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On K-AP And K-Avoid-AP Van Der Waerden Game

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Abstract. Let $w(k;2)$ be the van der Waerden number such that for every 2-colouring of $[1, w(k;2)]$ there is a monochromatic k-term arithmetic progression (AP). Consider the following two 2-players games: k-AP game and k-AVOID-AP game. These are two different games between two players, Player 1 and Player 2, on a sequence of integers $[1, n]$ where $n \in \mathbb{Z}^+$. Each player's aim is to obtain or avoid forming a monochromatic k-term arithmetic progression. The player who first obtains a monochromatic k-term arithmetic progression wins or loses, thus ending the game. In this paper, we investigate these two games and propose a new parameter: the minimum number of turns $\hat{w}_n(k)$ (and $\tilde{w}_n(k)$) needed for any of the player to win in k-AP (and k-AVOID-AP game respectively). We propose the winning strategies for Player 1 and Player 2 and hence show that $\hat{w}_n(3) = 5, \hat{w}_n(4) = 7$ and $\tilde{w}_n(3) = 9$. We also have shown that in a k-AVOID-AP game on $[1, n]$, where n is sufficiently large, Player 2 always has a winning strategy if n is even.

Key words and Phrases: van der Waerden number, k-term arithmetic progression, k-AP game, k-AVOID-AP game

1. Introduction

For an arithmetic progression $A = \{a + ld : 0 \leq l \leq k-1\}$, we say that A is a *k-term arithmetic progression (AP) with difference d*, or we write $\{a + ld\}_{0 \leq l \leq k-1}$. For a positive integer t, we denote the set $\{1, 2, 3, \ldots, t\}$ by $[1, t]$. An *r-colouring* of a set S is a surjective function $\chi : S \to [1, r]$. A *monochromatic k-term arithmetic progression* refers to a k-term arithmetic progression such that all of its elements

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are of the same colour. One of the most fundamental Ramsey-type theorem is the van der Waerden theorem. The van der Waerden's theorem [1] states that there exists a least positive integer $w = w(k;r)$ such that for any $n \geq w$, every r-colouring of $[1, n]$ admits a monochromatic k-term arithmetic progression. For more results on van der Waerden numbers, see [2, 3, 4, 5, 6, 7].

Theorem 1.1. [1] *(van der Waerden's Theorem) Let $k, r \geq 2$ be integers. There exists a least positive integer $w = w(k;r)$ such that for any $n \geq w$, every r-colouring of $[1, n]$ admits a monochromatic k-term arithmetic progression.*

Note that exact values of $w(k;2)$ are only known for small value of k where $w(3;2) = 9, w(4;2) = 35, w(5;2) = 178$ and $w(6;2) = 1132$, see [5]. The exact value of $w(k;r)$ is not known in general. We refer the readers to [4, 3, 8, 9, 6, 7].

There are many variants of Ramsey games that have been investigated, see [10, 11, 12, 13, 14, 15, 16, 17, 18]. Consider one of the games here, which is inspired by Ramsey Theory: Player 1 (blue) and Player 2 (red) alternate colouring edges of a complete graph K_m with their colours. Let H be a subgraph of K_m. The first player to avoid a monochromatic H in his colour wins. This motivates us to investigate a similar van der Waerden game as follows.

A k-AP van der Waerden game (or k-AP game) is a game between two players, Player 1 and Player 2, on the set of integers $[1, n]$. Initially, all the integers $1, 2, \ldots, n$ are uncoloured. Player 1 will first select an integer and colour it with red. Player 2 will then select another integer and colour it with blue. The two players continue this alternating colouring process on uncoloured integers. Each player's aim is to form a monochromatic k-term arithmetic progression as soon as possible. The player who first obtains a monochromatic k-term arithmetic progression wins, thus ending the game. We assume all players play optimally where both players never purposefully avoid a monochromatic k-term arithmetic progression and at the same time, they would like to prevent the opponent from forming a monochromatic k-term arithmetic progression.

A k-AVOID-AP van der Waerden game (or k-AVOID-AP game) is a game opposite to that of the k-AP van der waerden game where both players' aims are to avoid forming a monochromatic k-term arithmetic progression. The player who first obtains a monochromatic k-term arithmetic progression loses, thus ending the game. We assume all players play optimally where both players never purposefully form a monochromatic k-term arithmetic progression and at the same time, they would like to "force" a monochromatic k-term arithmetic progression for the opponent.

We begin by explaining some notation. A *turn* of the game is an action of colouring by either Player 1 or Player 2. If they are playing on the set of integers $[1, n]$ and $n \geq w(k;2)$, then by Theorem 1.1, one of the players will win in k-AP game when all the integers are coloured. Let T_i be the ith turn. Player 1 makes his move on the odd-numbered turns, whereas Player 2 makes his move on the even-numbered turns. Let I_i denote the status of the integer i where $i \in [1, n]$. Specifically, $I_i = 1$ means that the number i has been coloured by Player 1 with red and $I_i = 2$ means that the number i has been coloured by Player 2 with blue. A player is said to have a winning strategy if he will win the game by choosing certain integers at each of his turn regardless of what integer the other player chooses at the other player's turn.

In a k-AP game on the set of integers $[1, n]$, let $\hat{w}_n(k)$ to be the minimum number of turns for any of the players to win the game (i.e. the minimum number of turns to form a k-term arithmetic progression). This means that if $\hat{w}_n(k)$ is odd, then Player 1 wins and if $\hat{w}_n(k)$ is even, then Player 2 wins. In a k-AVOID-AP game on the set of integers $[1, n]$, let $\tilde{w}_n(k)$ to be the minimum number of turns for any of the players to win the game (i.e. the minimum number of turns for the opponent to form a k-term arithmetic progression). So, if $\tilde{w}_n(k)$ is odd, then Player 1 loses and if $\tilde{w}_n(k)$ is even, then Player 2 loses. Clearly, if $n = w(k; 2)$, then $\hat{w}_n(k) \le w(k; 2)$ and $\tilde{w}_n(k) \le w(k; 2)$.

2. k-term arithmetic progression in a k-AP game

In a k-AP game, there are three possibilities:

(1) Player 1 has a winning strategy.
(2) Player 2 has a winning strategy.
(3) Both players have no winning strategy.

First, we show that only (1) and (3) are possible.

Theorem 2.1. *In a k-AP game on the set of integers $[1, n]$, either Player 1 has a winning strategy or both players have no winning strategy.*

Proof. Suppose Player 2 has a winning strategy δ_n on $[1, n]$. At T_1, Player 1 colours 1 with red. By using the strategy δ_n, Player 2 will need to colour $b_1 \in [2, n]$ at T_2 with blue. Now, Player 1 shall assume that he had not coloured 1 with red at T_1, and Player 2 has coloured b_1 at turn T_1' with blue. Basically, Player 1 assume himself as the second player and assume Player 2 as the first player. Using the strategy δ_n, Player 1 will need to colour $a_1 \in [1, n] \setminus \{b_1\}$ at T_2' with red. If $a_1 = 1$, then Player 1 simply chooses another uncoloured integer, say a_1', and colours it with red. If $a_1 \neq 1$, then Player 1 will colour a_1 with red.

At T_4, Player 2 colours $b_2 \in [2, n] \setminus \{a_1\}$ with blue according to the strategy δ_n. At T_4', Player 1 will need to colour $a_2 \in [1, n] \setminus \{b_1, a_1, b_2\}$ with red. We consider two cases:

Case 1: Suppose $a_1 \neq 1$. If $a_2 = 1$, then Player 1 simply choose another uncoloured integer, say a_2' and colour it with red. If $a_2 \neq 1$, then Player 1 will colour a_2 with red.

Case 2: Suppose $a_1 = 1$. Then both a_1' and 1 are already coloured by red. In this scenario, if $a_2 = a_1'$, then Player 1 simply choose another uncoloured integer, say a_3' and colour it with red. If $a_2 \neq a_1'$, then Player 1 will colour a_2 with red. The two players will continue this process until one of them wins.

Since Player 2 is using the strategy δ_n, at T_{2m}, Player 2 would have coloured m distinct integers $b_1, b_2, \ldots, b_m$ and there is a k-term arithmetic progression among them. At T_{2m-1}, Player 1 have coloured m distinct integers $1, a_1, a_2, \ldots, a_{m-1}$. Note that there should not be any k-term arithmetic progressions in $\{1, a_1, b_2, \ldots, a_{m-1}\}$.

On the other hand, Player 1 is using the same strategy δ_n to win. So, at T'_{2m-2}, Player 1 have coloured $m-1$ distinct integers $a_1, a_2, \ldots, a_{m-1}$. Since there is no k-term arithmetic progressions in $\{a_1, a_2, \ldots, a_{m-1}\}$, at T'_{2m-1}, Player 2 could not have won. But at T'_{2m-1}, Player 2 have coloured m distinct integers $b_1, b_2, \ldots, b_m$ and there is a k-term arithmetic progression in $\{b_1, b_2, \ldots, b_m\}$, a contradiction. Hence, Player 2 does not have a winning strategy and the theorem follows. □

We will show that in a k-AP game where $k = 3, 4$, Player 1 will have a winning strategy. Furthermore, we show that $\hat{w}_n(3) = 5$ in $[1, n]$ where $n = w(k; 2) = 9$ and $\hat{w}_n(4) = 7$ in $[1, n]$ where $n = 25 < w(4; 2) = 35$.

Lemma 2.2. *In a k-AP game on the set of integers* $[1, n]$*,* $\hat{w}_n(k) \geq 2k - 1$.

Proof. In order to form a monochromatic k-term arithmetic progression in $[1, n]$, the winner has to choose at least k integers. If Player 2 wins the game, then Player 1 must have chosen at least k integers; hence $\hat{w}_n(k) \geq k + k = 2k$. If Player 1 wins the game, then Player 2 has to choose $k - 1$ integers and hence $\hat{w}_n(k) \geq k + k - 1 = 2k - 1$. The result follows. □

Lemma 2.3. *Suppose there is a sequence of increasing integers*

$$t_0 \quad t_1 \quad t_2 \quad \underline{t_3} \quad t_4 \quad \underline{t_5} \quad t_6 \quad \underline{t_7} \quad t_8,$$

where $t_{i+1} - t_i = t_{j+1} - t_j$ *for all* i, j*. If at turn* T_{2i} *(Player 2's turn),* t_4 *and* t_6 *have been coloured with red and* t_i *has not been coloured for all* $i \notin \{1, 4, 6\}$*, then regardless of Player 2's choices at* T_{2i} *or* T_{2i+2}*, Player 1 will obtain a* 4*-term arithmetic progression with red at* T_{2i+3}*.*

Proof. Assume that at T_{2i}, Player 2 colours b_i with blue. Suppose $b_i \in \{t_3, t_5, t_7\}$. Then at T_{2i+1}, Player 1 colours t_2 with red. Now, t_0, t_2, t_4, t_6 and t_2, t_4, t_6, t_8 are two 4-term arithmetic progressions and all of these integers are coloured with red except t_0 and t_8, which are not coloured. So, regardless of any choice Player 2 makes at T_{2i+2}, Player 1 will obtain a 4-term arithmetic progression with red at T_{2i+3} by colouring t_0 or t_8 with red.

Suppose $b_i \notin \{t_3, t_5, t_7\}$. Then at T_{2i+1}, Player 1 colours t_5 with red. Now, t_3, t_4, t_5, t_6 and t_4, t_5, t_6, t_7 are two 4-term arithmetic progressions and all of these integers are coloured with red except t_3 and t_7, which are not coloured. So, regardless of any choice Player 2 makes at T_{2i+2}, Player 1 will obtain a 4-term arithmetic progression with red at T_{2i+3} by colouring t_3 or t_7 with red. □

Theorem 2.4. *Consider a k-AP game on* $[1, n]$.

(a) *If* $k = 3$ *and* $n \geq 5$*, then Player 1 has a winning strategy. Furthermore,* $\hat{w}_n(3) = 5$.

(b) *If* $k = 4$ *and* $n \geq 25$*, then Player 1 has a winning strategy. Furthermore,* $\hat{w}_n(4) = 7$*.*

Proof. (a) By Lemma 2.2, it is clear that $\hat{w}_n(3) \geq 5$. At T_1, Player 1 colours 3 with red. Suppose Player 2 colours 1 with blue T_2. At T_3, Player 1 colours 4 with red. A 3-term arithmetic progression with red colour is formed if 2 or 5 is coloured by Player 1 at T_5. So, regardless of which integer is coloured by Player 2 at T_4, Player 1 will win.

Suppose Player 2 colours 2 with blue at T_2. At T_3, Player 1 colours 5 with red. A 3-term arithmetic progression with red colour is formed if 1 or 4 is coloured by Player 1 at T_5. So, regardless of which integer is coloured by Player 2 at T_4, Player 1 will also win.

Suppose Player 2 colours an integer $a \geq 4$ with blue at T_2. Player 1 colours 1 with red if $a = 4$ and colours 2 with red if $a \neq 4$. In either case, it is not hard to see that Player 1 will win at T_5 regardless of any choice Player 2 makes at T_4.

(b) By Lemma 2.2, it is clear that $\hat{w}_n(4) \geq 7$. To show that the equality holds, we shall show that Player 1 has a winning strategy and will win at T_7, regardless of the choices made by Player 2 at T_2, T_4 and T_6. At T_1, Player 1 colours 13 with red. At T_2, Player 2 colours b_1 with blue. Now, we look at the following two different sequences:

$$1 \quad 7 \quad \underline{10} \quad 13 \quad \underline{16} \quad 19 \quad \underline{22} \quad 25$$

$$5 \quad 9 \quad \underline{11} \quad 13 \quad \underline{15} \quad 17 \quad \underline{19} \quad 21.$$

Suppose $b_1 \in \{1, 7, 10, 16, 22, 25\}$. Then $b_1 \notin \{5, 9, 11, 15, 17, 19, 21\}$. At T_3, Player 1 colours 17 with red. At T_4, in the following sequence of integers

$$5 \quad 7 \quad 9 \quad \underline{11} \quad 13 \quad \underline{15} \quad 17 \quad \underline{19} \quad 21,$$

we see that integers 13 and 17 are coloured with red and the integers $5, 9, 11, 15, 19, 21$ are not coloured. By Lemma 2.3, Player 1 will obtain a 4-term arithmetic progression with red at T_7.

Suppose $b_1 = 19$. Then at T_3, Player 1 colours 9 with red. At T_4, in the following sequence of integers

$$1 \quad 3 \quad 5 \quad \underline{7} \quad 9 \quad \underline{11} \quad 13 \quad \underline{15} \quad 17,$$

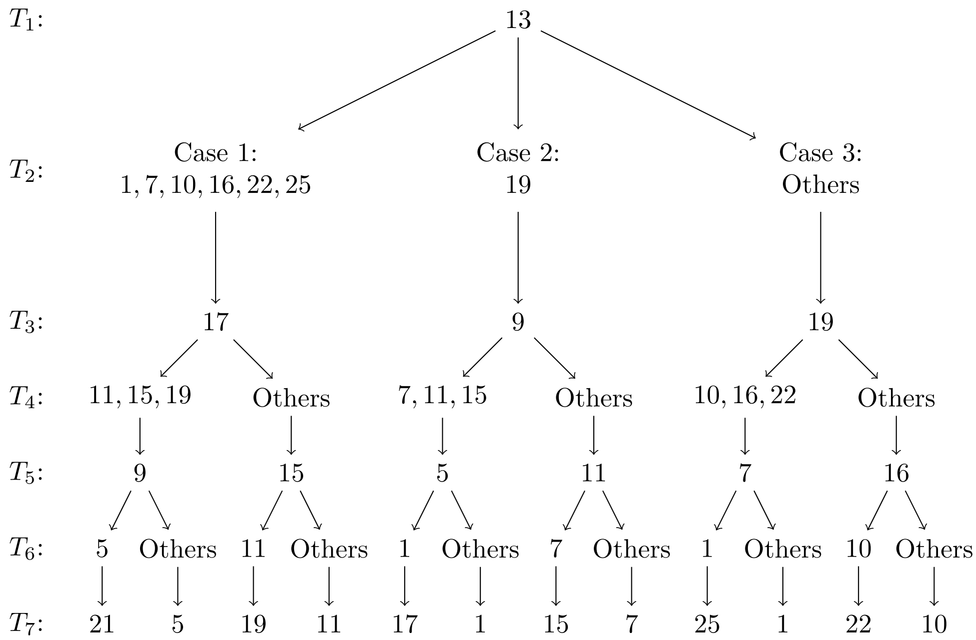
we see that integers 9 and 13 are coloured with red and the integers $1, 5, 7, 11, 15, 17$ are not coloured. By Lemma 2.3, Player 1 will obtain a 4-term arithmetic progression with red at T_7.

Suppose $b_1 \notin \{1, 7, 10, 16, 19, 22, 25\}$. Then at T_3, Player 1 colours 19 with red. At T_4, in the following sequence of integers

$$1 \quad 4 \quad 7 \quad \underline{10} \quad 13 \quad \underline{16} \quad 19 \quad \underline{22} \quad 25,$$

we see that integers 13 and 19 are coloured with red and the integers $1, 7, 10, 16, 22, 25$ are not coloured. By Lemma 2.3, Player 1 will obtain a 4-term arithmetic progression with red at T_7.

To better describe Player 1's strategy, we use the following graph for better illustrations.



□

3. k-term arithmetic progression in k-AVOID-AP game

In this section, we shall consider k-AVOID-AP games.

Theorem 3.1. *Let $n \geq w(k;2)$ and set $N = n$ if n is even, otherwise set $N = n+1$. Then, in a k-AVOID-AP game on $[1, N]$, Player 2 always has a winning strategy.*

Proof. Note that N is always even. We can illustrate the interval $[1, N]$ as an array below:

$$\begin{array}{cccccc} 1 & 2 & 3 & \dots & \frac{N}{2}-1 & \frac{N}{2} \\ N & N-1 & N-2 & \dots & \frac{N}{2}+2 & \frac{N}{2}+1 \end{array} \tag{1}$$

Consider any turn T_i where i is odd. If Player 1 chooses an integer $a_i \in [1, N]$ at T_i, then Player 2 is able to choose the integer $N - a_i + 1$ on the following turn T_{i+1}. Referring to our array (1) above, this means that Player 2 is choosing the integer that is mirrored by the separation between the first and second rows. For example, if Player 1 chooses 3, then Player 2 chooses $N-2$, which is directly below 3 in array (1). The same is true if Player 1 chooses a number in the second row of the array; Player 2 will choose the number directly above it. Hence, Player 2 will always have an available 'mirrored' number to choose from after Player 1's turns. As long as Player 1 does not obtain a k-term arithmetic progression, Player 2 will also not obtain a k-term arithmetic progression. By van der Waerden's theorem, because $N \geq n \geq w(k;2)$, an arithmetic progression will eventually be formed. Since Player 1 always makes his colouring choices a turn earlier, Player 1 will obtain a monochromatic k-term arithmetic progression before Player 2 does, thus losing the game. Hence, Player 2 will always win. □

Corollary 3.2. *In a k-AVOID-AP game on $[1,n]$, if $n \geq w(k;2)$ and n is even, then $\tilde{w}_n(k) \leq n-1$.*

Proof. By Theorem 3.1, if n is even, then it is clear that Player 2 will win. Furthermore, even with an optimum strategy used by Player 1, Player 2 may keep avoiding a monochromatic k-term arithmetic progression in his colour until his second last turn. Hence Player 1 will form a k-term arithmetic progression in his colour in his last turn before Player 2 chooses the last integer. Hence, $\tilde{w}_n(k) \leq n-1$. □

Lemma 3.3. *If Player 2 has a strategy to avoid k-term arithmetic progressions on $[1,n]$, then Player 1 has a strategy to avoid $(k+1)$-term arithmetic progressions on $[1,n+1]$. Hence, $\tilde{w}_{n+1}(k+1) \leq \tilde{w}_n(k)+1$.*

Proof. Suppose Player 2 has a strategy δ to avoid k-term arithmetic progressions on $[1,n]$. Then, in $[1,n+1]$, Player 1 colours integer $n+1$ at T_1 with red, then in the rest of his turns, Player 1 can use strategy δ to avoid monochromatic k-term arithmetic progressions on $[1,n]$. As a result, Player 1 may avoid $k+1$-term arithmetic progressions in the game even with the integer $n+1$ coloured by him. □

In the next subsection, we will show that when $n = w(3;2) = 9$, Player 2 still wins (Corollary 3.7).

3.1. **3-AVOID-AP game on $[1,9]$.**

Lemma 3.4. *In a 3-AVOID-AP game on $[1,9]$, Player 1 has a strategy to avoid defeat (or avoid forming a monochromatic 3-term arithmetic progression) until T_7.*

Proof. Let $C_i = \{2i-1, 2i\}$ for $i = 1,2,3$ and $C_4 = \{7,8,9\}$. At T_1, Player 1 will colour 8 with red. If Player 2 colours 9 with blue at T_2, then Player 1 colours 7 with red at T_3. If Player 2 did not colour 9 at T_2, then Player 1 will colour 9 with red at T_3. Hence, after T_3, we have the following two cases.

Case 1. $(I_7, I_8, I_9) = (1,1,2)$. Player 2 colours an integer at T_4. By the pigeonhole principle, at T_5, we let Player 1 colour 1, or otherwise colour 2 if 1 has been coloured by Player 2. If Player 1 colours 1 at T_5, then at T_7, Player 1 can colour 2, 3 or 5. Any of these choices will not form a red monochromatic 3-term arithmetic progression. If Player 1 colours 2 at T_5, then after T_5, $(I_1, I_2, I_7, I_8, I_9) = (2,1,1,1,2)$. Hence, Player 1 can colour 3 or 4 at T_7 without forming a red monochromatic 3-term arithmetic progression.

Case 2. $(I_8, I_9) = (1, 1)$. Suppose at T_2 and T_4, Player 2 colours integers in C_i and C_j respectively (i may be equal to j). After T_4, integers in C_k are not coloured for some $k = 1$ or 2. At T_5, Player 1 colours an uncoloured integer in C_k with red. Now, C_k has two integers and one of them is coloured with red by Player 1 at T_5. If the other integer has not been coloured by Player 2 at T_6, then Player 1 colours the uncoloured integer in C_k at T_7. We are done. If Player 2 colours the integer in C_k with blue at T_6, then at T_7, Player 1 colours 5 or 6 (in which case does not result in a red monochromatic 3-term arithmetic progression) if both of these integers are uncoloured yet, or else colours any other uncoloured integers in C_1 or C_2.

For all cases, Player 1 does not form a monochromatic 3-term arithmetic progression until T_7. This completes the proof. □

Before we move to the next result, we introduce the parameter λ which refers to the number of possible moves available to Player 2 to avoid a monochromatic 3-term arithmetic progression after T_6. Below we give an example:

$$2\ 1\ 1\ \_\ \_\ 2\ \_\ 1\ 2$$

The above represents the state of a 3-AVOID-AP game on $[1, 9]$ after Player 2 colours a number during T_6. Instead of writing the numbers $1, \ldots, 9$ for the interval, we instead represent the coloured numbers with the numbers 1 and 2 to represent each Players' choices so far, and the uncoloured numbers are left as blank spaces. The above represents the state where $I_2 = I_3 = I_8 = 1$ and $I_1 = I_6 = I_9 = 2$. From here, we see that Player 2 has two winning moves for the upcoming T_8, by colouring either 4 or 7. In other words, either $I_4 = 2$ or $I_7 = 2$ at T_8 would guarantee a win for P_2 because neither of these choices lead to an arithmetic progression in blue. Hence, $\lambda = 2$.

Note that we only calculate the value of λ after P_2's choice at the end of T_6. Meaning that in order for Player 2 to win, λ must be of value at least 2. If $\lambda = 1$, then Player 2 can only avoid blue 3-term arithmetic progression by colouring a particular integer i at T_8. However, this may lead to the scenario where Player 1 colours i at T_7 and thus forcing Player 2 to lose (forming a blue 3-term arithmetic progression at T_8).

Lemma 3.5. *If $\lambda \geq 2$, Player 2 wins.*

Proof. The parameter $\lambda \geq 2$ implies that there exist integers $i_1, \ldots, i_\lambda$ such that when coloured by Player 2 at T_8, he can always avoid a monochromatic arithmetic progression and thus leads to the defeat of Player 1, i.e., Player 1 will form a red monochromatic 3-term arithmetic progression (because $w(3; 2) = 9$). Given that Player 1 can only colour a single integer at T_7, there is still at least one uncoloured integer amongst $i_1, \ldots, i_\lambda$ for P_2 to choose at T_8 to remain undefeated. □

We are now well-equipped to prove our main result.

Theorem 3.6. *In a* 3*-AVOID-AP game on* $[1, 9]$*, Player 2 always has a winning strategy.*

Proof. Suppose Player 1 colour integers with red and Player 2 colour integers with blue. In each of the cases below, if Player 2 did not form a blue 3-term arithmetic progression after his turn in T_8, then by van der Waerden theorem where $w(3; 2) = 9$, P_1 will definitely form a red 3-term arithmetic progression after T_9 which is the last turn. Hence, to show that P_2 has a winning strategy, we want to show that $\lambda \geq 2$ after T_6, then Player 2 will have at least two possible choices in T_8 and remain undefeated (i.e. not form a blue 3-term arithmetic progression). Here, we investigate the possible choices of the two players in each turn.

Turn 1 (T_1): Player 1 picks an integer and colours it with red.

Turn 2 (T_2): Player 2 colours 1 with blue. If 1 was coloured by Player 1, then Player 2 colours 9 with blue. Now, relabel those integers $i \to 10 - i$ for $1 \leq i \leq 9$. After relabelling, 1 is coloured with blue. Note that the relabelling can be reversed. If a monochromatic 3-term arithmetic progression after relabelling is formed after some turns, then by reversing the relabelling, we still have a monochromatic 3-term arithmetic progression after the reversing process. So, we may assume that 1 is coloured with blue.

Turn 3 (T_3): Player 1 colours his second integer.

Turn 4 (T_4): Before Player 2 colours his second integer, two integers are already coloured by Player 1 with red. We first consider the following cases that depend on the choices of Player 1:

Case 4.1: $I_4 = I_7 = 1$. Player 2 colours 8, and then at T_6, Player 2 colours 6 if it is uncoloured or 9, otherwise. By checking the uncoloured integers after T_6, we see that $\lambda \geq 2$.
Case 4.2: $I_7 = I_8 = 1$. Player 2 colours 4, and then at T_6 colours 9. This gives us $\lambda \geq 2$. Note that at T_6, 9 must be uncoloured otherwise Player 1 loses.
Case 4.3: $I_4 = I_8 = 1$. Player 2 colours 7, and then at T_6 colours 6. This gives us $\lambda \geq 2$. Note that at T_6, 6 must be uncoloured otherwise Player 1 loses.
Case 4.4: $I_3 = I_6 = 1$. Player 2 colours 2, and then at T_6, Player 2 colours 4 if it is uncoloured or 7, otherwise. By checking the uncoloured integers after T_6, we see that $\lambda \geq 2$.
Case 4.5: $I_2 = I_3 = 1$. Player 2 colours 9, then in T_6, Player 2 colours 6 if it is uncoloured or 8 otherwise. In either case, Player 2 colours 4 in T_8.
Case 4.6: $I_2 = I_6 = 1$. Player 2 colours 3, and then at T_6 colours 8 if it is uncoloured or 9, otherwise. This gives us $\lambda \geq 2$.
Case 4.7: $I_2 = I_9 = 1$. Player 2 colours 3, and then at T_6 colours 6 if it is uncoloured or 8, otherwise. This gives us $\lambda \geq 2$.
Case 4.8: $I_3 = I_9 = 1$. Player 2 colours 2, and then at T_6 colours 6. This gives us $\lambda \geq 2$. Note that at T_6, 6 must be uncoloured otherwise Player 1 loses.

Case 4.9: $I_6 = I_9 = 1$. Player 2 colours 8 at T_4. If Player 1 colours 2 at T_5, then Player 2 colours 3 at T_6. If Player 1 colours 4 at T_5, then Player 2 colours 7 at T_6. If Player 1 colours 5 at T_5, then Player 2 colours 3 at T_6. If Player 1 colours 7 at T_5, then Player 2 colours 4 at T_6. In either case, $\lambda \geq 2$.
Case 4.10: $I_4 = I_9 = 1$. Player 2 colours 7, and then at T_6 colours 2 if it is uncoloured or 3, otherwise. This gives us $\lambda \geq 2$.
Case 4.11: $I_7 = I_9 = 1$. Player 2 colours 4, and then at T_6 colours 8. This gives us $\lambda \geq 2$. Note that at T_6, 8 must be uncoloured otherwise Player 1 loses.
Case 4.12: $I_8 = I_9 = 1$. Player 2 colours 6 at T_4. If Player 1 colours 2 at T_5, then Player 2 colours 3 at T_6. If Player 1 colours 3 at T_5, then Player 2 colours 2 at T_6. If Player 1 colours 4 at T_5, then Player 2 colours 7 at T_6. If Player 1 colours 5 at T_5, then Player 2 colours 7 at T_6. In either case, $\lambda \geq 2$.
Case 4.13: $I_5 = I_9 = 1$. Player 2 colours 6 at T_4. If Player 1 picks 2 at T_5, then Player 2 colours 3 at T_6. If Player 1 colours 3 at T_5, then Player 2 colours 2 at T_6. If Player 1 colours 4 at T_5, then Player 2 colours 7 at T_6. If Player 1 colours 8 at T_5, then Player 2 colours 3 at T_6. In either case, $\lambda \geq 2$.

For the remaining cases, we may assume that

(i) any two or all integers in $\{4, 7, 8\}$ remain uncoloured by Player 1 before T_4,
(ii) any two or all integers in $\{2, 3, 6\}$ remain uncoloured by Player 1 before T_4, and
(iii) 9 is uncoloured by Player 1 before T_4.

So, at T_4, Player 2 colours 9 with blue. At T_5, Player 1 picks his third digit. In this scenario, P_2 would have to avoid integer 5 in all the remaining turns (i.e. T_6 and T_8). Before T_6, Player 1 has coloured 3 integers and these three integers cannot be $\{2, 3, 4\}$ or $\{6, 7, 8\}$ or $\{4, 6, 8\}$.

Furthermore, at least one of the integers in $\{4, 7, 8\}$ and at least one of the integers in $\{2, 3, 6\}$ are uncoloured before T_6. Suppose that after T_5, 4,6,7 are coloured with red. Now, at T_6, Player 2 colours 8 with blue. Since 2,3 are still uncoloured after T_6, we have $\lambda \geq 2$. So, we may assume that before T_6, the three integers coloured by Player 1 cannot be $\{2, 3, 4\}$ or $\{6, 7, 8\}$ or $\{4, 6, 8\}$ or $\{4, 6, 7\}$. Now, at T_6, Player 2 colours 2 with blue if it has not been coloured by Player 1.

If two of the integers in $\{4, 7, 8\}$ are uncoloured, then $\lambda \geq 2$ because $(I_1, I_2, I_9) = (2, 2, 2)$ after T_6. If $7, 8$ are coloured with red, then after T_6, 4 and 6 are uncoloured, thus $\lambda \geq 2$. If $4, 8$ are coloured with red, then after T_6, 6 and 7 are uncoloured, thus $\lambda \geq 2$. If $4, 7$ are coloured with red, then after T_6, 6 and 8 are uncoloured, thus $\lambda \geq 2$.

Suppose 2 has been coloured by Player 1 but 3 has not. If two of the integers in $\{4, 7, 8\}$ are uncoloured, then at T_6, Player 2 colours 3 with blue. We have $\lambda \geq 2$ because $(I_1, I_3, I_9) = (2, 2, 2)$ after T_6. Suppose that after T_5, 2,7,8 are coloured with red. Now, at T_6, Player 2 colours 4 with blue. Since 3,6 are still uncoloured after T_6, we have $\lambda \geq 2$. If 2,4,8 are coloured with red after T_5, then at T_6, Player 2 colours 7 with blue. If 2,4,7 are coloured with red after T_5, then at T_6, Player 2 colours 8 with blue. In either case, 3 and 6 are still uncoloured after T_6, so we have $\lambda \geq 2$.

Suppose 2 and 3 have been coloured by Player 1, then this case as in Case 4.5 where Player will have a winning strategy.

This completes the proof.

□

Theorem 3.7. *In a* 3*-AVOID-AP game on* $[1, 9]$*, Player 2 always wins and* $\tilde{w}_9(3) = 9$.

Proof. This follows immediately from Lemma 3.4 and Theorem 3.6. □

4. Conclusions

In this paper, we have found $\hat{w}_n(3) = 5$ and $\hat{w}_n(4) = 7$. We also have shown that in a k-AVOID-AP game on $[1, n]$, where n is sufficiently large, Player 2 always has a winning strategy if n is even. New techniques might need to be developed in order to find the winning strategy for either Player 1 or 2 when n is odd. Besides that, $\hat{w}_n(k)$ for $k \geq 5$ and $\tilde{w}_n(k)$ are also remained unknown. For k-AP game, it will be interesting to determine if there is a strategy for Player 1 to win the game.

We only consider 2-players games in this paper. So, it will be interesting to consider r-players games where each player is given one colour. Hence, this will be an r-colouring of $[1, n]$. Questions that can be asked are whether some of the results in this paper can be extended to r-players games.

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