m+n} - (2x+1)(nx^2+nx+1)(x+1)^{m-2}.
\end{aligned}$$

□

Corollary 3.19. Let $m \geq 3$ and $n \geq 2$. Then, the monophonic polynomial of $P_m + \overline{K_n}$ is

$$\begin{aligned}
M(P_m + \overline{K_n}, x) &= (x+1)^{m+n} - (2x+1)(nx+1)(x+1)^{m-2}\\
&\quad + \big[(m-1)x^2+mx+1\big]\left[(x^n+nx+1)-(x+1)^n\right].
\end{aligned}$$

Proof. Note that $M(\overline{K_n}, x) = x^n$. By Theorem 2.2, $M(P_n, x) = x^2(x+1)^{n-2}$. By Example 3.7, $\kappa(P_n, x) = (n-1)x^2+nx$. By Example 3.9, $\kappa(\overline{K_n}, x) = nx$. By Example 3.11, $\pi(P_n, x) = (x+1)^n - \big[(n-1)x^2+nx+1\big]$. By Example 3.13, $\pi(\overline{K_n}, x) = (x+1)^n - (nx+1)$. Theorem 3.15, we have the following:

$$\begin{aligned}
M(P_m + \overline{K_n}, x) &= M(P_m, x)\big[\kappa(\overline{K_n}, x)+1\big] + M(\overline{K_n}, x)\left[\kappa(P_m, x)+1\right] + \pi(P_m, x)\pi(\overline{K_n}, x)\\
&= x^2(x+1)^{m-2}(nx+1) + x^n\big[(m-1)x^2+mx+1\big]\\
&\quad + \big[(x+1)^m - [(m-1)x^2+mx+1]\big]\left[(x+1)^n-(nx+1)\right]\\
&= x^2(x+1)^{m-2}(nx+1) + x^n\big[(m-1)x^2+mx+1\big] + (x+1)^{m+n}\\
&\quad - (nx+1)(x+1)^m - [(m-1)x^2+mx+1](x+1)^n\\
&\quad + (nx+1)[(m-1)x^2+mx+1]\\
&= (x+1)^{m+n} + (nx+1)(x+1)^{m-2}\big[x^2-(x+1)^2\big]
\end{aligned}$$

$$+\left[(m-1)x^2+mx+1\right]\left[x^n+(nx+1)-(x+1)^n\right]$$
$$=(x+1)^{m+n}-(2x+1)(nx+1)(x+1)^{m-2}$$
$$+\left[(m-1)x^2+mx+1\right]\left[(x^n+nx+1)-(x+1)^n\right].$$

□

Corollary 3.20. Let $m \geq 4$ and $n \geq 2$. Then, the monophonic polynomial of $C_m + \overline{K_n}$ is

$$M(C_m+\overline{K_n},x)=(x+1)^{m+n}+(mx^2+mx+1)\left[x^n-(x+1)^n\right].$$

Proof. By Theorem 2.3, $M(C_n,x) = (x+1)^n - (nx^2+nx+1)$. By Example 3.8, $\kappa(C_n,x) = nx^2+nx$. By Example 3.9, $\kappa(\overline{K_n},x) = nx$. By Example 3.12, $\pi(C_n,x) = (x+1)^n - (nx^2+nx+1)$. By Example 3.13, $\pi(\overline{K_n},x) = (x+1)^n-(nx+1)$. Thus, Theorem 3.15, we have the following:

$$\begin{aligned}
M(C_m+\overline{K_n},x) &= M(C_m,x)\left[\kappa(\overline{K_n},x)+1\right]+M(\overline{K_n},x)\left[\kappa(C_m,x)+1\right]+\pi(C_m,x)\pi(\overline{K_n},x)\\
&=\left[(x+1)^m-(mx^2+mx+1)\right](nx+1)+x^n(mx^2+mx+1)\\
&\quad+\left[(x+1)^m-(mx^2+mx+1)\right]\left[(x+1)^n-(nx+1)\right]\\
&=(nx+1)(x+1)^m-(nx+1)(mx^2+mx+1)+x^n(mx^2+mx+1)\\
&\quad+(x+1)^{m+n}-(nx+1)(x+1)^m-(mx^2+mx+1)(x+1)^n\\
&\quad+(nx+1)(mx^2+mx+1)\\
&=(x+1)^{m+n}+(mx^2+mx+1)\left[x^n-(x+1)^n\right].
\end{aligned}$$

□

In the next result, we provide another proof of Theorem 2.5 using Theorem 3.15.

Another Proof of Theorem 2.5

Let $2 \leq m \leq n$. Then

$$\begin{aligned}
M(K_{m,n},x) &= (x+1)^{m+n}+(mx+1)\left[x^n-(x+1)^n\right]\\
&\quad+(nx+1)\left[x^m-(x+1)^m\right]+(mx+1)(nx+1).
\end{aligned}$$

Proof. Let $2 \leq m \leq n$. Note that $K_{m,n} \cong \overline{K_m} + \overline{K_n}$. Thus, by Theorem 3.14, we have the following:

$$\begin{aligned} M(K_{m,n}, x) &= M(\overline{K_m} + \overline{K_n}) \\ &= M(\overline{K_m}, x)[\kappa(\overline{K_n}, x) + 1] + M(\overline{K_n}, x)[\kappa(\overline{K_m}, x) + 1] \\ &\quad + \pi(\overline{K_m}, x)\pi(\overline{K_n}, x) \\ &= x^m(nx + 1) + x^n(mx + 1) \\ &\quad + [(x + 1)^m - (mx + 1)][(x + 1)^n - (nx + 1)] \\ &= x^m(nx + 1) + x^n(mx + 1) + (x + 1)^{m+n} - (mx + 1)(x + 1)^n \\ &\quad - (nx + 1)(x + 1)^m] + (mx + 1)(nx + 1) \\ &= (x + 1)^{m+n} + (mx + 1)\left[x^n - (x + 1)^n\right] \\ &\quad + (nx + 1)\left[x^m - (x + 1)^m\right] + (mx + 1)(nx + 1). \end{aligned}$$

□

Acknowledgement. The author would like to thank Caraga State University for granting Sabbatical Leave, which provided opportunity to dedicate time and effort in conducting this research.

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