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On the Non-commuting Graph Associated to a Finite-dimensional Lie Algebra

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Abstract. In this paper, we define the non-commuting graph associated to a Lie algebra L and obtain some basic graph properties such as connectivity, diameter, girth, Hamiltonian and Eulerian. Moreover, planarity, outer planarity and isomorphism between two such graphs are also discussed in the paper.

Key words and Phrases: Non-commuting graph, Lie algebra.

1. INTRODUCTION

Research topics in algebraic structures and graph theory have led to several exciting questions and results. Some papers have been written about assigning a graph to a group or ring and determining its algebraic properties. One can associate a graph to a group G by various ways. For example, zero-divisor graph of a commutative ring, non-commuting graph of a group and relative non-commuting graph of a group are defined(see [1, 2, 3]). One of the associated graphs to a group G is the non-commuting graph, whose vertices are $G \setminus Z(G)$ and two distinct vertices x and y are adjacent if and only if $xy \neq yx$. Some results in group theory may be extended to Lie algebra. These conclusions are not always similar and even they are different in many situations. Recently, Lie algebras have attracted some authors due to wide applications in other sciences, especially physics. Understanding the structure of fundamental particles and the formulation of quantum theories in high-energy physics makes it inevitable for physicists to be familiar with Lie algebras.

In this paper, a graph G_L is associated to a Lie algebra L and the properties of this kind of graph are investigated. Let L be a Lie algebra and $Z(L)$ be its

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center. We defined G_L as follows: Take $L \setminus Z(L)$ the vertices of G_L and join two distinct vertices x and y whenever $[x, y] \neq 0$. We call G_L the non-commuting graph of a Lie algebra L. All graphs are considered undirected and simple (with no loops or multiple edges). For every graph, we denote the set of vertices and the edges of G_L by $V(G_L)$ and $E(G_L)$, respectively. If L is abelian, then G_L is null graph. Our main goal is to detect the properties of a Lie algebra L with its associated graph G_L and vice verse.

In section 2, we study some properties of G_L for a non-abelian Lie algebra. In addition, some results are obtained about connectivity, Hamiltonian, Eulerian properties, the girth and diameter of G_L. In section 3, we determine under what conditions of L the graph G_L is planar and outerplanar. In the last section, we consider isomorphism between two non-commuting graphs and show that under some conditions two Lie algebras have equal elements. In the rest of this section, we recall some concepts of graphs and Lie algebras that are used in the next sections.

Let Γ be an arbitrary graph. The degree of a vertex $v \in V(\Gamma)$ denoted by $\deg(v)$, is the number of edges, which are incident to v. The minimum and maximum degrees are be denoted by $\delta(\Gamma)$ and $\Delta(\Gamma)$, respectively. Also, $|V(\Gamma)|$ is called the size of Γ. If the degree of all vertices Γ is equal to $|V(\Gamma)| - 1$, then Γ is a complete graph. This graph with n vertices is denoted by K_n. A graph Γ is regular if the degrees of all vertices of Γ are the same. A path of length n consists of vertices from v_0 to v_n and a sequence of distinct e_i such that $e_i = \{v_{i-1}, v_i\}$ and $1 \leq i \leq n$. A cycle is a path in a graph that starts and ends at the same vertex. A graph is connected when there is at least a path for every pair of vertices. If v and w are two vertices of Γ, then $d(v, w)$ is the length of the shortest path between v and w. The longest distance between all pairs of vertices is called the diameter of Γ and denoted by $\mathrm{diam}(\Gamma)$. The girth of Γ, $\mathrm{girth}(\Gamma)$, is the length of the shortest cycle in graph Γ. In graph theory, a planar graph is a graph that can be drawn such that none of the edges cross each other. The following theorem is an important tool in proving some theorems related to the planar graphs in section 3.

Theorem 1.1. [4, Corollary 9.5.3] *If Γ is a simple and planar, then $\delta(\Gamma) \leq 5$.*

If a graph can be drawn in the plane without any edge crossings, such that all vertices lie on the boundary of the outer face is called an outerplanar graph. Every outerplanar graph is planar, but the converse of fact is not true. For example, the complete graph with four vertices is the smallest planar graph that is not outerplanar.

Theorem 1.2. [5, Chaper 10] *Every subgraph of an outerplanar contains a vertex with degree at most 2.*

An Eulerian cycle is cycle that visits every edge of a graph exactly once. If graph G has an Eulerian cycle, then it is called Eulerian. The next theorem states a condition equivalent to an Eulerian graph.

Theorem 1.3. [4, Theorem 4.1] *A connected non-empty graph is Eulerian if and only if it has no vertices of odd degree.*

A Hamiltonian cycle is a cycle that visits each vertex exactly once. Every graph that has a Hamiltonian cycle is called Hamiltonian graph. The following theorem is an important equipment for detecting Hamiltonian graphs.

Theorem 1.4. *(Dirac's Theorem)*[4, Theorem 4.3] *Let Γ be a simple graph of order n such that $n \geq 3$ and the degree of every vertex is at least $\frac{n}{2}$. Then Γ is Hamiltonian.*

Recall that a bipartite graph is a graph whose vertex set can be divided into two non-empty and disjoint subsets U and V, every vertex in U is adjacent to every vertex in V and two vertices within the same set are not adjacent. If $|U| = m$ and $|V| = n$, then a bipartite graph is denoted $K_{m,n}$. The next theorem is used to detect a planar graph.

Theorem 1.5. *(Kuratowski's Theorem)*[4, Theorem 9.10] *A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

The neighbourhood of a vertex $x \in V(\Gamma)$ is the set of all vertices which are adjacent to x. A dominating set D for a graph Γ is a subset of $V(\Gamma)$ such that every vertices of Γ is either D or neighbourhood of D. The domination number $\gamma(\Gamma)$ is the size of smallest dominating set of Γ.
A Lie algebra L is a vector space over the field $\mathbb{F}$ together with a bilinear map called the Lie bracket $L \times L \to L$ with $(x, y) \mapsto [x, y]$, satisfying the following axioms:

$$\begin{aligned} &\text{(i)} \ [x,x] = 0 \quad \text{for all} \quad x \in L, \\ &\text{(ii)} \ [x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0 \quad \text{for all} \quad x,y,z \in L. \end{aligned}$$

The Lie bracket $[x, y]$ is often referred to as the commutator of x and y. The second condition is known as the Jacobi identity. Since the Lie bracket $[-,-]$ is bilinear, we have $0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y]$. Hence the first condition implies $[x, y] = -[y, x]$ for all $x, y \in L$. For example, every vector space L with $[x, y] = 0$ for all $x, y \in L$ is a Lie algebra. This is called an abelian Lie algebra. A subalgebra of L is a subspace K such that is close under the Lie bracket. In other words, $[x, y] \in K$ for all $x, y \in K$. For instance, $C_L(x) = \{y \in L | [x, y] = 0\}$ is the centralizer of element x in L and a subalgebra of L. An ideal of L is a subalgebra I with this condition $[x, y] \in I$ for all $x \in L$ and $y \in I$. Let I and J are two ideals of a Lie algebra L. It is obvious that $[I, J] = \langle [x, y] \mid x \in I, y \in J \rangle$ is an ideal of L. In particular, the derived subalgebra L^2 is the ideal generates by all $[x, y]$ such that $x, y \in L$. Another important ideal is the centre of a Lie algebra L and defined as $Z(L) = \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$. If L is a Lie algebra over the field $\mathbb{F}_q$ such that $\dim L = n$ and $\{x_1, \ldots, x_n\}$ is a basis of L, then every element x of L are of the form $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$ and the number of elements is equal to q^n. All notations and terminologies are standard and reader can refer to [6, 4]. Moreover, from now on all Lie algebras are considered finite-dimensional over the field $\mathbb{F}_q$ unless otherwise stated.

2. Basic results

In this section, we repeat the definition of non-commuting graph of Lie algebra L over the field $\mathbb{F}_q$ and then investigate about connectivity, diameter, girth, Hamiltonian, Eulerian, and domination number of this graph. First, we state the definition of the new graph as the following.

Definition 2.1. *Let L be a Lie algebra and $Z(L)$ be the centre of L. The non-commuting graph of L, denoted by G_L, is an undirected simple graph whose vertices are all elements in $L \setminus Z(L)$ and two distinct vertices x and y are adjacent if and only if $[x, y] \neq 0$.*

Lemma 2.2. *Let L be a Lie algebra and G_L be the non-commuting graph associated to L. Then $\deg(x) = |L| - |C_L(x)|$ for every $x \in V(G_L)$.*

Proof. Since $\deg(x) = |\{y \in V(G_L) | [x, y] \neq 0\}|$ and $V(G_L) = L \setminus Z(L)$, we have

$$\begin{aligned}\deg(x) &= |V(G_L) \setminus C_L(x)| = (|L \setminus Z(L)|) - |C_L(x)| \\ &= |L| - |C_L(x) \cup Z(L)| = |L| - |C_L(x)|.\end{aligned}$$

Hence $\deg(x) = |L| - |C_L(x)|$ for all $x \in V(G_L)$.

Lemma 2.3. *Let L be a Lie algebra. Then G_L does not have any isolated vertex.*

Proof. On the contrary, let x be an isolated vertex of G_L. Then $\deg(x) = 0$. We know that $\deg(x) = |L| - |C_L(x)|$, by Lemma 2.2. Therefore $L = C_L(x)$ and so $x \in Z(L)$. It is a contradiction with $x \in V(G_L)$.

In the following proposition, we show that the non-commuting graph G_L is connected.

Proposition 2.4. *Let L be a Lie algebra. Then G_L is connected.*

Proof. Let x and y be two arbitrary distinct vertices of G_L. If they are adjacent, then G_L is connected. Suppose that x and y are not adjacent. Then $[x, y] = 0$. Since $x, y \in V(G_L)$, there exist $x_1, y_1 \in V(G_L)$ such that $[x, x_1] \neq 0$ and $[y, y_1] \neq 0$, by Lemma 2.3. If $[x, y_1] \neq 0$ or $[y, x_1] \neq 0$, then $x \sim y_1 \sim y$ or $x \sim x_1 \sim y$. Therefore G_L is connected. Now, let $[x, y_1] = [y, x_1] = 0$. Then $[x, x_1+y_1] = [x, x_1]+[x, y_1] \neq 0$ and $[y, x_1 + y_1] = [y, x_1] + [y, y_1] \neq 0$. So, $x_1 + y_1 \in V(G_L)$ and $x \sim x_1 + y_1 \sim y$ is a path. Hence G_L is connected.

The next proposition states the girth of non-commuting graph G_L is 3.

Proposition 2.5. *Let L be a Lie algebra. Then girth$(G_L) = 3$.*

Proof. Assume that x is an arbitrary vertex of G_L. Then there exists $y \in V(G_L)$ such that x and y are adjacent, by Lemma 2.3. Since $[x, y] \neq 0$, we have $[x+y, y] \neq 0$ and $[x + y, x] \neq 0$. So, $x + y \in V(G_L)$ and $x + y$, x and y are adjacent. Hence there is a cycle of length 3 among the vertices $\{x, y, x+y\}$ and the girth of G_L is equal to 3.

In the next proposition, we obtain an upper bound for the diameter of non-commuting graph G_L.

Proposition 2.6. *Let L be a Lie algebra. Then diam$(G_L) \leq 2$.*

PROOF. Assume that $x, y \in V(G_L)$. We show that $d(x, y) \leq 2$. If x and y are adjacent, then $d(x, y) = 1 < 2$. Suppose that x and y are not adjacent. Thus there are $x_1, y_1 \in V(G_L)$ such that $[x, x_1] \neq 0$ and $[y, y_1] \neq 0$, by Lemma 2.3. Now, we have the following cases:
Case 1. x is adjacent to y_1. In this case, $d(x, y) = d(x, y_1) + d(y_1, y) = 2 \leq 2$.
Case 2. y is adjacent to x_1. By a similar way, one can see $d(x, y) \leq 2$.
Case 3. x and y are not adjacent to y_1 and x_1, respectively. So, $[x, y_1] = [y, x_1] = 0$. Since $[x, x_1 + y_1] \neq 0$ and $[y, x_1 + y_1] \neq 0$, we have $x_1 + y_1 \in V(G_L)$ and

$$d(x, y) = d(x, x_1 + y_1) + d(x_1 + y_1, y) = 2 \leq 2.$$

Hence $\mathrm{diam}(G_L) \leq 2$, by the previous cases.

In the following, we investigate under what condition $\mathrm{diam}(G_L) = 2$. First of all, we need to prove the next theorem.

Theorem 2.7. *Let L be a Lie algebra over the field $\mathbb{F}_q$ and G_L be complete. Then $|Z(L)| = 1$ and $q = 2$.*

PROOF. Since G_L is complete, then $\deg(x) = |V(G_L)| - 1$ and so $\deg(x) = |L| - |Z(L)| - 1$ for all $x \in V(G_L)$. On the other hand, we know that $\deg(x) = |L| - |C_L(x)|$, by Lemma 2.2. Therefore $|C_L(x)| = |Z(L)| + 1$. As x is a vertex of G_L and $x \in C_L(x)$, we have $Z(L) \subsetneq C_L(x)$. Also, $Z(L)$ is a subalgebra of $C_L(x)$ thus $\dim Z(L) \leq \dim C_L(x) - 1$. Hence $|Z(L)| \leq |C_L(x)|/q$ for some $q \geq 2$ and so $|Z(L)|$ attains the maximum size if $q = 2$. It implies that $2|Z(L)| \leq |C_L(x)|$ and then

$$0 = |C_L(x)| - |Z(L)| - 1 \geq 2|Z(L)| - |Z(L)| - 1 = |Z(L)| - 1.$$

Hence $|Z(L)| = 1$. On the contrary, let $q \geq 3$. If $x \in V(G_L)$, then $A = \{2x, 3x, \ldots, (q-1)x\}$ is a non-empty subset of $V(G_L)$. Also, $[x, \alpha x] = 0$ for all $1 \leq \alpha \leq q-1$ and so none of the vertices in A is adjacent to x. Therefore $\deg(x) < |V(G_L)| - 1$, which is a contradiction. Hence $q = 2$.

Corollary 2.8. *Let L be a Lie algebra over the field $\mathbb{F}_q$ such that $|Z(L)| \neq 1$ or $q \neq 2$. Then diam$(G_L) = 2$.*

In the next lemma and propositions, we obtain some properties of the non-commuting graph G_L.

Lemma 2.9. *Let L be a Lie algebra. Then* $\deg(x) \geq 2$ *for all $x \in V(G_L)$.*

Proof. First, we show that G_L has at least three vertices. Since L is non-abelian, then $\dim L/Z(L) \geq 2$. Hence $|L|/|Z(L)| \geq q^2 \geq 4$ and so $|L| \geq 4|Z(L)|$. Also, we know that $|V(G_L)| = |L| - |Z(L)|$ thus $|V(G_L)| \geq 3$. On the other hand, $\deg(x) = |L| - |C_L(x)|$ for all $x \in V(G_L)$ by Lemma 2.2. Moreover, x is a vertex of G_L and so $x \notin Z(L)$. We know that $C_L(x) \neq L$ when $x \notin Z(L)$. This leads to the inequality $\dim C_L(x) \leq \dim L - 1$. Hence $\frac{|L|}{q} \geq |C_L(x)|$ for some $q \geq 2$, The centralizer $|C_L(x)|$ attains its maximal possible size when $q = 2$, yielding the sharpest bound $\frac{|L|}{2} \geq |C_L(x)|$. In addition, the fact that $0, x \in C_L(x)$ and $\frac{|L|}{2} \geq |C_L(x)|$ ensures that $\deg(x) \geq 2$.

Corollary 2.10. *Let L be a Lie algebra and G_L be the non-commuting graph of L. Then G_L is neither tree nor star graph.*

Proposition 2.11. *Let L be a Lie algebra. Then G_L is Hamiltonian.*

Proof. We know that $\deg(x) = |L| - |C_L(x)|$ for all $x \in V(G_L)$, by Lemma 2.2 and $|V(G_L)| \geq 3$, by the proof of Lemma 2.9. Since $q|C_L(x)| \leq |L|$ for all $x \in V(G_L)$ and $q \geq 2$, then

$$2\deg(x) = 2(|L| - |C_L(x)|) \geq |L| > |L| - |Z(L)| = |V(G_L)|.$$

Therefore $\deg(x) > |V(G_L)|/2$ for all $x \in V(G_L)$ and so G_L is Hamiltonian, by Theorem 1.4.

Proposition 2.12. *Let L be a Lie algebra. Then G_L is an Eulerian graph.*

Proof. Suppose that x is an arbitrary vertex of G_L, $|L| = q^n$. Since $C_L(x)$ is a subalgebra of L, it follows that $|C_L(x)| = q^m$ where $n > m$, and the order of $C_L(x)$ divides the order of L. On the other hand $\deg(x) = |L| - |C_L(x)|$, by Lemma 2.2. Thus $\deg(x) = q^n - q^m$. Hence if q is either even or odd, then in both cases $\deg(x)$ is even. Therefore the degree of all vertices of G_L is even. It implies that G_L is Eulerian, by Theorem 1.3.

Proposition 2.13. *Let L be an n-dimensional Lie algebra over the field $\mathbb{F}_q$ with the derived subalgebra of dimension 1. Then G_L is $(q^n - q^{n-1})$-regular.*

Proof. We claim that $\dim C_L(x) = n - 1$ for all $x \in V(G_L)$. By Rank-Nullity Theorem, we have $\dim L = \dim \ker ad_x + \dim \operatorname{Im} ad_x$, where $ad_x : L \to L$ is a linear transformation given by $x \mapsto [x, y]$ for every $y \in L$. Since $\ker ad_x = C_L(x)$ and $\dim L = n$, then $\dim C_L(x) + \dim \operatorname{Im} ad_x = n$. On the other hand, $\operatorname{Im} ad_x \subseteq L^2$ and $\dim L^2 = 1$ thus $\dim \operatorname{Im} ad_x = 1$ for all $x \in V(G_L)$. Therefore $\dim C_L(x) = n - 1$ for all $x \in V(G_L)$. Hence $deg(x) = |L| - |C_L(x)| = q^n - q^{n-1}$ for all $x \in V(G_L)$ and so G_L is $(q^n - q^{n-1})$-regular.

Proposition 2.14. *Let L be a Lie algebra. Then G_L is not complete bipartite.*

Proof. On the contrary, let G_L be complete bipartite and the vertex set of G_L be divided to two distinct parts A and B such that $|A| \leq |B|$. Then

$$2|A| \leq |A| + |B| = |V(G_L)| = |L| - |Z(L)|$$

and so $|A| \leq (|L| - |Z(L)|)/2$. Assume that $x \in V(G_L)$ such that $x \in B$. Then $\deg(x) = |A|$ and we have $|L| - |C_L(x)| = \deg(x) = |A| \leq (|L| - |Z(L)|)/2$. Therefore

$$|L| \leq 2|C_L(x)| - |Z(L)|. \tag{1}$$

Since $Z(L) \subsetneqq C_L(x)$, we may assume that $|C_L(x)|/|Z(L)| = q^t$ for some $t \geq 1$. Hence $|L|/|C_L(x)| \leq 2 - |Z(L)|/|C_L(x)| = 2 - \frac{1}{q^t} < 2$, by (1). It implies that $L = C_L(x)$. Hence $x \in Z(L)$, which is a contradiction.

In the following proposition, we prove that if the domination number of non-commuting graph G_L associated to a Lie algebra L over the field $\mathbb{F}_q$ is 1, then the center of Lie algebra L is zero and $q = 2$.

Proposition 2.15. *Let L be a Lie algebra over the field $\mathbb{F}_q$ and G_L be a non-commuting graph. If $\gamma(G_L) = 1$, then $|Z(L)| = 1$ and $q = 2$.*

Proof. Since $\gamma(G_L) = 1$, there exists $x \in V(G_L)$ such that $deg(x) = |V(G_L)| - 1$. Also, there is a vertex y such that $[x, y] \neq 0$, by Lemma 2.3. On the contrary, let $|Z(L)| \geq 2$. Then we can consider a non-zero element $z \in Z(L)$. On the other hand, $[x + z, y] \neq 0$ thus $x + z \in V(G_L)$, but $[x, x + z] = 0$. It is a contradiction when $\deg(x) = |V(G_L)| - 1$. Therefore $|Z(L)| = 1$. Now, we prove $q = 2$. On the contrary, let $q \geq 3$. Since x is a vertex and $q \geq 3$, we have $A = \{2x, 3x, \ldots, (q-1)x\} \subseteq V(G_L)$. Also, none of the vertices in A is adjacent to x. So, $\deg(x) < |V(G_L)| - 1$ and it is a contradiction. Hence $q = 2$.

Note that one can prove Theorem 2.7 in another way by the above proposition. In the following example, we show that the converse of previous proposition is not true.

Example 2.16. *Let $L = \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1] = x_1, [x_2, y_2] = x_2 \rangle$ is a Lie algebra over the field $\mathbb{F}_2$. It is obvious $|Z(L)| = 1$ and $|L| = 16$. So, $|V(G_L)| = 15$ and $V(G_L) = L \setminus \{0\}$. According to the representation of L and the vertices of G_L, we can see that every vertex is not adjacent to at least a vertex besides itself. Hence $\gamma(G_L) \neq 1$.*

Theorem 2.17. *Let L be a Lie algebra and G_L be the non-commuting graph associated to L. Then $\gamma(G_L) = 1$ if and only if there is $x \in V(G_L)$ such that $|C_L(x)| = 2$.*

Proof. Let $\gamma(G_L) = 1$. Then there is $x \in V(G_L)$ such that $\deg(x) = |V(G_L)| - 1$. On the other hand, $\deg(x) = |L| - |C_L(x)|$, by Lemma 2.2 and $|V(G_L)| = |L| - |Z(L)|$, we have $|L| - |C_L(x)| = |L| - |Z(L)| - 1$. So, $|C_L(x)| = |Z(L)| + 1$. Also, $|Z(L)| = 1$, by Proposition 2.15, hence $|C_L(x)| = 2$. Conversely, let $|C_L(x)| = 2$ for some $x \in V(G_L)$. Then $C_L(x) = \{0, x\}$. Hence x is adjacent to all vertex of G_L besides itself and $\gamma(G_L) = 1$.

3. PLANARITY AND OUTERPLANARITY OF THE NON-COMMUTING GRAPH G_L

In this section, we classify all planar and outerplanar non-commuting graphs of a Lie algebra L. First, we prove the following propositions that play an important role to determine planar graphs.

Lemma 3.1. *There is no* 3*-dimensional Lie algebra over the field* $\mathbb{F}_2$ *with the derived subalgebra of dimension* 1 *and* $Z(L) = 0$.

PROOF. On the contrary, let L be a 3-dimensional Lie algebra over the field $\mathbb{F}_2$ such that $\dim L^2 = 1$ and $Z(L) = 0$. Assume that $\{x, y, z\}$ is a basis of L. Then the set $\{[x, y], [x, z], [y, z]\}$ generates L^2. Without loss of generality, let $\langle [x, y] \rangle$ be a basis of L^2. Then $[x, z] = \alpha[x, y]$ and $[y, z] = \beta[x, y]$ for some $\alpha, \beta \in \mathbb{F}_2$. Since $Z(L) = 0$, then at least one of the coefficients α and β is non-zero. If $\alpha = 1$ and $\beta = 0$, then we can check $y + z \in Z(L)$. It is a contradiction. Also, $x + z \in Z(L)$ when $\alpha = 0$ and $\beta = 1$ and so we have a contradiction. Finally, if $\alpha = \beta = 1$, then $x + y + z \in Z(L)$. Again, it is a contradiction. Hence there is no such Lie algebra.

Proposition 3.2. *Let* L *be a* 3*-dimensional Lie algebra over the field* $\mathbb{F}_2$ *with* $Z(L) = 0$ *and* $\dim L^2 = 2$. *Then* G_L *is isomorphic to the following graph:*

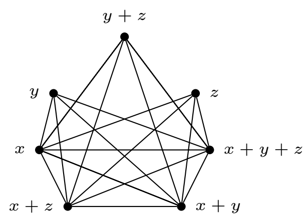

FIGURE 1.

PROOF. First, let the set $\{y, z\}$ be a basis of L^2 and expand it to $\{x, y, z\}$ of a basis of L. Then $\{[x, y], [x, z], [y, z]\}$ generates L^2. Since L^2 is abelian by [6, Lemma 3.3], we have $[y, z] = 0$. On the other hand, $\dim L^2 = 2$ thus $\{[x, y], [x, z]\}$ is another basis of L^2. Hence the set $\{[x, y], [x, z]\}$ is linearly independent. Also, we know that $V(G_L) = \{x, y, z, x + y, x + z, y + z, x + y + z\}$. So, one see that the brackets $[x, y], [x, z], [x, x + y], [x, x + z], [x, y + z]$ and $[x, x + y + z]$ are non-zero. Therefore x is adjacent to $y, z, x + y, x + z, y + z$ and $x + y + z$. By a similar method, the adjacency of other vertices can be found. Hence G_L is isomorphic to Figure 1.

Proposition 3.3. *Let L be a 3-dimensional Lie algebra over the field $\mathbb{F}_2$ with $Z(L) = 0$ and $\dim L^2 = 3$. Then G_L is isomorphic to the following graph:*

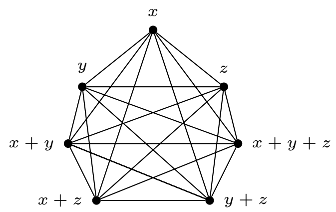

FIGURE 2.

PROOF. Let $\{x, y, z\}$ be a basis of L. Since $Z(L) = 0$, so $V(G_L) = \{x, y, z, x + y, x + z, y + z, x + y + z\}$. Also, $\dim L^2 = 3$ and $B = \{[x, y], [x, z], [y, z]\}$ generates L^2, then B is a basis of L^2. Hence all elements of $V(G_L)$ are adjacent and G_L is isomorphic to Figure 2.

Proposition 3.4. *Let L be a 3-dimensional Lie algebra over the field $\mathbb{F}_2$ with $\dim Z(L) = 1$. Then $\dim L^2 = 1$ and G_L is isomorphic to the following graph:*

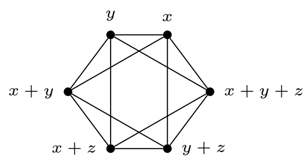

FIGURE 3.

PROOF. Suppose that $Z(L) = \langle z \rangle$ thus we expand $\{z\}$ to a basis $\{x, y, z\}$ of L. Thus $\{[x, y], [x, z], [y, z]\}$ generates L^2. On the other hand, $z \in Z(L)$ and so $[x, z] = [y, z] = 0$. It implies $L^2 = \langle [x, y] \rangle$ and so $[x, y] \neq 0$. Additionally, $V(G_L) = \{x, y, x+y, x+z, y+z, x+y+z\}$. Since $[x, y], [x, x+y], [x, y+z]$ and $[x, x+y+z]$ are non-zero, then x is adjacent to the vertices $y, x + y, y + z$ and $x + y + z$. By a similar way, the adjacency of other vertices are obtained . Hence G_L is isomorphic to Figure 3.

Theorem 3.5. *Let L be a 3-dimensional Lie algebra over the field $\mathbb{F}_2$. Then G_L is isomorphic to one of the Figures 1, 2 or 3.*

Proof. Since L is non-abelian, we have $\dim L^2 \neq 0$ and $\dim L/Z(L) \geq 2$. Also, $\dim L = 3$ thus $1 \leq \dim L^2 \leq 3$ and $\dim Z(L) \leq 1$. Now, the proof is completed by Lemma 3.1, Propositions 3.2, 3.3 and 3.4.

Theorem 3.6. *Let G_L be the non-commuting associated graph to a Lie algebra L. Then G_L is planar if and only if it is isomorphic to one of the following graphs:*

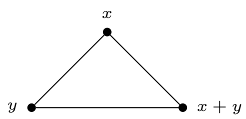

Figure 4.

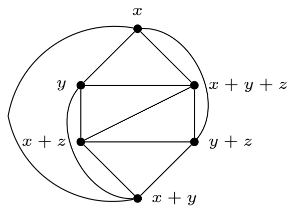

Figure 5.

Proof. Suppose that G_L is a planar graph thus there is $x \in V(G_L)$ such that $deg(x) \leq 5$, by Theorem 1.1. Since $\deg(x) = |L| - |C_L(x)|$, by Lemma 2.2 and $|C_L(x)| \leq |L|/q$, then $|L| \leq 5q/(q-1)$. On the other hand, $f(q) = 5q/(q-1)$ for all $q \geq 2$ is a descending function and thus $|L| \leq 10$. It is obvious that $|L|$ is 2^2, 2^3 or 3^2. Consider the following cases:

Case 1. $|L| = 2^2$. Since L is a non-abelian Lie algebra of dimension 2, we have $\dim L/Z(L) \geq 2$ and so $|Z(L)|$ can be 1. Let $\{x, y\}$ be a basis of L. Then $[x, y] \neq 0$ and $L^2 = \langle [x, y] \rangle$. Then $V(G_L) = \{x, y, x + y\}$, $[x, y] \neq 0$, $[x, x + y] \neq 0$ and $[y, x + y] \neq 0$. Hence G_L is isomorphic to Figure 4.

Case 2. $|L| = 3^2$. By a similar method of case 1, we can see that $|Z(L)|$ is 1 or 3. Let $\{x, y\}$ be a basis of L such that $Z(L) = \{0\}$. Since L is non-abelian, then $[x, y] \neq 0$ and so $L^2 = \langle [x, y] \rangle$. Then $V(G_L) = \{x, 2x, y, 2y, x+y, 2x+y, x+2y, 2x+2y\}$. So, the graph G_L contains the subgraph $K_{3,3}$ as following:

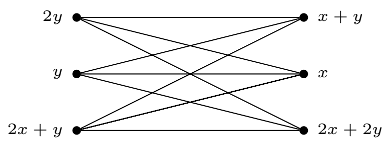

Figure 6.

Hence G_L is not planar, by Theorem 1.5.

Case 3. $|L| = 2^3$. In this case, G_L is isomorphic to Figures 1, 2 or 3, by Theorem 3.5. If G_L is isomorphic to Figure 3, then we can draw the graph by another way and see that it is planar. So, G_L is isomorphic to Figure 5. Suppose that G_L is isomorphic to Figure 1. Then the graph has the following subgraph that is isomorphic to $K_{3,3}$.

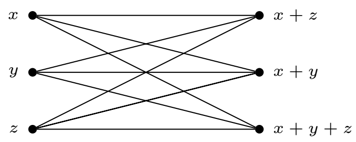

FIGURE 7.

Hence G_L is not planar, by Theorem 1.5. Again, G_L is not planar, by Theorem 1.5 when it is isomorphic to Figure 2. Therefore G_L is isomorphic to Figures 4 or 5.

In the next theorem, we study the existence of non-commuting outerplanar graph associated to a Lie algebra L.

Theorem 3.7. *Let L be a Lie algebra and G_L be the non-commuting associated graph to L. Then G_L is outerplanar if and only if G_L is isomorphic to Figure 4.*

PROOF. Let G_L be an outerplanar graph. Then there is a vertex x such that $deg(x) \leq 2$, by Theorem 1.2. On the other hand, we know that $deg(x) = |L| - |C_L(x)| \leq 2$ and $|C_L(x)| \leq |L|/q$. It implies that $|L| \leq 2q/(q-1)$ for all $q \geq 2$. Since $f(q) = 2q/(q-1)$ for all $q \geq 2$ is descending function, we have $|L| \leq 4$. So, $|L|$ is 2, 3, or 2^2. If $|L| = 2$, or 3, then L is abelian and it is a contradiction. Therefore $|L| = 2^2$. Now, G_L is isomorphic to Figure 4 ,by a similar way in the case 1 of the proof of Theorem 3.6. The converse is obvious.

4. ISOMORPHISM OF THE NON-COMMUTING GRAPHS

In this section, we are going to see that if $G_{L_1} \cong G_{L_2}$ for two Lie algebras L_1 and L_2, then what properties can be deduced from this isomorphism. In the following theorem, we show that if there exists a vertex of degree prime power number in G_{L_1} and non-commuting graphs of two Lie algebras L_1 and L_2 are isomorphic, then their fields have the same characteristic.

Theorem 4.1. *Let L_1 and L_2 be two finite-dimensional Lie algebras over fields $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$, respectively. Also, suppose that $G_{L_1} \cong G_{L_2}$ and there exists a vertex $x \in V(G_{L_1})$ such that $\deg(x)$ is a prime power number. Then $q_1 = q_2$.*

PROOF. Since $G_{L_1} \overset{\varphi}{\cong} G_{L_2}$, we have $|V(G_{L_1})| = |V(G_{L_2})|$ and

$$|L_1| - |C_{L_1}(x)| = \deg(x) = \deg(\varphi(x)) = |L_2| - |C_{L_2}(\varphi(x))|,$$

by Lemma 2.2. Let $|L_1| = q_1^{n_1}$, $|L_2| = q_2^{n_2}$, $|C_{L_1}(x)| = q_1^{m_1}$ and $|C_{L_2}(x)| = q_2^{m_2}$, where $n_1 > m_1$ and $n_2 > m_2$. Then $q_1^{m_1}(q_1^{n_1-m_1} - 1) = q_2^{m_2}(q_2^{n_2-m_2} - 1)$. On the other hand, $\deg(x) = p^n$ for some prime p and $n \geq 1$ thus $q_1 | p^n$ and $q_2 | p^n$, which imply that $q_1 = p = q_2$ and the proof follows.

Theorem 4.2. *Let L_1 and L_2 be Lie algebras and $G_{L_1} \cong G_{L_2}$. If there exists a vertex $x \in V(G_{L_1})$ such that $\deg(x) = q$, where q is a prime number, then $|L_1| = |L_2|$.*

Proof. We know that L_1 and L_2 are the Lie algebras over the $\mathbb{F}_p$, by the previous theorem. Since $G_{L_1} \overset{\varphi}{\cong} G_{L_2}$, we have $q = \deg(x) = \deg(\varphi(x))$ and so $|C_{L_1}(x)|(|L_1|/|C_{L_1}(x)| - 1) = |C_{L_2}(\varphi(x))|(|L_2|/|C_{L_2}(\varphi(x))| - 1) = q$. One can see that $|C_{L_1}(x)| \geq 2$ and $|C_{L_2}(\varphi(x))| \geq 2$, so $|C_{L_1}(x)| = |C_{L_2}(\varphi(x))| = q$. On the other hand, $Z(L_1) \lneqq C_{L_1}(x)$ and $Z(L_2) \lneqq C_{L_2}(\varphi(x))$ imply that $|Z(L_1)| = |Z(L_2)| = 1$. In addition, $|V(G_{L_1})| = |V(G_{L_2})|$, by the existence of isomorphism between G_{L_1} and G_{L_2} and $|L_1| - |Z(L_1)| = |L_2| - |Z(L_2)|$. Therefore $|L_1| = |L_2|$, as required.

Theorem 4.3. *Let L_1 and L_2 be Lie algebras and $G_{L_1} \cong G_{L_2}$. If there exists a vertex $x \in V(G_{L_1})$ such that $\deg(x) = pq$, where p and q are distinct prime numbers and $p > q$, then $L_1 \cong L_2$.*

Proof. By a similar way of the previous theorem, we have

$$|C_{L_1}(x)|(|L_1|/|C_{L_1}(x)| - 1) = |C_{L_2}(\varphi(x))|(|L_2|/|C_{L_2}(\varphi(x))| - 1) = pq$$

and so there are three possibilities:
(i) $|C_{L_1}(x)| = p$ and $|L_1|/|C_{L_1}(x)| - 1 = q$. In this case, $|L_1| = pq + p = (q+1)p$. Thus the only possibility is when $p = 3$ and $q = 2$. Therefore $|L_1| = 3^2$.
(ii) $|C_{L_1}(x)| = q$ and $|L_1|/|C_{L_1}(x)| - 1 = p$. Then $|L_1| = pq + q = (p+1)q$ and so $p = 2$ and $q = 3$. Since $p > q$, then it is impossible.
(iii) $|C_{L_1}(x)| = pq$ and $|L_1|/|C_{L_1}(x)| - 1 = 1$. Thus $|L_1| = 2pq$, which is not possible.
Hence the only possibility is case (i) when $|C_{L_1}(x)| = p$. Similar to the proof of previous theorem, $|C_{L_2}(\varphi(x))| = p$, where φ is an isomorphism from $C_{L_1}(x)$ to $C_{L_2}(\varphi(x))$. Thus $|Z(L_1)| = |Z(L_2)| = 1$, which implies that $|L_1| = |L_2| = 3^2$. Since up to isomorphism there is a unique 2-dimensional non-abelian Lie algebra, then $L_1 \cong L_2$.

Theorem 4.4. *Let L_1 and L_2 be Lie algebras and $G_{L_1} \cong G_{L_2}$. If there exists a vertex $x \in V(G_1)$ such that $\deg(x) = p^2q$, where p and q are prime numbers and $p > q$. Then $L_1 \cong L_2$.*

Proof. We know that $p^2q = \deg(x) = |L_1| - |C_{L_1}(x)| = |C_{L_1}(x)|(|L_1|/|C_{L_1}(x)| - 1)$. Thus there are the following possibilities:
(i) $|C_{L_1}(x)| = p$. We have $|L_1|/|C_{L_1}(x)| - 1 = pq$ and so $|L_1| = p^2q + p = p(pq+1)$. Since $|L_1|$ must be a prime power number, this is impossible.
(ii) $|C_{L_1}(x)| = q$. Then $|L_1| = p^2q + p = q(p^2+1)$ and again it is impossible.
(iii) $|C_{L_1}(x)| = pq$. Since $|C_{L_1}(x)|$ is not a prime power, this case does not occur.
(iv) $|C_{L_1}(x)| = p^2$. We have $|L_1| = p^2q + p = p^2(q+1)$ and the only possibility is when $p = 3$ and $q = 2$. Then $|L_1| = 3^2$. Now, if φ is an isomorphism between G_{L_1} and G_{L_2}, then $p^2q = \deg(x) = \deg(\varphi(x)) = |L_2| - |C_{L_2}(\varphi(x))| =$

$|C_{L_2}(\varphi(x))|(|L_2|/|C_{L_2}(\varphi(x))|-1)$. So,$|C_{L_2}(\varphi(x))| = p^2$ and $|L_2| = 3^2$, by considering the above cases. Hence $|L_1| = |L_2| = 3^2$. Since, there is only a 2-dimensional non-abelian Lie algebra, by [6, Theorem 3.1], then $L_1 \cong L_2$.

Corollary 4.5. *Let L_1 and L_2 be Lie algebras and $G_{L_1} \cong G_{L_2}$. If there is a vertex $x \in V(G_{L_1})$ such that $\deg(x) = p^n q$, where p and q are prime numbers, $p > q$ and $n \geq 1$, then $|L_1| = |L_2|$.*

Proof. The proof is similar to the above theorems.

Proposition 4.6. *Let L_1 and L_2 be Lie algebras over the field $\mathbb{F}_2$ such that $G_{L_1} \cong G_{L_2}$. Then $|L_1| = |L_2|$.*

Proof. We know that $|V(G_{L_1})| = |V(G_{L_2})|$ when $G_{L_1} \cong G_{L_2}$. Then $|L_1| - |Z(L_1)| = |L_2| - |Z(L_2)|$. Suppose that $|L_1| = 2^{n_1}$, $|L_2| = 2^{n_2}$, $|Z(L_1)| = 2^{m_1}$ and $|Z(L_2)| = 2^{m_2}$. Thus $2^{n_1} - 2^{m_1} = 2^{n_2} - 2^{m_2}$. We can assume that $m_1 \geq m_2$. So,

$$2^{m_1-m_2}(2^{n_1}-1) = 2^{n_2}-1.$$

On the other hand, $2^{n_2}-1$ is odd and so $2^{m_1-m_2}(2^{n_1}-1)$ is odd only if $2^{m_1-m_2} = 1$. Hence $m_1 = m_2$ and so $|Z(L_1)| = |Z(L_2)|$. It implies that $|L_1| = |L_2|$.

It is a common question to ask what properties of Lie algebras are preserved under isomorphism between the non-commuting graphs of them. The next example shows that it is not true for the property of nilpotency.

Example 4.7. *Assume that $L_1 = \langle x, y, z | [x, y] = z\rangle$ and $L_2 = \langle x, y, z | [x, y] = x\rangle$. We can see that the non-commuting graph associated to L_1 and L_2 is isomorphic to Figure 3. Also, L_1 is nilpotent, but L_2 does not have such property. So, the nilpotency property is not kept under isomorphism of the non-commuting graphs.*

It is interesting to see that under what conditions we will have $G_{L_1} \cong G_{L_2}$ if and only if $|L_1| = |L_2|$. We left it as a conjecture here.

Conjecture. Find conditions such that $G_{L_1} \cong G_{L_2}$ if and only if $|L_1| = |L_2|$.

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Competing interests

The authors declare no conflict of interest.

REFERENCES

[1] I. Beck, "Coloring of commutative rings," *Journal of algebra*, vol. 116, no. 1, pp. 208–226, 1988. `https://doi.org/10.1016/0021-8693(88)90202-5`.

[2] A. Abdollahi, S. Akbari, and H. Maimani, "Non-commuting graph of a group," *Journal of algebra*, vol. 298, no. 2, pp. 468–492, 2006. `https://doi.org/10.1016/j.jalgebra.2006.02.015`.

[3] B. Tolue and A. Erfanian, "Relative non-commuting graph of a finite group," *Journal of Algebra and its Applications*, vol. 12, no. 02, p. 1250157, 2013. `https://doi.org/10.1142/S0219498812501575`.

[4] J. A. Bondy, U. S. R. Murty, *et al.*, *Graph theory with applications*, vol. 290. Macmillan London, 1976. `https://www.iro.umontreal.ca/~hahn/IFT3545/GTWA.pdf`.

[5] R. Diestel, *Graph Theory, 5th Ed.*, vol. 290. Springer, 2017.

[6] K. Erdmann and M. J. Wildon, *Introduction to Lie algebras*, vol. 122. Springer, 2006. `https://link.springer.com/book/10.1007/1-84628-490-2`.