

Multi-Decomposition of Product Graphs into Kites and Stars on Four Edges

Jothimani Velusamy^{1*}, Hemalatha Palanisamy²

¹Department of Mathematics, Kongu Engineering College, India, jothi18597@gmail.com

²Department of Mathematics, Vellalar College for Women, India,
dr.hemalatha@gmail.com²

Abstract. A decomposition of a graph G is a set of edge-disjoint subgraphs $H_1, H_2, \dots, H_r$ of G such that every edge of G belongs to exactly one H_i . If all the subgraphs in the decomposition of G are isomorphic to a graph H then we say that G is H -decomposable. The graph G has an $\{H_1^\alpha, H_2^\beta\}$ -decomposition, if α copies of H_1 and β copies of H_2 decompose G , where α and β are non-negative integers. In this paper, we have obtained the decomposition of $K_m \times K_n$ into α kites and β stars on four edges for some of the admissible pairs (α, β) , whenever $mn(m-1)(n-1) \equiv 0 \pmod{8}$, for $m \geq 3$ and $n \geq 4$. Also, we have obtained the decomposition of $K_m \otimes \overline{K_n}$ into α kites and β stars on four edges for some of the admissible pairs (α, β) , whenever $m(m-1)n^2 \equiv 0 \pmod{8}$, for $m \geq 3$ and $n \geq 4$. Here $K_m \times K_n$ and $K_m \otimes \overline{K_n}$ respectively denotes the tensor and wreath product of complete graphs.

Keywords: Kite, star, tensor product, wreath product, multi-decomposition.

1. INTRODUCTION

All graphs considered here are finite, simple, and undirected. Let $S_4 (= K_{1,4})$ denote a *star* on 4 edges. A *Kite* $\mathcal{K}$ is a graph in which one edge is attached to a vertex of the triangle C_3 . Let $K_{n,n} - I$ denote a complete bipartite graph minus one factor. For two graphs G_1 and G_2 , their *tensor product* $G_1 \times G_2$ and *wreath product* $G_1 \otimes G_2$ have the same vertex set $V(G_1) \times V(G_2)$, and the corresponding edge sets are defined as follows: $E(G_1 \times G_2) = \{(g_1, g_2)(g'_1, g'_2) | g_1 g'_1 \in E(G_1) \text{ and } g_2 g'_2 \in E(G_2)\}$ and $E(G_1 \otimes G_2) = \{(g_1, g_2)(g'_1, g'_2) | g_1 g'_1 \in E(G_1) \text{ or } g_1 = g'_1, g_2 g'_2 \in E(G_2)\}$. In general, if G is the product of two graphs G_1 and G_2 of order m and n , respectively, then G can be considered a multi-partite graph with mn vertices where $V(G) = \{v_i^j | i \in \{1, 2, \dots, m\} \text{ and } j \in \{1, 2, \dots, n\}\}$ and the edge set of G

*Corresponding author

2020 Mathematics Subject Classification: 05C51, 05C76

Received: 04-04-2024, accepted: 11-05-2025.

is defined according to the product considered. For a graph G , if $E(G)$ can be partitioned into subsets $E_1, \dots, E_k$ such that the subgraph of G induced by E_i is H_i for each $1 \leq i \leq k$, we write $G = H_1 \oplus \dots \oplus H_k$. For $1 \leq i \leq k$, if $H_i \cong H$, we say that G has an H -decomposition. If G can be decomposed into α copies of H_1 and β copies of H_2 , then we say that G has an $\{H_1^\alpha, H_2^\beta\}$ -decomposition or an (H_1, H_2) -multi-decomposition. If the necessary conditions are satisfied for the existence of a $\{H_1^\alpha, H_2^\beta\}$ -decomposition of G by pairs (α, β) , then we say that (α, β) is an admissible pair. Note that if G_1 and G_2 have an (H_1, H_2) -multi-decomposition, then the (vertex or edge) disjoint union of G_1 and G_2 also has a decomposition of the same type.

The study of $\{H_1^\alpha, H_2^\beta\}$ -decomposition has been introduced by Abueida and Daven [1]. Moreover, Abueida and O'Neil [2] have settled the existence of $\{H_1^\alpha, H_2^\beta\}$ -decomposition of $K_m(\lambda)$ when $\{H_1, H_2\} = \{S_n, C_n\}$ for $n = 3, 4, 5$. In [3], Yizhe Gao and Dan Roberts obtained the multi-designs for the graph pair formed by the 6-cycle and 3-prism. In [4], Pauline Ezhilarasi et al. proved the existence of multi-decomposition in the tensor product of complete graphs into cycles and stars with four edges. In [5], Pauline Ezhilarasi and Muthusamy have obtained the multi-decomposition of some product graphs into paths and stars with three edges. In [6], Jeevadoss and Muthusamy proved the existence of multi-decomposition of some product graphs into paths and cycles of length four. In [7], Ilayaraja, Sowndharya, et al. studied the multi-decomposition of some product graphs into paths and stars on five vertices. All these studies motivated us to study the multi-decomposition of simple graphs into kites and stars on four edges. In [8], we have obtained the necessary and sufficient conditions for the existence of a multi-decomposition of complete graphs into kites and stars on four edges. Also, the multi-decomposition of the cartesian product of complete graphs into kites and stars has been studied in [9]. In this paper, we have obtained a $\{K^\alpha, S_4^\beta\}$ -decomposition of $K_m \times K_n$ and $K_m \otimes \overline{K_n}$ for some admissible pairs (α, β) , whenever $m \geq 3$ and $n \geq 4$.

Notations:

- (1) In a multi-partite graph G , we define F as a 1 -factor of distance p if $F = F_1 \cup F_2$, where $F_1 = \bigcup_{i=1}^m v_{i+p}^j v_i^j$ and $F_2 = \bigcup_{j=1}^n v_i^{j+p} v_i^j$ and the additions in the subscripts and the superscripts are taken modulo m and n respectively, with residues $0, 1, 2, \dots, m-1$ and $0, 1, 2, \dots, n-1$.
- (2) A kite in a multi-partite graph G is of the form $(v_i^j, v_k^l, v_p^q; v_x^y)$, where v_i^j, v_k^l, v_p^q is a triangle C_3 and $v_p^q v_x^y$ is the pendant edge e attached to the vertex v_p^q of C_3 . It is denoted by $\mathcal{K}$.
- (3) A star S_4 in a multi-partite graph G is of the form $(v_p^q; v_i^j, v_k^l, v_m^n, v_x^y)$, where v_p^q is the center vertex (of degree 4) and $v_i^j, v_k^l, v_m^n, v_x^y$ are the end vertices (pendant vertices).
- (4) The n stars with the same end vertices $v_a^u, v_b^w, v_c^x, v_d^y$ and different center vertices $a_i^1, a_m^2, \dots, a_p^n$ are denoted by $(a_i^1, a_m^2, \dots, a_p^n; v_a^u, v_b^w, v_c^x, v_d^y)$.

1.1. Preliminary results.

In this section, we will present some of the known lemmas and theorems which will be used in proving our main results.

Lemma 1.1. [8] *The graph K_8 admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ - decomposition if and only if $\alpha + \beta = 7$, where $\alpha, \beta \geq 0$.*

Lemma 1.2. [8] *The graph K_9 admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ - decomposition if and only if $\alpha + \beta = 9$, where $\alpha, \beta \geq 0$.*

Theorem 1.3. [8] *For all even $t \geq 4$, if $n = 4t$ or $n = 4t + 1$, then K_n admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ - decomposition for the pairs (α, β) satisfying $0 \leq \alpha \leq \frac{9t-4}{2}$, $\beta = \frac{1}{4}\binom{n}{2} - \alpha$.*

Theorem 1.4. [10] *Let k, m and $n \in \mathbb{Z}^+$ with $m \leq n$. Then there exists an S_k -decomposition of $K_{m,n}$ if and only if one of the following holds:*

- (i) $k \leq m$ and $mn \equiv 0 \pmod{k}$
- (ii) $m < k \leq n$ and $n \equiv 0 \pmod{k}$.

Observation 1.1. *If $G = K_{n,n} - I$, where $n \equiv 1 \pmod{4}$, then the degree of every vertex in G is a multiple of 4 and hence G admits an S_4 -decomposition in this case, by Theorem 1.4.*

Observation 1.2. *The graph $P_3 \times K_n$ admits an S_4 -decomposition when n is odd.*

2. $\{\mathcal{K}^\alpha, S_4^\beta\}$ -DECOMPOSITION OF $K_m \times K_n$

In this section, the necessary and sufficient conditions for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition in the tensor product of complete graphs have been discussed.

Theorem 2.1. *Let $m \geq 3$ and $n \geq 4$ be the given integers. Then the necessary condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_m \times K_n$ is $mn(m-1)(n-1) \equiv 0 \pmod{8}$.*

Proof. Obvious from the edge divisibility condition required for the desired decomposition in $K_m \times K_n$. $\square$

The following lemmas are useful in proving our main result on the sufficient condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition in $K_m \times K_n$ for $m \geq 3$ and $n \geq 4$.

Lemma 2.2. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_3 \times K_4$ for all admissible pairs (α, β) such that $\alpha + \beta = 9$.*

Proof. The decomposition of $K_3 \times K_4$ into α kites (i.e, the set K) and β stars (i.e, the set S) on 4 edges for all admissible pairs (α, β) such that $\alpha + \beta = 9$ is discussed in the following cases:

Case 1: $(\alpha, \beta) = (9, 0)$.

$K = \{(v_2^2, v_3^3, v_1^1, v_4^4), (v_1^2, v_3^4, v_2^3, v_1^1), (v_3^1, v_1^3, v_2^2, v_4^4), (v_1^2, v_2^1, v_3^3, v_4^4), (v_1^3, v_2^2, v_4^4, v_3^3), (v_2^1, v_4^4, v_3^2, v_1^1), (v_2^1, v_3^3, v_4^4, v_2^2), (v_1^2, v_3^4, v_2^3, v_1^1), (v_3^1, v_4^4, v_2^3, v_3^2)\}$.

Case 2: $(\alpha, \beta) = (8, 1)$.

$K = \{(v_2^2, v_3^3, v_1^1, v_2^2), (v_1^2, v_3^4, v_2^3, v_1^1), (v_3^1, v_1^3, v_2^2, v_4^4), (v_2^1, v_2^2, v_3^3, v_4^4), (v_1^3, v_2^2, v_4^4, v_1^1), (v_2^1, v_4^4, v_3^2, v_3^3), (v_2^1, v_3^3, v_4^4, v_1^1), (v_1^2, v_4^4, v_3^3, v_2^2)\}$ and $S = \{(v_1^4, v_2^2, v_3^1, v_3^3, v_3^3)\}$.

Case 3: $(\alpha, \beta) = (7, 2)$.

$K = \{(v_1^1, v_2^2, v_3^3, v_4^4), (v_2^2, v_3^4, v_1^2, v_4^4), (v_2^2, v_3^1, v_4^3, v_2^2), (v_2^1, v_3^3, v_1^2, v_3^3), (v_1^3, v_2^2, v_4^4, v_1^1), (v_2^1, v_4^4, v_3^2, v_3^3)\}$ and $S = \{(v_1^1, v_2^2, v_3^4, v_2^2, v_2^2), (v_1^4, v_2^2, v_3^1, v_3^3, v_3^3)\}$.

Case 4: $(\alpha, \beta) = (6, 3)$.

$K = \{(v_1^1, v_2^2, v_3^3, v_4^4), (v_1^2, v_3^4, v_2^3, v_1^1), (v_3^1, v_1^3, v_2^2, v_4^4), (v_2^2, v_1^4, v_2^3, v_1^1), (v_2^1, v_3^3, v_2^2, v_3^3), (v_3^1, v_2^2, v_4^4, v_1^1)\}$ and $S = \{(v_1^4, v_2^2, v_2^2, v_3^3, v_3^3), (v_2^4, v_3^3, v_1^2, v_1^1, v_1^1), (v_2^3, v_2^2, v_1^1, v_3^3, v_2^2)\}$.

Case 5: $(\alpha, \beta) = (5, 4)$.

$K = \{(v_2^2, v_3^3, v_1^1, v_2^2), (v_2^2, v_3^4, v_1^2, v_3^3), (v_2^2, v_3^1, v_4^3, v_2^2), (v_1^4, v_2^2, v_3^3, v_3^3), (v_2^2, v_1^4, v_3^3, v_2^2)\}$ and $S = \{(v_1^4, v_2^2, v_2^2, v_3^3, v_3^3), (v_2^4, v_3^3, v_1^2, v_1^1, v_1^1), (v_2^3, v_1^1, v_2^2, v_3^3, v_4^4), (v_4^4, v_3^3, v_1^1, v_2^2, v_2^2)\}$.

Case 6: $(\alpha, \beta) = (4, 5)$.

$K = \{(v_1^1, v_2^2, v_3^3, v_2^2), (v_2^2, v_3^4, v_1^2, v_2^2), (v_2^2, v_3^1, v_4^3, v_2^2), (v_1^3, v_2^2, v_4^4, v_2^2)\}$ and $S = \{(v_1^1, v_2^2, v_2^2, v_3^3, v_3^3), (v_3^4, v_2^2, v_2^2, v_3^3, v_3^3), (v_1^4, v_2^2, v_2^2, v_3^3, v_2^2), (v_3^1, v_2^2, v_2^2, v_3^3, v_2^2), (v_3^4, v_2^2, v_2^2, v_2^2, v_2^2)\}$.

Case 7: $(\alpha, \beta) = (3, 6)$.

$K = \{(v_1^1, v_2^2, v_3^3, v_2^2), (v_1^3, v_2^2, v_2^2, v_3^3), (v_1^4, v_2^2, v_3^3, v_2^2)\}$ and $S = \{(v_1^1, v_2^2, v_2^2, v_3^3, v_3^3), (v_3^3, v_2^2, v_2^2, v_4^4, v_4^4), (v_1^4, v_2^2, v_2^2, v_3^3, v_2^2), (v_3^1, v_2^2, v_2^2, v_3^3, v_2^2), (v_4^4, v_1^2, v_2^2, v_2^2, v_2^2), (v_1^2, v_2^2, v_2^2, v_2^2, v_3^3)\}$.

Case 8: $(\alpha, \beta) = (2, 7)$.

$K = \{(v_1^1, v_2^2, v_3^3, v_4^4), (v_2^2, v_3^4, v_1^2, v_2^2)\}$ and $S = \{(v_1^1, v_2^2, v_2^2, v_3^3, v_3^3), (v_3^3, v_2^2, v_4^4, v_3^3, v_3^3), (v_3^1, v_2^2, v_2^2, v_4^4, v_2^2), (v_4^4, v_2^2, v_2^2, v_3^3, v_1^1), (v_1^2, v_2^2, v_2^2, v_3^3, v_2^2), (v_4^4, v_1^2, v_2^2, v_3^3, v_2^2), (v_3^3, v_2^2, v_2^2, v_4^4, v_2^2)\}$.

Case 9: $(\alpha, \beta) = (1, 8)$.

$K = \{(v_1^1, v_2^2, v_4^4, v_1^1)\}$ and $S = \{(v_1^1, v_2^2, v_2^2, v_3^3, v_3^3), (v_1^3, v_2^2, v_2^2, v_3^3, v_3^3), (v_3^1, v_2^2, v_2^2, v_3^3, v_2^2), (v_3^4, v_2^2, v_2^2, v_1^1), (v_2^2, v_2^2, v_2^2, v_4^4, v_3^3), (v_4^4, v_1^2, v_2^2, v_2^2, v_3^3), (v_3^3, v_2^2, v_2^2, v_1^1, v_2^2), (v_2^2, v_2^2, v_2^2, v_3^3, v_1^1)\}$.

Case 10: $(\alpha, \beta) = (0, 9)$.

$S = \{(v_1^1, v_2^2, v_4^4, v_3^3, v_3^3), (v_2^2, v_1^3, v_3^3, v_2^2, v_1^1), (v_2^2, v_1^2, v_4^4, v_1^1, v_3^3), (v_2^2, v_1^2, v_3^3, v_4^4, v_2^2), (v_3^1, v_2^2, v_3^3, v_2^2, v_2^2), (v_4^4, v_1^2, v_3^3, v_2^2, v_2^2), (v_3^3, v_2^2, v_2^2, v_1^1, v_2^2), (v_2^2, v_1^3, v_3^3, v_4^4, v_1^1)\}$. $\square$

Lemma 2.3. *There exists a $\{K^\alpha, S_4^\beta\}$ -decomposition of $K_3 \times K_5$ for all admissible pairs (α, β) such that $\alpha + \beta = 15$.*

Proof. The decomposition $K \cup S$ of $K_3 \times K_5$ into α kites and β stars on 4 edges for all admissible pairs (α, β) such that $\alpha + \beta = 15$ is discussed in the following cases:

Case 1: $(\alpha, \beta) = (15, 0)$.

$K = \{(v_1^5, v_3^3, v_4^4, v_1^1), (v_2^3, v_2^2, v_1^4, v_2^2), (v_1^3, v_2^2, v_3^1, v_2^2), (v_2^1, v_2^2, v_3^5, v_1^1), (v_3^4, v_2^2, v_1^1, v_2^2), (v_2^4, v_1^3, v_3^5, v_4^4), (v_1^2, v_2^2, v_3^4, v_1^1), (v_2^2, v_3^3, v_1^1, v_2^2), (v_2^2, v_1^5, v_2^2, v_3^4), (v_1^4, v_2^2, v_3^1, v_2^2), (v_1^5, v_3^1, v_2^2, v_3^5), (v_1^5, v_4^4, v_2^2, v_3^5), (v_2^5, v_1^3, v_2^2, v_2^2), (v_2^5, v_3^3, v_1^2, v_2^2), (v_1^4, v_3^3, v_1^2, v_1^1)\}$.

Case 2: $(\alpha, \beta) = (14, 1)$.

$K = \{(v_1^5, v_2^4, v_3^3, v_2^2), (v_2^3, v_2^2, v_1^4, v_2^2), (v_1^3, v_2^2, v_3^1, v_2^2), (v_2^1, v_2^2, v_3^5, v_1^1), (v_3^4, v_2^2, v_1^1, v_2^2), (v_2^4, v_1^3, v_3^5, v_4^4), (v_1^2, v_2^2, v_3^4, v_1^1), (v_2^2, v_3^3, v_1^1, v_2^2), (v_2^2, v_1^5, v_2^2, v_3^4), (v_1^4, v_2^2, v_3^1, v_2^2), (v_1^5, v_3^1, v_2^2, v_3^5), (v_1^5, v_4^4, v_2^2, v_3^5), (v_2^5, v_3^3, v_1^2, v_2^2), (v_1^4, v_3^3, v_1^2, v_1^1)\}$ and $S = \{(v_1^1, v_2^2, v_3^3, v_2^2, v_2^2)\}$.

Case 3: $(\alpha, \beta) = (13, 2)$.

$K = \{(v_1^5, v_2^4, v_3^3, v_2^2), (v_2^3, v_2^2, v_1^4, v_2^2), (v_1^3, v_2^2, v_3^1, v_2^2), (v_3^4, v_2^2, v_1^1, v_2^2), (v_2^4, v_1^3, v_3^5, v_4^4), (v_1^2, v_2^2, v_3^4, v_1^1), (v_2^2, v_3^3, v_1^1, v_2^2), (v_2^2, v_1^5, v_2^2, v_3^4), (v_1^4, v_2^2, v_3^1, v_2^2), (v_1^5, v_3^1, v_2^2, v_3^5), (v_1^5, v_4^4, v_2^2, v_3^5), (v_2^5, v_3^3, v_1^2, v_2^2), (v_1^4, v_3^3, v_1^2, v_1^1)\}$.

$(v_2^4), (v_1^5, v_3^1, v_2^3, v_3^5), (v_1^5, v_3^4, v_2^2, v_3^5), (v_2^5, v_3^1, v_2^3, v_2^4), (v_2^5, v_3^3, v_2^2, v_2^4), (v_1^4, v_3^3, v_2^1, v_3^3), (v_2^3, v_3^4, v_2^1, v_2^1), (v_2^4, v_3^5, v_1^3, v_3^4)\}$ and $S=\{(v_1^1; v_2^2, v_3^3, v_2^3, v_2^4), (v_3^5; v_1^1, v_4^1, v_2^1, v_2^1)\}$.

Case 4: $(\alpha, \beta) = (12, 3)$.

$K=\{(v_1^5, v_2^4, v_3^3; v_2^2), (v_2^3, v_2^3, v_4^1; v_2^5), (v_1^3, v_3^1, v_2^2; v_4^1), (v_3^4, v_2^5, v_1^1; v_2^3), (v_2^3, v_2^5, v_1^2; v_4^3), (v_1^5, v_1^3, v_2^3; v_3^5), (v_1^5, v_3^4, v_2^2, v_3^5), (v_2^5, v_3^3, v_2^2; v_4^1), (v_2^5, v_3^3, v_2^2; v_4^1), (v_1^4, v_3^3, v_2^1; v_3^1), (v_2^3, v_3^4, v_2^1; v_2^1), (v_2^4, v_3^5, v_1^3; v_3^4)\}$ and $S=\{(v_1^1; v_2^2, v_3^3, v_2^3, v_2^4), (v_3^5; v_1^1, v_4^1, v_2^1, v_2^1), (v_1^3; v_2^5, v_4^1, v_2^4, v_2^1)\}$.

Case 5: $(\alpha, \beta) = (11, 4)$.

$K=\{(v_1^5, v_2^4, v_3^3; v_4^1), (v_1^4, v_2^3, v_2^2; v_1^1), (v_1^3, v_3^1, v_2^2; v_4^1), (v_2^4, v_3^3, v_2^5; v_1^2), (v_3^4, v_2^2, v_3^3; v_3^5), (v_3^3, v_2^2, v_1^1; v_3^5), (v_3^5, v_1^4, v_2^2; v_5^1), (v_2^5, v_1^3, v_2^2; v_4^1), (v_1^2, v_3^1, v_2^4; v_2^3), (v_3^4, v_3^3, v_2^1; v_3^5), (v_3^3, v_2^1, v_2^1; v_4^1)\}$ and $S=\{(v_1^1; v_2^2, v_3^3, v_2^3, v_2^4), (v_2^5; v_4^1, v_1^2, v_3^3, v_3^4), (v_1^1; v_2^5, v_2^4, v_3^3, v_3^4), (v_3^3; v_2^5, v_1^5, v_2^3, v_4^1)\}$.

Case 6: $(\alpha, \beta) = (10, 5)$.

$K=\{(v_1^5, v_2^3, v_4^2; v_1^2), (v_1^4, v_2^3, v_2^2; v_1^1), (v_1^3, v_2^2, v_3^1; v_5^1), (v_2^4, v_3^3, v_2^5; v_2^2), (v_2^3, v_2^1, v_4^3; v_2^2), (v_3^3, v_2^2, v_1^1; v_4^2), (v_1^4, v_3^3, v_2^2; v_2^1), (v_1^3, v_4^4, v_2^2; v_2^3), (v_1^2, v_3^3, v_2^2; v_3^3), (v_2^5, v_3^3, v_2^2; v_2^2)\}$ and $S=\{(v_1^5; v_4^4, v_2^3, v_2^2, v_2^2), (v_3^5; v_1^2, v_1^1, v_2^3, v_2^2), (v_1^3; v_2^1, v_1^1, v_2^3, v_2^2), (v_1^1; v_2^5, v_2^2, v_4^4, v_2^2), (v_1^4; v_2^5, v_3^3, v_2^2, v_2^2)\}$.

Case 7: $(\alpha, \beta) = (9, 6)$.

$K=\{(v_2^4, v_3^4, v_1^5; v_3^1), (v_1^4, v_2^3, v_2^2; v_2^1), (v_2^2, v_3^1, v_3^3; v_2^1), (v_2^4, v_3^3, v_2^5; v_2^2), (v_2^3, v_2^1, v_4^3; v_2^2), (v_3^3, v_2^2, v_1^1; v_4^2), (v_1^4, v_3^3, v_2^2; v_2^1), (v_2^5, v_3^3, v_2^2; v_2^1), (v_3^3, v_2^1, v_2^2; v_4^2)\}$ and $S=\{(v_1^5; v_4^4, v_2^3, v_2^2, v_2^2), (v_3^5; v_2^1, v_1^1, v_2^3, v_2^2), (v_1^3; v_2^1, v_1^1, v_2^3, v_2^2), (v_1^1; v_2^5, v_2^2, v_4^4, v_2^2), (v_1^4; v_2^5, v_3^3, v_2^2, v_2^2)\}$.

Case 8: $(\alpha, \beta) = (8, 7)$.

$K=\{(v_2^4, v_3^3, v_1^5; v_3^1), (v_1^4, v_2^3, v_2^2; v_2^1), (v_2^2, v_3^1, v_3^3; v_2^1), (v_2^4, v_3^3, v_2^5; v_2^2), (v_2^3, v_3^4, v_2^1; v_4^2), (v_3^3, v_2^2, v_1^1; v_4^2), (v_1^4, v_3^3, v_2^2; v_2^1), (v_1^2, v_3^3, v_2^2; v_3^1), (v_3^3, v_2^1, v_2^2; v_4^2)\}$ and $S=\{(v_1^5; v_4^4, v_2^3, v_2^2, v_2^2), (v_3^5; v_2^1, v_1^1, v_2^3, v_2^2), (v_1^3; v_2^1, v_1^1, v_2^3, v_2^2), (v_1^1; v_2^5, v_2^2, v_4^4, v_2^2), (v_1^4; v_2^5, v_3^3, v_2^2, v_2^2)\}$.

Case 9: $(\alpha, \beta) = (7, 8)$.

$K=\{(v_1^5, v_2^4, v_3^3; v_2^1), (v_1^4, v_2^3, v_2^2; v_2^1), (v_1^3, v_2^2, v_3^1; v_5^1), (v_2^4, v_3^3, v_2^5; v_2^2), (v_2^3, v_2^1, v_4^3; v_2^2), (v_3^3, v_2^2, v_1^1; v_4^2), (v_1^4, v_3^3, v_2^2; v_2^1), (v_1^2, v_3^3, v_2^2; v_3^1), (v_3^3, v_2^1, v_2^2; v_4^2)\}$ and $S=\{(v_1^5; v_4^4, v_2^3, v_2^2, v_2^2), (v_3^5; v_2^1, v_1^1, v_2^3, v_2^2), (v_1^3; v_2^1, v_1^1, v_2^3, v_2^2), (v_1^1; v_2^5, v_2^2, v_4^4, v_2^2), (v_1^4; v_2^5, v_3^3, v_2^2, v_2^2)\}$.

Case 10: $(\alpha, \beta) = (6, 9)$.

$K=\{(v_1^5, v_2^4, v_3^3; v_2^1), (v_1^4, v_2^3, v_2^2; v_3^1), (v_1^3, v_2^2, v_3^1; v_2^1), (v_2^4, v_3^3, v_2^5; v_2^2), (v_2^3, v_2^1, v_4^3; v_2^1), (v_3^3, v_2^2, v_1^1; v_4^2), (v_1^4, v_3^3, v_2^2; v_2^1), (v_1^2, v_3^3, v_2^2; v_3^1), (v_3^3, v_2^1, v_2^2; v_4^2)\}$ and $S=\{(v_1^5; v_2^2, v_3^3, v_2^2, v_2^1), (v_1^4; v_3^3, v_3^1, v_2^2, v_2^1), (v_2^5; v_1^4, v_1^3, v_1^2, v_1^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1)\}$.

Case 11: $(\alpha, \beta) = (5, 10)$.

$K=\{(v_1^5, v_2^4, v_3^3; v_2^1), (v_1^4, v_2^3, v_2^2; v_3^1), (v_1^3, v_2^2, v_3^1; v_2^1), (v_2^4, v_3^3, v_2^5; v_2^2), (v_2^3, v_2^1, v_4^3; v_2^1), (v_3^3, v_2^2, v_1^1; v_4^2), (v_1^4, v_3^3, v_2^2; v_2^1), (v_1^2, v_3^3, v_2^2; v_3^1), (v_3^3, v_2^1, v_2^2; v_4^2)\}$ and $S=\{(v_1^5; v_2^2, v_3^3, v_2^2, v_2^1), (v_1^4; v_3^3, v_3^1, v_2^2, v_2^1), (v_2^5; v_1^4, v_1^3, v_1^2, v_1^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1)\}$.

Case 12: $(\alpha, \beta) = (4, 11)$.

$K=\{(v_1^5, v_2^4, v_3^3; v_2^1), (v_1^4, v_2^3, v_2^2; v_3^1), (v_1^3, v_2^2, v_3^1; v_2^1), (v_2^4, v_3^3, v_2^5; v_2^2), (v_2^3, v_2^1, v_4^3; v_2^1), (v_3^3, v_2^2, v_1^1; v_4^2), (v_1^4, v_3^3, v_2^2; v_2^1), (v_1^2, v_3^3, v_2^2; v_3^1), (v_3^3, v_2^1, v_2^2; v_4^2)\}$ and $S=\{(v_1^5; v_2^2, v_3^3, v_2^2, v_2^1), (v_1^4; v_3^3, v_3^1, v_2^2, v_2^1), (v_2^5; v_1^4, v_1^3, v_1^2, v_1^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1)\}$.

Case 13: $(\alpha, \beta) = (3, 12)$.

$K=\{(v_3^3, v_2^4, v_1^5; v_3^1), (v_1^4, v_2^3, v_2^2; v_3^1), (v_3^5, v_4^2, v_1^3; v_3^1)\}$ and $S=\{(v_1^5; v_2^2, v_2^2, v_2^1, v_3^1), (v_1^4; v_3^3, v_3^1, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1), (v_2^5; v_4^4, v_3^3, v_2^2, v_2^1)\}$.

Case 14: $(\alpha, \beta) = (2, 13)$.

$K = \{(v_1^5, v_2^4, v_3^3, v_4^1), (v_3^5, v_2^4, v_1^3, v_4^1)\}$ and $S = \{(v_1^5, v_2^3, v_2^2, v_2^1, v_3^1), (v_1^4, v_3^2, v_1^1, v_2^2, v_2^1), (v_2^5, v_4^1, v_1^3, v_2^1, v_1^1), (v_2^5, v_4^1, v_3^3, v_2^2, v_1^1), (v_2^4, v_1^2, v_1^1, v_3^1, v_2^2), (v_2^1, v_2^2, v_1^3, v_3^5, v_3^3), (v_3^5, v_4^1, v_2^1, v_3^3, v_1^1), (v_4^1, v_5^1, v_1^3, v_2^2, v_2^1), (v_3^2, v_1^5, v_4^1, v_3^1, v_2^1), (v_1^1, v_3^2, v_2^2, v_4^1, v_3^3), (v_2^2, v_1^3, v_1^1, v_3^5, v_3^3), (v_1^2, v_4^1, v_3^3, v_3^1, v_3^1), (v_3^2, v_4^1, v_3^3, v_3^1, v_3^1)\}$.

Case 15: $(\alpha, \beta) = (1, 14)$.

$K = \{(v_3^5, v_2^4, v_1^3, v_4^1)\}$ and $S = \{(v_1^5, v_2^2, v_2^3, v_3^1, v_4^1), (v_1^4, v_2^1, v_2^2, v_3^1, v_3^2), (v_2^1, v_4^1, v_2^3, v_3^1, v_2^1), (v_1^1, v_2^2, v_2^3, v_3^3), (v_2^1, v_3^1, v_5^1, v_3^5, v_3^3), (v_2^3, v_2^1, v_1^1, v_3^1, v_5^1), (v_3^3, v_2^4, v_1^5, v_4^1, v_2^1), (v_4^1, v_5^1, v_1^3, v_2^2, v_2^1), (v_3^5, v_4^1, v_1^2, v_3^2, v_1^1), (v_2^5, v_4^1, v_3^1, v_2^1, v_1^1), (v_2^5, v_4^1, v_3^3, v_2^2, v_3^1), (v_4^1, v_2^1, v_1^1, v_3^1, v_2^2), (v_3^2, v_4^1, v_3^3, v_3^2, v_3^1), (v_2^2, v_1^3, v_3^3, v_3^1)\}$.

Case 16: $(\alpha, \beta) = (0, 15)$.

$S = \{(v_1^5, v_3^4, v_2^3, v_2^2, v_2^1), (v_1^4, v_3^4, v_2^5, v_2^3, v_2^2, v_2^1), (v_1^3, v_3^3, v_2^5, v_2^4, v_2^2, v_2^1), (v_1^2, v_2^2, v_2^5, v_2^4, v_2^3, v_2^2), (v_1^1, v_3^1, v_2^5, v_2^4, v_2^3, v_2^2), (v_1^5, v_3^4, v_2^3, v_2^2, v_2^1), (v_1^4, v_3^4, v_2^5, v_2^3, v_2^2, v_2^1), (v_1^3, v_3^3, v_2^5, v_2^4, v_2^2, v_2^1), (v_1^2, v_2^2, v_2^5, v_2^4, v_2^3, v_2^2), (v_1^1, v_3^1, v_2^5, v_2^4, v_2^3, v_2^2)\}$. $\square$

Lemma 2.4. *There exists a $\{K^\alpha, S_4^\beta\}$ -decomposition of $K_4 \times K_4$ for all admissible pairs (α, β) such that $\alpha + \beta = 18$.*

Proof. The decomposition $K \cup S$ of $K_4 \times K_4$ into α kites and β stars on 4 edges for all admissible pairs (α, β) such that $\alpha + \beta = 18$ is discussed in the following cases:

Case 1: $(\alpha, \beta) = (18, 0)$.

$K = \{(v_1^4, v_2^3, v_3^2, v_4^1), (v_2^4, v_1^3, v_3^2, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^1, v_3^3, v_2^4, v_4^1), (v_2^1, v_3^3, v_2^4, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^1, v_3^3, v_2^4, v_4^1), (v_2^1, v_3^3, v_2^4, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^1, v_3^3, v_2^4, v_4^1), (v_2^1, v_3^3, v_2^4, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^1, v_3^3, v_2^4, v_4^1), (v_2^1, v_3^3, v_2^4, v_4^1)\}$.

Case 2: $(\alpha, \beta) = (17, 1)$.

$K = \{(v_2^5, v_3^4, v_4^3, v_1^2), (v_3^5, v_2^4, v_4^3, v_1^2), (v_1^5, v_2^4, v_3^3, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1)\}$ and $S = \{(v_1^1, v_2^2, v_2^3, v_3^4)\}$.

Case 3: $(\alpha, \beta) = (16, 2)$.

$K = \{(v_1^4, v_2^3, v_3^2, v_4^1), (v_3^4, v_2^3, v_1^2, v_4^1), (v_3^1, v_2^2, v_4^3, v_1^1), (v_2^1, v_2^2, v_3^3, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^1, v_2^2, v_3^3, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^1, v_2^2, v_3^3, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^1, v_2^2, v_3^3, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^1, v_2^2, v_3^3, v_4^1)\}$ and $S = \{(v_1^1, v_2^2, v_2^3, v_3^4), (v_1^4, v_2^3, v_3^2, v_4^1)\}$.

Case 4: $(\alpha, \beta) = (15, 3)$.

$K = \{(v_2^5, v_3^4, v_1^3, v_4^1), (v_2^4, v_3^3, v_1^2, v_4^1), (v_3^4, v_2^3, v_1^2, v_4^1), (v_2^4, v_3^3, v_1^2, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1)\}$ and $S = \{(v_1^1, v_2^2, v_2^3, v_3^4), (v_3^4, v_2^3, v_1^2, v_4^1), (v_3^1, v_2^2, v_4^3, v_1^1)\}$.

Case 5: $(\alpha, \beta) = (14, 4)$.

$K = \{(v_2^5, v_3^4, v_1^3, v_4^1), (v_2^4, v_3^3, v_1^2, v_4^1), (v_3^4, v_2^3, v_1^2, v_4^1), (v_2^4, v_3^3, v_1^2, v_4^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1), (v_1^4, v_2^3, v_3^2, v_4^1), (v_1^2, v_3^3, v_4^1, v_2^1)\}$ and $S = \{(v_1^1, v_2^2, v_2^3, v_3^4), (v_3^4, v_2^3, v_1^2, v_4^1), (v_3^1, v_2^2, v_4^3, v_1^1), (v_4^1, v_2^3, v_3^2, v_4^1)\}$.

Case 6: $(\alpha, \beta) = (13, 5)$.

$$K = \{(v_2^3, v_3^3, v_1^1; v_4^3), (v_2^4, v_3^3, v_1^1; v_4^4), (v_4^3, v_2^4, v_1^1; v_3^2), (v_2^4, v_3^3, v_1^2; v_2^1), (v_2^1, v_4^4, v_3^3; v_2^4), (v_2^1, v_2^3, v_4^3; v_2^1), (v_2^1, v_4^4, v_1^3; v_2^3), (v_2^2, v_3^3, v_1^3; v_2^4), (v_4^3, v_4^4, v_1^3; v_2^4), (v_2^3, v_3^2, v_4^4; v_1^4), (v_2^1, v_2^4, v_3^3; v_1^4), (v_2^1, v_4^4, v_3^3; v_2^2), (v_4^3, v_4^4, v_2^2; v_1^4), (v_2^3, v_4^4, v_3^3; v_1^4)\} \text{ and } S = \{(v_4^1, v_2^2, v_3^2, v_4^2, v_3^2), (v_4^3, v_2^1, v_4^1, v_2^4, v_1^3), (v_4^4, v_2^1, v_2^2, v_3^3, v_3^3), (v_4^4, v_1^4, v_3^3, v_2^4, v_1^3), (v_4^4, v_2^1, v_2^3, v_2^3, v_1^4)\}.$$

Case 7: $(\alpha, \beta) = (12, 6)$.

$$K = \{(v_2^2, v_3^3, v_1^1; v_4^3), (v_2^2, v_3^3, v_1^1; v_4^4), (v_3^4, v_2^4, v_1^1; v_2^3), (v_2^4, v_3^3, v_1^2; v_2^1), (v_2^1, v_4^4, v_3^3; v_2^4), (v_2^1, v_2^3, v_3^4; v_2^1), (v_2^1, v_4^4, v_1^3; v_3^2), (v_3^1, v_4^3, v_4^1; v_1^4), (v_2^3, v_2^3, v_4^1; v_4^1), (v_2^1, v_2^4, v_3^3; v_4^1), (v_2^1, v_3^4, v_3^3; v_2^3), (v_3^4, v_3^4, v_2^2; v_1^3), (v_3^2, v_4^4, v_3^3; v_1^4)\} \text{ and } S = \{(v_4^1, v_2^2, v_2^3, v_4^2, v_2^3), (v_3^4, v_1^2, v_4^1, v_4^2, v_3^3), (v_4^4, v_1^2, v_2^2, v_2^3, v_3^3), (v_2^4, v_1^4, v_2^3, v_4^2, v_4^3), (v_1^1, v_2^1, v_2^2, v_2^3, v_2^3), (v_3^1, v_2^2, v_3^3, v_2^4, v_4^2)\}.$$

Case 8: $(\alpha, \beta) = (11, 7)$.

$$K = \{(v_2^2, v_3^3, v_1^1; v_4^4), (v_2^2, v_3^3, v_1^1; v_4^3), (v_3^2, v_4^3, v_1^1; v_4^4), (v_2^2, v_3^1, v_3^3; v_4^4), (v_4^2, v_2^3, v_1^3; v_4^3), (v_4^1, v_3^3, v_4^4; v_4^2), (v_3^2, v_2^3, v_1^1; v_4^3), (v_3^2, v_4^3, v_2^1; v_4^4), (v_3^1, v_4^4, v_2^2; v_4^1), (v_3^3, v_4^2, v_2^4; v_4^3), (v_3^1, v_4^4, v_2^3; v_4^1)\} \text{ and } S = \{(v_1^1; v_2^2, v_3^2, v_4^2, v_3^2), (v_1^2; v_3^1, v_4^1, v_4^3, v_4^2), (v_3^1; v_2^1, v_4^1, v_2^4, v_4^3), (v_4^1; v_2^1, v_2^2, v_3^2, v_4^3), (v_4^2, v_1^1, v_3^1, v_4^1, v_4^3), (v_4^4; v_2^2, v_3^1, v_2^3, v_3^3), (v_4^4; v_1^1, v_2^1, v_3^1, v_2^2)\}.$$

Case 9: $(\alpha, \beta) = (10, 8)$.

$$K = \{(v_1^1, v_2^2, v_3^3, v_4^4), (v_2^1, v_3^3, v_2^1, v_4^4), (v_1^1, v_2^3, v_3^4, v_1^1), (v_2^2, v_1^1, v_3^1, v_4^4), (v_3^1, v_2^3, v_4^2, v_4^4), (v_2^3, v_2^3, v_1^4, v_4^4), (v_2^3, v_4^4, v_2^1, v_4^4), (v_3^4, v_4^4, v_2^2, v_4^4), (v_3^1, v_4^2, v_2^3, v_4^4), (v_4^2, v_4^4, v_3^3, v_4^4)\} \text{ and } S = \{(v_1^1, v_2^2, v_3^2, v_4^2, v_3^2), (v_2^1, v_3^1, v_4^1, v_4^3, v_2^4), (v_3^1, v_2^1, v_4^1, v_4^2, v_4^3), (v_4^1, v_2^1, v_2^2, v_3^2, v_3^4), (v_4^2, v_1^1, v_3^1, v_4^1, v_4^3), (v_4^4, v_2^2, v_3^1, v_2^3, v_3^3), (v_4^4, v_1^1, v_2^1, v_1^3, v_2^2), (v_4^4, v_1^1, v_2^1, v_4^1, v_4^4)\}.$$

Case 10: $(\alpha, \beta) = (9, 9)$.

$$K = \{(v_1, v_3, v_2, v_4), (v_2, v_1, v_3, v_4), (v_2, v_4, v_2, v_4), (v_3, v_2, v_3, v_2), (v_3, v_3, v_4, v_2), (v_2, v_3, v_1, v_3), (v_3, v_4, v_2, v_4), (v_2, v_4, v_3, v_3), (v_4, v_3, v_2, v_3)\} \text{ and } S = \{(v_1, v_2, v_2, v_4, v_3), (v_2, v_1, v_1, v_4, v_4), (v_3, v_2, v_1, v_2, v_4), (v_4, v_1, v_2, v_3, v_3), (v_4, v_1, v_3, v_4, v_4), (v_4, v_2, v_3, v_3, v_3), (v_4, v_1, v_1, v_1, v_2), (v_4, v_1, v_1, v_1, v_4), (v_4, v_3, v_4, v_2, v_4)\}.$$

Case 11: $(\alpha, \beta) = (8, 10)$.

$$K = \{(v_1^1, v_2^2, v_3^3; v_4^4), (v_1^1, v_1^1, v_3^3; v_4^4), (v_2^2, v_4^4, v_1^1; v_4^4), (v_3^1, v_2^2, v_3^3; v_2^2), (v_3^1, v_3^2, v_4^4; v_3^3), (v_2^2, v_3^3, v_4^4; v_2^2), (v_3^3, v_4^4, v_2^2; v_4^4), (v_2^2, v_4^4, v_3^3; v_3^3)\} \text{ and } S = \{(v_1^1; v_2^2, v_3^2, v_4^4, v_2^3), (v_2^2; v_3^1, v_4^1, v_4^3, v_2^4), (v_3^3; v_1^2, v_4^1, v_4^2, v_3^4), (v_4^4; v_1^2, v_2^2, v_3^3, v_3^4), (v_2^2; v_1^1, v_3^1, v_4^1, v_4^3), (v_4^4; v_2^2, v_3^1, v_3^3, v_3^3), (v_4^4; v_1^1, v_2^1, v_3^1, v_3^2), (v_4^4; v_1^1, v_2^1, v_4^1, v_2^4), (v_4^4; v_1^1, v_4^1, v_3^2, v_4^2), (v_2^2; v_3^2, v_2^4, v_3^1, v_3^3)\}.$$

Case 12: $(\alpha, \beta) = (7, 11)$.

$$K = \{(v_1^1, v_2^2, v_3^3; v_4^4), (v_2^2, v_3^3, v_2^2; v_4^4), (v_3^3, v_4^4, v_1^1; v_2^2), (v_1^1, v_2^2, v_3^3; v_2^2), (v_1^1, v_2^2, v_3^3; v_3^3), (v_2^2, v_3^3, v_4^4; v_1^1)\} \text{ and } S = \{(v_1^1, v_2^2, v_2^2, v_4^4, v_3^3), (v_2^2, v_3^3, v_1^1, v_4^4, v_2^2), (v_3^3; v_1^1, v_1^1, v_4^4, v_3^3), (v_4^4; v_2^2, v_2^2, v_3^3, v_3^3), (v_4^4; v_1^1, v_1^1, v_3^3, v_3^3), (v_4^4; v_2^2, v_2^2, v_3^3, v_3^3), (v_4^4; v_1^1, v_1^1, v_3^3, v_3^3), (v_4^4; v_2^2, v_2^2, v_3^3, v_3^3), (v_4^4; v_1^1, v_1^1, v_3^3, v_3^3), (v_4^4; v_2^2, v_2^2, v_3^3, v_3^3)\}.$$

Case 13: $(\alpha, \beta) = (6, 12)$.

$$K = \{(v_1^1, v_2^2, v_3^3, v_4^4), (v_2^2, v_3^3, v_1^1, v_2^4), (v_2^2, v_3^3, v_1^3, v_4^4), (v_1^1, v_3^3, v_2^4, v_3^3), (v_3^3, v_4^4, v_1^2, v_4^4), (v_2^2, v_4^4, v_3^4, v_1^2)\} \text{ and } S = \{(v_1^1, v_2^2, v_2^2, v_2^4, v_3^3), (v_1^2, v_1^3, v_1^4, v_3^4, v_4^4), (v_1^3, v_1^2, v_1^4, v_2^4, v_3^4), (v_1^4, v_1^2, v_2^2, v_3^3, v_3^4), (v_2^4, v_1^1, v_1^3, v_4^4, v_4^3), (v_4^4, v_2^2, v_1^3, v_3^3, v_3^3), (v_4^4, v_1^1, v_1^2, v_1^3, v_2^2), (v_3^4, v_1^1, v_1^2, v_1^4, v_2^4), (v_4^4, v_1^3, v_1^4, v_2^2, v_2^4), (v_2^4, v_3^2, v_2^4, v_1^3, v_3^3), (v_1^4, v_2^2, v_2^3, v_2^3, v_3^3), (v_2^2, v_1^2, v_1^3, v_2^3, v_4^3)\}.$$

Case 14: $(\alpha, \beta) = (5, 13)$.

$$K = \{(v_1^1, v_2^2, v_3^3; v_4^4), (v_2^2, v_3^3, v_1^1; v_4^4), (v_1^1, v_3^3, v_4^4; v_2^2), (v_2^2, v_4^4, v_1^1; v_3^3), (v_1^1, v_3^3, v_2^2; v_4^4)\} \text{ and } S = \{(v_1^1; v_2^2, v_3^3, v_4^4), (v_2^2; v_1^1, v_3^3, v_4^4), (v_3^3; v_1^1, v_2^2, v_4^4), (v_4^4; v_1^1, v_2^2, v_3^3), (v_2^2; v_1^1, v_3^3, v_4^4), (v_4^4; v_1^1, v_2^2, v_3^3), (v_3^3; v_1^1, v_2^2, v_4^4), (v_4^4; v_1^1, v_2^2, v_3^3), (v_2^2; v_1^1, v_3^3, v_4^4), (v_3^3; v_1^1, v_2^2, v_4^4), (v_4^4; v_1^1, v_2^2, v_3^3)\}.$$

Case 15: $(\alpha, \beta) = (4, 14)$.

$K = \{(v_1^1, v_2^2, v_3^3; v_4^4), (v_1^2, v_3^3, v_2^1; v_4^4), (v_1^3, v_3^1, v_2^2; v_4^3), (v_2^1, v_4^3, v_3^2; v_1^3)\}$ and $S = \{(v_1^1; v_2^2, v_3^3, v_4^4, v_2^3), (v_2^1; v_3^3, v_4^4, v_2^2, v_3^4), (v_3^1; v_2^2, v_4^4, v_2^3, v_3^4), (v_4^1; v_2^2, v_3^3, v_2^4, v_3^4), (v_2^2; v_1^3, v_3^1, v_4^4, v_3^4), (v_4^4; v_2^2, v_1^3, v_3^2, v_3^3), (v_3^3; v_4^1, v_2^1, v_3^1, v_2^3), (v_4^4; v_1^1, v_2^1, v_3^1, v_2^3), (v_1^1; v_3^3, v_4^1, v_2^3, v_4^4), (v_2^2; v_3^3, v_4^1, v_1^3, v_3^3), (v_4^4; v_2^2, v_3^3, v_2^3, v_3^3), (v_2^3; v_1^3, v_3^2, v_3^4), (v_3^3; v_2^1, v_3^1, v_2^2, v_4^4), (v_4^4; v_2^1, v_3^1, v_2^2, v_3^3)\}$.

Case 16: $(\alpha, \beta) = (3, 15)$.

$K = \{(v_1^1, v_3^3, v_2^2; v_1^3), (v_2^1, v_2^2, v_3^3; v_4^4), (v_2^1, v_3^3, v_4^4; v_2^2)\}$ and $S = \{(v_1^1; v_2^2, v_3^3, v_4^4, v_2^3), (v_2^1; v_4^4, v_3^4, v_2^4, v_3^4), (v_3^1; v_2^2, v_4^4, v_2^3, v_3^4), (v_4^1; v_2^2, v_3^3, v_2^4, v_3^4), (v_2^2; v_1^3, v_3^1, v_4^4, v_3^4), (v_4^4; v_2^2, v_1^3, v_2^3, v_3^3), (v_4^4; v_1^1, v_2^1, v_3^1, v_2^3), (v_4^4; v_1^1, v_2^1, v_4^4, v_2^3), (v_1^1; v_3^3, v_4^1, v_2^3, v_4^4), (v_2^2; v_3^3, v_4^1, v_1^3, v_3^3), (v_4^4; v_2^2, v_3^3, v_2^3, v_3^3), (v_3^3; v_2^1, v_3^1, v_2^2, v_4^4), (v_4^4; v_2^1, v_3^1, v_2^2, v_3^3), (v_3^3; v_2^1, v_3^1, v_2^2, v_3^3)\}$.

Case 17: $(\alpha, \beta) = (2, 16)$.

$K = \{(v_1^1, v_2^2, v_3^3; v_1^3), (v_2^1, v_3^3, v_2^2; v_4^4)\}$ and $S = \{(v_1^1; v_2^2, v_3^3, v_4^4, v_2^3), (v_2^1; v_4^4, v_3^4, v_2^4, v_3^4), (v_3^1; v_2^2, v_4^4, v_2^3, v_3^4), (v_4^1; v_2^2, v_3^3, v_2^4, v_3^4), (v_2^2; v_1^3, v_3^1, v_4^4, v_3^4), (v_4^4; v_2^2, v_1^3, v_2^3, v_3^3), (v_4^4; v_1^1, v_2^1, v_3^1, v_2^3), (v_3^3; v_4^1, v_2^1, v_3^1, v_2^3), (v_4^4; v_3^3, v_4^1, v_2^3, v_4^4), (v_1^1; v_3^3, v_4^1, v_2^3, v_4^4), (v_2^2; v_3^3, v_4^1, v_1^3, v_3^3), (v_1^1; v_2^2, v_3^3, v_2^3, v_3^3), (v_3^3; v_2^1, v_3^1, v_2^2, v_4^4), (v_4^4; v_2^1, v_3^1, v_2^2, v_3^3), (v_3^3; v_2^1, v_3^1, v_2^2, v_3^3)\}$.

Case 18: $(\alpha, \beta) = (1, 17)$.

$K = \{(v_2^2, v_4^4, v_1^3; v_2^3)\}$ and $S = \{(v_1^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_2^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_1^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_2^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_1^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_2^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_1^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_2^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_1^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_2^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_1^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_2^1; v_3^3, v_2^2, v_4^4, v_2^3)\}$.

Case 19: $(\alpha, \beta) = (0, 18)$.

$S = \{(v_1^1; v_2^2, v_3^3, v_4^4, v_2^3), (v_2^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_3^1; v_2^2, v_4^4, v_2^3, v_3^4), (v_4^1; v_2^2, v_3^3, v_2^4, v_3^4), (v_1^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_2^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_3^1; v_2^2, v_4^4, v_2^3, v_3^4), (v_4^1; v_2^2, v_3^3, v_2^4, v_3^4), (v_1^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_2^1; v_3^3, v_2^2, v_4^4, v_2^3), (v_3^1; v_2^2, v_4^4, v_2^3, v_3^4), (v_4^1; v_2^2, v_3^3, v_2^4, v_3^4)\}$.

□

Lemma 2.5. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_4 \times K_5$ for some admissible pairs (α, β) such that $\alpha + \beta = 30$.*

Proof. We can write, $K_4 \times K_5 = K_4 \times K_4 \oplus 12K_{1,4}$. By Lemma 2.4, the graph $K_4 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. Then the graph $K_4 \times K_5$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some of the admissible pairs (α, β) such that $\alpha + \beta = 30$. In this case we have, $0 \leq \alpha \leq 18$ and $12 \leq \beta \leq 30$ such that $\alpha + \beta = 30$. □

Lemma 2.6. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_4 \times K_6$ for some admissible pairs (α, β) such that $\alpha + \beta = 45$.*

Proof. We can write, $K_4 \times K_6 = 2(K_3 \times K_4) \oplus 27K_{1,4}$. By Lemma 2.2, the graph $K_3 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 9$. Then the graph $K_4 \times K_6$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some of the admissible pairs (α, β) such that $\alpha + \beta = 45$. In this case we have, $0 \leq \alpha \leq 18$ and $27 \leq \beta \leq 45$ such that $\alpha + \beta = 45$. □

Lemma 2.7. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_4 \times K_7$ for some admissible pairs (α, β) such that $\alpha + \beta = 63$.*

Proof. We can write, $K_4 \times K_7 = K_3 \times K_4 \oplus K_4 \times K_4 \oplus 36K_{1,4}$. By Lemma 2.2, the graph $K_3 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs such that $\alpha + \beta = 9$. By Lemma 2.4, the graph $K_4 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs such that $\alpha + \beta = 18$. Hence in this case we have, $0 \leq \alpha \leq 27$ and $36 \leq \beta \leq 63$ such that $\alpha + \beta = 63$. Thus the graph $K_4 \times K_7$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some of the admissible pairs (α, β) such that $\alpha + \beta = 63$. $\square$

Lemma 2.8. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_5 \times K_5$ for some admissible pairs (α, β) such that $\alpha + \beta = 50$.*

Proof. Let $K_5 \times K_5 = K_4 \times K_4 \oplus 32K_{1,4}$, then by Lemma 2.4, the graph $K_4 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. Hence in this case we have, $0 \leq \alpha \leq 18$ and $32 \leq \beta \leq 50$ such that $\alpha + \beta = 50$. Thus the graph $K_5 \times K_5$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some of the admissible pairs (α, β) such that $\alpha + \beta = 50$. $\square$

Lemma 2.9. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_5 \times K_6$ for some admissible pairs (α, β) such that $\alpha + \beta = 75$.*

Proof. Let $K_5 \times K_6 = K_4 \times K_4 \oplus 57K_{1,4}$. By Lemma 2.4, the graph $K_4 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. Hence in this case we have, $0 \leq \alpha \leq 18$ and $57 \leq \beta \leq 75$ such that $\alpha + \beta = 75$. Thus the graph $K_5 \times K_6$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some of the admissible pairs (α, β) such that $\alpha + \beta = 75$. $\square$

Remark 2.1. *The necessary condition given in Theorem 2.1 can be classified into 3 cases according to the values of m and n as stated below:*

Case 1. *Both m and n are even.*

(i) $m, n \equiv 0 \pmod{4}$

(ii) $m \equiv 0 \pmod{4}, n \equiv 2 \pmod{4}$

Case 2. *m is even and n is odd.*

(i) m is even, $n \equiv 1 \pmod{4}$

(ii) $m \equiv 0 \pmod{4}, n \equiv 3 \pmod{4}$

Case 3. *Both m and n are odd.*

(i) $m \equiv 1 \pmod{4}, n \equiv 1 \pmod{4}$

For other values of m and n , the necessary condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition in $K_m \times K_n$ will not be satisfied.

Lemma 2.10. *If $m, n \equiv 0 \pmod{4}$, then there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_m \times K_n$.*

Proof. Let $m = 4p$ and $n = 4q$, where $p, q \geq 1$. Then we can write, $K_{4p} \times K_{4q} = pq(K_4 \times K_4) \oplus 2pq(p+q-2)(K_{4,12}) \oplus 2pq(p-1)(q-1)(K_{4,16})$. From Lemma 2.4, the graph $K_4 \times K_4$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. Also from Theorem 1.4, the graphs $K_{4,12}$ and $K_{4,16}$ have an S_4 -decomposition. Thus, the graph $K_m \times K_n$ has the desired decomposition for some admissible pairs (α, β) . $\square$

Lemma 2.11. *If $m \equiv 0(\text{mod } 4)$ and $n \equiv 2(\text{mod } 4)$, then there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_m \times K_n$.*

Proof. Let $m = 4p$ and $n = 4q + 2$, where $p, q \geq 1$. Then we can write, $K_{4p} \times K_{4q+2} = p(q-1)(K_4 \times K_4) \oplus p(K_4 \times K_6) \oplus 4p(q-1)(K_{4,18}) \oplus 2q(p-1)(q-1)(K_{4,12}) \oplus 3p(p-1)(K_{4,20}) \oplus 4p(p-1)(q-1)(K_{4,24})$. From Lemma 2.4, the graph $K_4 \times K_4$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. By Lemma 2.6, the graph $K_4 \times K_6$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition such that $\alpha + \beta = 45$. Also from Theorem 1.4, the graphs $K_{4,8}$, $K_{4,12}$, K_{20} and $K_{4,24}$ admits an S_4 -decomposition. Thus, the graph $K_m \times K_n$ has the desired decomposition for some admissible pairs (α, β) . $\square$

Lemma 2.12. *If m is even and $n \equiv 1(\text{mod } 4)$, then there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_m \times K_n$.*

Proof. The proof of this theorem follows from two cases.

Case 1: $m = 0(\text{mod } 4)$ and $n \equiv 1(\text{mod } 4)$.

Let $m = 4p$ and $n = 4q + 1$, where $p, q \geq 1$. Then we can write, $K_{4p} \times K_{4q+1} = p(q-1)(K_4 \times K_4) \oplus p(K_4 \times K_5) \oplus 4p(q-1)(K_{4,15}) \oplus 2p(p-1)(q-1)(K_{4,18}) \oplus \frac{p(p-1)}{2}(5K_{4,16}) \oplus 4p(p-1)(q-1)(K_{4,20})$. From Lemma 2.4, the graph $K_4 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. From Lemma 2.5, the graph $K_4 \times K_5$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 30$. Also, from Theorem 1.4, the graphs $K_{4,15}$, $K_{4,18}$, $K_{4,16}$ and $K_{4,20}$ admits an S_4 -decomposition and hence the desired decomposition exists in this case, for some of the admissible pairs (α, β) .

Case 2: $m \equiv 2(\text{mod } 4)$ and $n \equiv 1(\text{mod } 4)$.

Let $m = 4p+2$ and $n = 4q+1$, where $p, q \geq 1$. Then we can write, $K_{4p+2} \times K_{4q+1} = (p-1)(q-1)(K_4 \times K_4) \oplus (p-1)(K_4 \times K_5) \oplus 4(p-1)(q-1)(K_{4,15}) \oplus (q-1)(K_6 \times K_4) \oplus (K_6 \times K_5) \oplus 6(q-1)(K_{4,25}) \oplus 4(p-1)(q-1)(K_{4,18}) \oplus 5(p-1)(K_{4,24}) \oplus (p-1)(q-1)(5K_{4,24} \oplus 4K_{4,30})$. From Lemma 2.4, the graph $K_4 \times K_4$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. From Lemma 2.5, the graph $K_4 \times K_5$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 30$. From Lemma 2.6, the graph $K_6 \times K_4$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 45$. From Lemma 2.9, the graph $K_6 \times K_5$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 75$. Then from Theorem 1.4, the graphs $K_{4,15}$, $K_{4,25}$, $K_{4,18}$, $K_{4,24}$ and $K_{4,30}$ have an S_4 -decomposition and hence the desired decomposition exists in this case, for some of the admissible pairs (α, β) . $\square$

Lemma 2.13. *If $m \equiv 0(\text{mod } 4)$ and $n \equiv 3(\text{mod } 4)$, then there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_m \times K_n$.*

Proof. Let $m = 4p$ and $n = 4q + 3$, where $p \geq 1$ and $q \geq 0$. When $q = 0$, we can write, $K_{4p} \times K_3 = p(4p-1)(P_3 \times K_3)$. By Observation 1.2, the graph $P_3 \times K_3$ admits an S_4 -decomposition. When $q \neq 0$, we can write, $K_{4p} \times K_{4q+3} = p(q-1)(K_4 \times$

$K_4) \oplus p(K_4 \times K_7) \oplus 4p(q-1)(K_{4,21}) \oplus 2p(p-1)(q-1)(K_{4,12}) \oplus 7\frac{p(p-1)}{2}(K_{4,24}) \oplus 4p(p-1)(q-1)(K_{4,28})$. From Lemma 2.4, the graph $K_4 \times K_4$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. From Lemma 2.7, the graph $K_4 \times K_7$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 63$. Then from Theorem 1.4, the graphs $K_{4,21}$, $K_{4,12}$, $K_{4,24}$ and $K_{4,28}$ have an S_4 -decomposition and hence the desired decomposition exists in this case, for some of the admissible pairs (α, β) . $\square$

Lemma 2.14. *If $m \equiv 1 \pmod{4}$ and $n \equiv 1 \pmod{2}$, then there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_m \times K_n$.*

Proof. We discuss the proof in two cases.

Case 1: $m = 5$ and $n \equiv 1 \pmod{2}$.

Let $m = 5$ and $n = 2p + 1$, where $p \geq 1$. For $p = 1$, the graph $K_5 \times K_3$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) by Lemma 2.3. For $p > 1$, we can write, $K_5 \times K_{2p+1} = (p-2)(K_{5,5} - I) \oplus K_5 \times K_5 \oplus 5(p-2)(K_{2,20})$. From Observation 1.1, the graph $K_{5,5} - I$ admits an S_4 -decomposition. From Lemma 2.8, the graph $K_5 \times K_5$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 50$. Also, from Theorem 1.4, the graph $K_{2,20}$ has an S_4 -decomposition and hence the desired decomposition exists in this case, for some of the admissible pairs (α, β) .

Case 2: $m > 5$ and $n \equiv 1 \pmod{2}$.

Let $m = 4p + 1$ and $n = 2q + 1$, where $p \geq 2$ and $q \geq 1$. Then $K_{4p+1} \times K_{4q+1} = (2q+1)q(K_{4p+1,4p+1} - I)$, where I is a 1-factor of distance zero in $K_{4p+1,4p+1}$. From Observation 1.1, the graph $K_{4p+1,4q+1} - I$ admits an S_4 -decomposition and hence the proof. $\square$

Now we prove our main result, which is the sufficient condition for the existence of $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition in $K_m \times K_n$.

Theorem 2.15. *Let $m \geq 3$ and $n \geq 4$. The graph $K_m \times K_n$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some of the admissible pairs (α, β) , if $mn(m-1)(n-1) \equiv 0 \pmod{8}$.*

Proof. The proof follows from Lemmas 2.10 to 2.14. $\square$

3. $\{\mathcal{K}^\alpha, S_4^\beta\}$ -DECOMPOSITION OF $K_m \otimes \overline{K_n}$

In this section, the necessary and sufficient conditions for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition in $K_m \otimes \overline{K_n}$ have been discussed.

Theorem 3.1. *Let $m \geq 3$ and $n \geq 4$ be the given integers. Then the necessary condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_m \otimes \overline{K_n}$ is $m(m-1)n^2 \equiv 0 \pmod{8}$. $\square$*

The following lemmas are useful in proving our main result which is the sufficient condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition in $K_m \otimes \overline{K_n}$ for $m \geq 3$ and $n \geq 4$.

Lemma 3.2. *The graph $K_3 \otimes \overline{K_4}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 12$.*

Proof. The decomposition $K \cup S$ of $K_3 \otimes \overline{K_4}$ into α kites and β stars on 4 edges for all admissible pairs (α, β) such that $\alpha + \beta = 12$ is discussed in the following cases:

Case 1: $(\alpha, \beta) = (12, 0)$.

$K = \{(v_1^1, v_2^1, v_3^1, v_4^1), (v_1^2, v_2^2, v_3^2, v_4^2), (v_1^3, v_2^3, v_3^3, v_4^3), (v_1^4, v_2^4, v_3^4, v_4^4), (v_1^5, v_2^5, v_3^5, v_4^5), (v_1^6, v_2^6, v_3^6, v_4^6), (v_1^7, v_2^7, v_3^7, v_4^7), (v_1^8, v_2^8, v_3^8, v_4^8), (v_1^9, v_2^9, v_3^9, v_4^9), (v_1^{10}, v_2^{10}, v_3^{10}, v_4^{10}), (v_1^{11}, v_2^{11}, v_3^{11}, v_4^{11}), (v_1^{12}, v_2^{12}, v_3^{12}, v_4^{12})\}$.

Case 2: $(\alpha, \beta) = (11, 1)$.

$K = \{(v_1^1, v_2^1, v_3^1, v_4^1), (v_1^2, v_2^2, v_3^2, v_4^2), (v_1^3, v_2^3, v_3^3, v_4^3), (v_1^4, v_2^4, v_3^4, v_4^4), (v_1^5, v_2^5, v_3^5, v_4^5), (v_1^6, v_2^6, v_3^6, v_4^6), (v_1^7, v_2^7, v_3^7, v_4^7), (v_1^8, v_2^8, v_3^8, v_4^8), (v_1^9, v_2^9, v_3^9, v_4^9), (v_1^{10}, v_2^{10}, v_3^{10}, v_4^{10}), (v_1^{11}, v_2^{11}, v_3^{11}, v_4^{11})\}$ and $S = \{(v_1^{12}, v_2^{12}, v_3^{12}, v_4^{12})\}$.

Case 3: $(\alpha, \beta) = (10, 2)$.

$K = \{(v_1^1, v_2^1, v_3^1, v_4^1), (v_1^2, v_2^2, v_3^2, v_4^2), (v_1^3, v_2^3, v_3^3, v_4^3), (v_1^4, v_2^4, v_3^4, v_4^4), (v_1^5, v_2^5, v_3^5, v_4^5), (v_1^6, v_2^6, v_3^6, v_4^6), (v_1^7, v_2^7, v_3^7, v_4^7), (v_1^8, v_2^8, v_3^8, v_4^8), (v_1^9, v_2^9, v_3^9, v_4^9), (v_1^{10}, v_2^{10}, v_3^{10}, v_4^{10})\}$ and $S = \{(v_1^{11}, v_2^{11}, v_3^{11}, v_4^{11}), (v_1^{12}, v_2^{12}, v_3^{12}, v_4^{12})\}$.

Case 4: $(\alpha, \beta) = (9, 3)$.

$K = \{(v_1^1, v_2^1, v_3^1, v_4^1), (v_1^2, v_2^2, v_3^2, v_4^2), (v_1^3, v_2^3, v_3^3, v_4^3), (v_1^4, v_2^4, v_3^4, v_4^4), (v_1^5, v_2^5, v_3^5, v_4^5), (v_1^6, v_2^6, v_3^6, v_4^6), (v_1^7, v_2^7, v_3^7, v_4^7), (v_1^8, v_2^8, v_3^8, v_4^8), (v_1^9, v_2^9, v_3^9, v_4^9)\}$ and $S = \{(v_1^{10}, v_2^{10}, v_3^{10}, v_4^{10}), (v_1^{11}, v_2^{11}, v_3^{11}, v_4^{11}), (v_1^{12}, v_2^{12}, v_3^{12}, v_4^{12})\}$.

Case 5: $(\alpha, \beta) = (8, 4)$.

$K = \{(v_1^1, v_2^1, v_3^1, v_4^1), (v_1^2, v_2^2, v_3^2, v_4^2), (v_1^3, v_2^3, v_3^3, v_4^3), (v_1^4, v_2^4, v_3^4, v_4^4), (v_1^5, v_2^5, v_3^5, v_4^5), (v_1^6, v_2^6, v_3^6, v_4^6), (v_1^7, v_2^7, v_3^7, v_4^7), (v_1^8, v_2^8, v_3^8, v_4^8)\}$ and $S = \{(v_1^9, v_2^9, v_3^9, v_4^9), (v_1^{10}, v_2^{10}, v_3^{10}, v_4^{10}), (v_1^{11}, v_2^{11}, v_3^{11}, v_4^{11}), (v_1^{12}, v_2^{12}, v_3^{12}, v_4^{12})\}$.

Case 6: $(\alpha, \beta) = (7, 5)$.

$K = \{(v_1^1, v_2^1, v_3^1, v_4^1), (v_1^2, v_2^2, v_3^2, v_4^2), (v_1^3, v_2^3, v_3^3, v_4^3), (v_1^4, v_2^4, v_3^4, v_4^4), (v_1^5, v_2^5, v_3^5, v_4^5), (v_1^6, v_2^6, v_3^6, v_4^6), (v_1^7, v_2^7, v_3^7, v_4^7)\}$ and $S = \{(v_1^8, v_2^8, v_3^8, v_4^8), (v_1^9, v_2^9, v_3^9, v_4^9), (v_1^{10}, v_2^{10}, v_3^{10}, v_4^{10}), (v_1^{11}, v_2^{11}, v_3^{11}, v_4^{11}), (v_1^{12}, v_2^{12}, v_3^{12}, v_4^{12})\}$.

Case 7: $(\alpha, \beta) = (6, 6)$.

$K = \{(v_1^1, v_2^1, v_3^1, v_4^1), (v_1^2, v_2^2, v_3^2, v_4^2), (v_1^3, v_2^3, v_3^3, v_4^3), (v_1^4, v_2^4, v_3^4, v_4^4), (v_1^5, v_2^5, v_3^5, v_4^5), (v_1^6, v_2^6, v_3^6, v_4^6)\}$ and $S = \{(v_1^7, v_2^7, v_3^7, v_4^7), (v_1^8, v_2^8, v_3^8, v_4^8), (v_1^9, v_2^9, v_3^9, v_4^9), (v_1^{10}, v_2^{10}, v_3^{10}, v_4^{10}), (v_1^{11}, v_2^{11}, v_3^{11}, v_4^{11}), (v_1^{12}, v_2^{12}, v_3^{12}, v_4^{12})\}$.

Case 8: $(\alpha, \beta) = (5, 7)$.

$K = \{(v_1^1, v_2^1, v_3^1, v_4^1), (v_1^2, v_2^2, v_3^2, v_4^2), (v_1^3, v_2^3, v_3^3, v_4^3), (v_1^4, v_2^4, v_3^4, v_4^4), (v_1^5, v_2^5, v_3^5, v_4^5)\}$ and $S = \{(v_1^6, v_2^6, v_3^6, v_4^6), (v_1^7, v_2^7, v_3^7, v_4^7), (v_1^8, v_2^8, v_3^8, v_4^8), (v_1^9, v_2^9, v_3^9, v_4^9), (v_1^{10}, v_2^{10}, v_3^{10}, v_4^{10}), (v_1^{11}, v_2^{11}, v_3^{11}, v_4^{11}), (v_1^{12}, v_2^{12}, v_3^{12}, v_4^{12})\}$.

Case 9: $(\alpha, \beta) = (4, 8)$.

$K = \{(v_1^1, v_2^1, v_3^1, v_4^1), (v_1^2, v_2^2, v_3^2, v_4^2), (v_1^3, v_2^3, v_3^3, v_4^3), (v_1^4, v_2^4, v_3^4, v_4^4)\}$ and $S = \{(v_1^5, v_2^5, v_3^5, v_4^5), (v_1^6, v_2^6, v_3^6, v_4^6), (v_1^7, v_2^7, v_3^7, v_4^7), (v_1^8, v_2^8, v_3^8, v_4^8), (v_1^9, v_2^9, v_3^9, v_4^9), (v_1^{10}, v_2^{10}, v_3^{10}, v_4^{10}), (v_1^{11}, v_2^{11}, v_3^{11}, v_4^{11}), (v_1^{12}, v_2^{12}, v_3^{12}, v_4^{12})\}$.

Case 10: $(\alpha, \beta) = (3, 9)$.

$K = \{(v_1^1, v_2^1, v_3^1, v_4^1), (v_1^2, v_2^2, v_3^2, v_4^2), (v_1^3, v_2^3, v_3^3, v_4^3)\}$ and $S = \{(v_1^4, v_2^4, v_3^4, v_4^4), (v_1^5, v_2^5, v_3^5, v_4^5), (v_1^6, v_2^6, v_3^6, v_4^6), (v_1^7, v_2^7, v_3^7, v_4^7), (v_1^8, v_2^8, v_3^8, v_4^8), (v_1^9, v_2^9, v_3^9, v_4^9), (v_1^{10}, v_2^{10}, v_3^{10}, v_4^{10}), (v_1^{11}, v_2^{11}, v_3^{11}, v_4^{11}), (v_1^{12}, v_2^{12}, v_3^{12}, v_4^{12})\}$.

$$\text{K}=\{(v_1^1, v_3^2, v_2^1, v_3^3), (v_1^1, v_3^1, v_2^2, v_3^4)\} \text{ and } \text{S}=\{(v_1^2, v_1^3, v_1^4, v_2^1, v_2^2, v_2^3, v_2^4), (v_3^1, v_3^4, v_1^2, v_1^3, v_1^4, v_2^1), \\ (v_2^3, v_3^3, v_1^2, v_1^3, v_1^4, v_2^2), (v_2^3, v_2^4, v_1^3, v_2^3, v_3^3, v_3^4), (v_1^1, v_2^3, v_2^4, v_3^3, v_3^4)\}.$$
$$K = \{(v_2^1, v_3^2, v_1^1, v_3^3)\} \text{ and } S = \{(v_1^1, v_3^3, v_2^2, v_2^3, v_4^3), (v_2^2, v_3^1, v_4^1; v_2^1, v_2^2, v_3^2, v_4^2), (v_2^2, v_3^2, v_4^2; v_3^1, v_2^3, v_3^3, v_4^3), (v_3^1, v_3^3, v_4^3; v_2^1, v_3^1, v_4^1, v_2^1), (v_3^2; v_2^1, v_3^1, v_4^1, v_4^2)\}.$$
$$S = \{(v_1^1, v_1^2, v_1^3, v_1^4; v_2^1, v_2^2, v_2^3, v_2^4), (v_1^1, v_1^2, v_1^3, v_1^4; v_3^1, v_3^2, v_3^3, v_3^4), (v_2^1, v_2^2, v_2^3, v_2^4; v_3^1, v_3^2, v_3^3, v_3^4)\}. \quad \square$$

For other values of m and n , the necessary condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition in $K_m \otimes \overline{K_n}$ will not be satisfied.

Proof. Case 1. For $m = 4$, we can write, $K_4 \otimes \overline{K_4} = K_3 \otimes \overline{K_4} \oplus K_{4,12}$. By Lemma 3.2, the graph $K_3 \otimes \overline{K_4}$ admits a $\{K^\alpha, S_4^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 12$. By Theorem 1.4, the graph $K_{4,12}$ admits an S_4 -decomposition. Then the graph $K_4 \otimes \overline{K_4}$ admits a $\{K^\alpha, S_4^\beta\}$ -decomposition for some of the admissible pairs (α, β) such that $\alpha + \beta = 24$. Hence in this case we have, $0 \leq \alpha \leq 12$ and $12 \leq \beta \leq 24$ such that $\alpha + \beta = 24$.

Lemma 3.4. *For all $m \geq 3$ and $n > 4$ is even, there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_m \otimes \overline{K_n}$.*

We can write, $K_m \otimes \overline{K_n} = q(K_m \otimes \overline{K_4}) \oplus \frac{q(q-1)}{2}m(m-1)K_{4,4}$. By Lemmas 3.2 and 3.3, the graph $K_m \otimes \overline{K_4}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) . By Theorem 1.4, the graph $K_{4,4}$ admits an S_4 -decomposition and hence the graph $K_m \otimes \overline{K_n}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) .

We can write, $K_m \otimes \overline{K_n} = q(K_m \otimes \overline{K_4}) \oplus \frac{q(q-1)}{2}m(m-1)K_{4,4} \oplus 2K_m \oplus K_{m,m} - I \oplus$

$2qm(m-1)K_{1,4}$. By Lemmas 3.2 and 3.3, the graph $K_m \otimes \overline{K_4}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) . By Theorem 1.3, the graph K_m admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) . By Theorem 1.4, the graphs $K_{4,4}$ and $K_{1,4}$ admits an S_4 -decomposition. By Observation 1.1, the graph $K_{m,m} - I$ admits an S_4 -decomposition. Hence the graph $K_m \otimes \overline{K_n}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some admissible pairs (α, β) . $\square$

Lemma 3.5. *If $m \equiv 0$ or $1 \pmod{8}$ and $n \equiv 1 \pmod{2}$, then there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition of $K_m \otimes \overline{K_n}$ for some admissible pairs (α, β) .*

Proof. We can write, $K_m \otimes \overline{K_n} = nK_n \oplus (K_m \times K_n)$. By Theorem 1.3 and 2.15, the graphs K_m and $K_m \times K_n$ have the desired decomposition for some admissible pairs (α, β) and hence the proof follows. $\square$

Theorem 3.6. *Let $m \geq 3$ and $n \geq 4$ be the given integers. Then the graph $K_m \otimes \overline{K_n}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition for some of the admissible pairs (α, β) , if $m(m-1)n^2 \equiv 0 \pmod{8}$.*

Proof. The proof follows from Lemmas 3.4 and 3.5. $\square$

4. CONCLUDING REMARKS

In this paper, it is proved that there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition in $K_m \times K_n$ for some of the admissible pairs (α, β) , whenever $mn(m-1)(n-1) \equiv 0 \pmod{8}$, for $m \geq 3$ and $n \geq 4$. Also, it is proved that there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$ -decomposition in $K_m \otimes \overline{K_n}$ for some admissible pairs (α, β) , whenever $m(m-1)n^2 \equiv 0 \pmod{8}$, for $m \geq 3$ and $n \geq 4$.

REFERENCES

- [1] A. A. Abueida and M. Daven, "Multi-designs for graph pairs of order 4 and 5," *Graphs Combin.*, vol. 19, no. 4, pp. 433 – 447, 2003. <https://doi.org/10.1007/s00373-003-0530-3>.
- [2] A. A. Abueida and T. O'Neil, "Multi-decomposition of λK_m into small cycles and claws," *Bull. Inst. Comb. Appl.*, vol. 49, pp. 32 – 40, 2007.
- [3] Y. Gao and D. Roberts, "Multi-designs for the graph pair formed by the 6-cycle and 3-prism," *Electron. J. Graph Theory Appl.*, vol. 8, no. 1, p. 133 – 143, 2020. <https://www.ejgta.org/index.php/ejgta/article/view/896>.
- [4] A. P. Ezhilarasi, M. Ilayaraja, and A. Muthusamy, "Decomposition of the tensor product of complete graphs into cycles and stars with four edges," *TWMS J. App. and Eng. Math.*, vol. 13, no. 2, pp. 626 – 634, 2023. <https://belgelik.isikun.edu.tr/xmlui/handle/iubelgelik/5488>.
- [5] A. P. Ezhilarasi and A. Muthusamy, "Decomposition of the product graphs into paths and stars with three edges," *Bull. Inst. Combin. Appl.*, vol. 87, pp. 47 – 74, 2019. <https://bica.the-ica.org/Volumes/87/Reprints/BICA2018-18-Main-Reprint.pdf>.

- [6] S. Jeevadoss and A. Muthusamy, "Decomposition of the product graphs into paths and cycles of length four," *Graphs Combin.*, vol. 32, pp. 199 – 223, 2016. <https://doi.org/10.1007/s00373-015-1564-z>.
- [7] M. Ilayaraja, K. Sowndhariya, and A. Muthusamy, "Decomposition of the product graphs into paths and stars on five vertices," *AKCE Int. J. Graphs and Combin.*, vol. 17, pp. 777 – 783, 2020. <https://doi.org/10.1016/j.akcej.2019.09.007>.
- [8] P. Hemalatha and V. Jothimani, "Multi-decomposition of complete graphs into kites and stars of size four," *Submitted to South East Asian Journal of Mathematics and Mathematical Sciences*, 2023.
- [9] V. Jothimani and P. Hemalatha, "Multi-decomposition of cartesian product of complete graphs into kites and stars of size four," *Indian J. Discrete Math.*, vol. 9, no. 2, p. 87 – 99, 2023.
- [10] S. Yamamoto, H. Ikeda, S. Shige-Eda, K. Ushio, and N. Hamada, "On claw decomposition of complete graphs and complete bipartite graphs," *Hiroshima Math J.*, vol. 5, no. 1, pp. 33 – 42, 1975. <https://projecteuclid.org/journals/hiroshima-mathematical-journal/volume-5/issue-1/On-claw-decomposition-of-complete-graphs-and-complete-bigraphs/10.32917/hmj/1206136782.pdf>.