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On Generalization of Unbounded Order Convergence in Riesz Spaces

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Abstract. Order convergence is a crucial concept in the theory of Riesz spaces. A generalization of order convergence, known as unbounded order convergence (or uo convergence), has been introduced. In this paper, we present a further generalization of uo convergence by using an arbitrary nonempty subset of the positive cone of the space. We then investigate the properties of this generalization, including uniqueness of limits, algebraic properties, and the relationships among these types of convergence. In particular, we show that the generalizations of uo convergence generated by two subsets are equivalent if and only if the bands generated by those subsets are equal.

Key words and Phrases: Riesz spaces, order convergence, unbounded order convergence.

1. INTRODUCTION

Functional analysis is one branch of mathematics that focuses on the study of vector spaces equipped with specific structures. One of the theories within it is the theory of Riesz spaces, first investigated independently by Frigyes Riesz, Leinid Kantorovich, and Hans Freudenthal around 1935. Riesz spaces consist of vector spaces equipped with a partially ordered relation (usually denoted by "$\leq$") that forms a lattice and is compatible with the structure of the vector space. Furthermore, there is a structure derived from Riesz spaces known as Banach lattices. This structure comprises Riesz spaces equipped with a norm and satisfying certain axioms.

One of the important concepts in the theory of Riesz spaces is the order convergence. This concept is significant because its definition involves only the order relation inherent in the Riesz space and does not depend on a specific topology.

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Recall that a net in a Riesz space X is a function from a directed set to X. A net $(x_\alpha)_{\alpha\in\Lambda}$ in the Riesz space X is said to converge in order to $x \in X$, denoted by $x_\alpha \xrightarrow{o} x$, if there exists a net $(z_\beta)_{\beta\in\Pi}$ in X with $z_\beta \downarrow 0$, and for each $\beta \in \Pi$, there exists $\alpha_0 \in \Lambda$ such that $|x_\alpha - x| \leq z_\beta$ for every $\alpha \in \Lambda$ with $\alpha \geq \alpha_0$. Throughout this paper, unless ambiguity arises, we write (x_α) in place of $(x_\alpha)_{\alpha\in\Lambda}$. The concept of order convergence has many applications not only in functional analysis but also in other fields such as financial mathematics.

Some researchers then attempted to generalize the concept of order convergence. In 1948, Hidegoro Nakano [1] introduced a concept of convergence that he called individual convergence. Subsequently, in 1964, Ralph DeMarr [2] defined the unbounded order convergence, abbreviated as uo convergence, which is essentially equivalent to individual convergence. The term uo convergence later became more popular and widely used to this day. A net (x_α) in X is said to converge uo to $x \in X$, denoted as $x_\alpha \xrightarrow{uo} x$, if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in X^+$. Over time, many researchers have examined the properties and applications of uo convergence, including Kaplan [3] investigating the relationship between uo convergence and weak units, Gao and Xanthos [4] discussing applications of uo convergence in martingales without probability, and Niushan Gao [5] studying uo convergence in dual spaces.

In this paper, we introduce a novel form of convergence that generalizes the concept of unbounded order convergence. This new convergence reduces the condition from "for all $u \in X^+$" to "for all $u \in A$", where $A \subseteq X^+$ is specified, and is referred to as A-unbounded order convergence, abbreviated as A-uo convergence. We will investigate some properties of this new concept, including the limit uniqueness and its relation to other convergences. Before that, we establish a new criterion for $A \subseteq X^+$, denoted order density on positive cones, inspired by the notion of order dense Riesz subspaces. It will become apparent later in the paper that this criterion is intricately linked to the uniqueness of the limit of A-uo convergent nets.

2. ORDER DENSE SETS ON POSITIVE CONES

Throughout this paper, X denotes a Riesz space. The set of all positive elements of X is denoted by X^+. Analogously, for all $A \subseteq X$, A^+ is the set of all positive elements of A. Moreover, we use the symbols I_A and $\langle A \rangle$ to represent the ideal and band generated by A, respectively. Definitions and fundamental concepts of Riesz spaces are referenced from [6].

We start this section by recalling the definition of order dense Riesz subspaces given by Aliprantis and Burkinshaw [7]. A Riesz subspace Y of X is said to be order dense in X if for any $u \in X^+$, there exist $y \in Y$ such that $0 \leq y \leq x$ and $y \neq 0$. For the rest of the paper, we write "$a < b$" to denote $a \leq b$ with $a \neq b$. Inspired by this definition, we propose the definition of order density on a positive cone as follows.

Definition 2.1. *A set $A \subseteq X^+$ is said to be order dense in X^+ if for any $u \in X^+$, there exist $y \in I_A$ such that $0 < y \leq u$.*

Clearly, this definition is related to the concept of order dense Riesz subspace as A is order dense in X^+ if and only if I_A is an order dense ideal. It is also important to notice that A is order dense in X^+ if and only if I_A^+ is also order dense in X^+.

We now present two lemmas of order dense sets on the positive cone that will be usefull in the next discussion.

Lemma 2.2. *Let X be an Archimedean Riesz space and $A \subseteq X^+$. Then, A is order dense in X^+ if and only if $\langle A \rangle = X$.*

Proof. Consider that

$$\begin{aligned}
& A \text{ is order dense set in } X^+ \\
\Longleftrightarrow\ & I_A \text{ is order dense ideal in } X \\
\Longleftrightarrow\ & \forall x \in X, |x| = \sup\{y \in I_A : 0 \leq y \leq |x|\} && \text{(by Theorem 1.34. in [7])} \\
\Longleftrightarrow\ & \forall x \in X, x \in \langle I_A \rangle && \text{(by Theorem 7.8 in [6])} \\
\Longleftrightarrow\ & X \subseteq \langle I_A \rangle = \langle A \rangle \\
\Longleftrightarrow\ & X = \langle A \rangle && \text{(since } \langle A \rangle \subseteq X \text{ always holds)}
\end{aligned}$$

Hence, it is true that A is order dense set in X^+ if and only if $\langle A \rangle = X$. □

Lemma 2.3. *Let I and J be order dense ideals in X. Then, $I^+ \cap J^+$ is an order dense set in X^+.*

Proof. Let $x \in X$ with $x > 0$ be arbitrary. Since I is order dense in X, then $0 < u \leq x$ for some $u \in I^+$. Next, since J is also order dense in X, we can find $v \in J^+$ such that $0 < v \leq u$. The fact that I is an ideal gives $v \in I$, so that $v \in (I \cap J)^+ = I^+ \cap J^+$, and we also have $0 < v \leq u \leq x$. Therefore, it can be concluded that $I^+ \cap J^+$ is an order dense set in X^+. □

We end this section by giving some examples.

Example 2.4.

(1) In the Riesz space $\mathbb{R}^2$ equipped with pointwise ordering, the set $\{(1,1)\}$ is order dense in $(\mathbb{R}^2)^+$ since the ideal generated by it is the whole $\mathbb{R}^2$. On the other hand, the set $\{(1,0)\}$ is not order dense in $(\mathbb{R}^2)^+$ since there is no $x \in I_{\{(1,0)\}} = \{(r,0) : r \in \mathbb{R}\}$ satisfying $(0,0) < x \leq (0,1)$.

(2) *Let $\mathbb{R}^{\mathbb{N}}$ be the Riesz space of all sequences in $\mathbb{R}$ with pointwise ordering. Denote by $E := \{e_n : n \in \mathbb{N}\}$ the standard basis of $\mathbb{R}^{\mathbb{N}}$. We are going to prove that E is an order dense set in $(\mathbb{R}^{\mathbb{N}})^+$. To do this, take any $x \in (\mathbb{R}^{\mathbb{N}})^+$ which means x has (at least) one positive term, say its k-th term with $k \in \mathbb{N}$. It is clear that $x(k)e_k \in I_E$ and furthermore, $0 < x(k)e_k \leq x$. It follows that E is order dense in $(\mathbb{R}^{\mathbb{N}})^+$.*

3. BASIC PROPERTIES OF A-UO CONVERGENCE AND LIMIT UNIQUENESS

After performing the idea in defining order dense sets on the positive cone, we continue our discussion to the new concept of convergence we mentioned before. We propose the definition as follows.

Definition 3.1. *Given a nonempty set $A \subseteq X^+$. A net (x_α) in X is said to be A-unbounded order convergent to $x \in X$, denoted by $x_\alpha \xrightarrow{A\text{-uo}} x$, if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in A$. Subsequently, the net (x_α) is said to be A-unbounded order convergence if there exists $x \in X$ such that $x_\alpha \xrightarrow{A\text{-uo}} x$. In this case, x is called the A-uo limit of (x_α).*

As mentioned before, we write "A-uo" as an abbreviation of "A-unbounded order". If $x_\alpha \xrightarrow{A\text{-uo}} x$, we call the element x as A-uo limit of (x_α). For $A = X^+$, the convergence provided in Definition 3.1 is equivalent to the definition of unbounded order presented in Section 1. So, we will use the term "uo" instead of "X^+-uo".

Before exploring some properties, it is important to notice the following equivalence:

$$\begin{aligned} x_\alpha \xrightarrow{A\text{-uo}} x &\iff |x_\alpha - x| \wedge u \xrightarrow{o} 0 \quad \text{for all } u \in A \\ &\iff ||x_\alpha - x| - 0| \wedge u \xrightarrow{o} 0 \quad \text{for all } u \in A \\ &\iff |x_\alpha - x| \xrightarrow{A\text{-uo}} 0. \end{aligned} \tag{1}$$

Thus, when studying some properties of A-uo convergence, it is sufficient to study the A-uo convergence of positive nets to zero.

We now discuss the first property. The set A in the definition of A-uo convergence can be "extended" to I_A^+ in the following sense.

Theorem 3.2. *Given $x \in X$ and a nonempty set $A \subseteq X^+$. For any net (x_α) in X, we have $x_\alpha \xrightarrow{A\text{-uo}} x$ if and only if $x_\alpha \xrightarrow{I_A^+\text{-uo}} x$.*

Proof. Since $A \subseteq I_A^+$, the backward implication is trivial. To prove the forward one, we may assume that (x_α) is in X^+ and $x = 0$ according to equivalence (1).

Now, take any $v \in I_A^+$. Then, there are $n \in \mathbb{N}$, $u_1, u_2, \ldots, u_n \in A$, and nonnegative numbers $r_1, r_2, \ldots, r_n \in \mathbb{R}$ such that

$$v \leq \sum_{k=1}^{n} r_k u_k.$$

From the assumption that $x_\alpha \xrightarrow{A\text{-uo}} 0$, we have $x_\alpha \wedge u_k \xrightarrow{\text{o}} 0$ for all $k \in \{1, 2, \ldots, n\}$, so

$$\begin{aligned} x_\alpha \wedge v &\leq x_\alpha \wedge \sum_{k=1}^{n} r_k u_k \leq \sum_{k=1}^{n} x_\alpha \wedge (r_k u_k) \\ &\leq \sum_{k=1}^{n} ((r_k+1)x_\alpha) \wedge ((r_k+1)u_k) = \sum_{k=1}^{n} (r_k+1)(x_\alpha \wedge u_k) \xrightarrow{\text{o}} 0. \end{aligned}$$

Hence, $x_\alpha \xrightarrow{I_A^+\text{-uo}} 0$ as desired. □

Theorem 3.2 provides convenience in proving the A-uo convergence of nets. If I is an ideal, instead of showing that $|x_\alpha - x| \wedge u \xrightarrow{\text{o}} 0$ for all $u \in I^+$, it is sufficient to show the convergence for all u in a set that generates I.

Example 3.3.

(1) Let c_{00} be the Riesz space of all sequences in $\mathbb{R}$ containing a finite number of nonzero terms equipped with pointwise ordering. Let E be the standard basis as in Example 2.4(2). As a subset of c_{00}, it is easy to see that the ideal generated by E is c_{00} itself.

Now, consider the sequence (ne_n) in c_{00}. For any $n, m \in \mathbb{N}$ with $n > m$, we have $(ne_n) \wedge e_m = \boldsymbol{0}$, where $\boldsymbol{0}$ is the zero sequence. This means the sequence $(ne_n \wedge e_m)_{n \in \mathbb{N}}$ converge to $\boldsymbol{0}$ in order for all $m \in \mathbb{N}$. Thus, it gives $ne_n \xrightarrow{E\text{-uo}} \boldsymbol{0}$ and it implies $ne_n \xrightarrow{uo} \boldsymbol{0}$ by Theorem 3.2.

(2) In the Riesz space $\mathbb{R}^2$ with pointwise ordering, consider the sequence $((0, n))$ and ideal $I = \{(r, 0) : r \in \mathbb{R}\}$. It's easy to see that $|(0, n) - (0, s)| \wedge (1, 0) \xrightarrow{\text{o}} (0, 0)$ for all $s \in \mathbb{R}$. Hence, since I is generated by $\{(0, 1)\}$, we have $(0, n) \xrightarrow{I^+\text{-uo}} (0, s)$ for any $s \in \mathbb{R}$.

Example 3.3(2) shows that the A-uo limit of a net is not unique in general. We provide a sufficient condition for the uniqueness in the theorem below together with some algebraic properties of A-uo limit.

Theorem 3.4. *Let $A \subseteq X^+$ be a nonempty set, (x_α) and (y_α) be two nets in X, and $x, y \in X$. If $x_\alpha \xrightarrow{A\text{-uo}} x$ and $y_\alpha \xrightarrow{A\text{-uo}} y$, then*

(1) if A is order dense in X^+ and $x_\alpha \xrightarrow{A\text{-uo}} z$ for some $z \in X$, then $z = x$;

(2) $rx_\alpha + sy_\alpha \xrightarrow{A\text{-}uo} rx + sy$ *for all* $r, s \in \mathbb{R}$*;*
(3) $|x_\alpha| \xrightarrow{A\text{-}uo} |x|, x_\alpha \vee y_\alpha \xrightarrow{A\text{-}uo} x \vee y, x_\alpha \wedge y_\alpha \xrightarrow{A\text{-}uo} x \wedge y, x_\alpha^+ \xrightarrow{A\text{-}uo} x^+,$ *and* $x_\alpha^- \xrightarrow{A\text{-}uo} x^-$*;*
(4) *if* A *is order dense in* X^+ *and* $x_\alpha \leq y_\alpha$ *for all index* α*, then* $x \leq y$*.*

Proof. According to Theorem 3.2, it can be assumed that $A = I^+$ for some ideal I in X.

(1) Suppose $|x - z| > 0$. Since A is order dense in X^+, there exists $v \in A$ satisfying $0 < v \leq |x - z|$. Next, since $x_\alpha \xrightarrow{A\text{-uo}} x$ and $x_\alpha \xrightarrow{A\text{-uo}} z$, we have
$$\begin{aligned} v = |x - z| \wedge v &\leq (|x_\alpha - x| + |x_\alpha - z|) \wedge v \\ &\leq |x_\alpha - x| \wedge v + |x_\alpha - z| \wedge v \xrightarrow{o} 0, \end{aligned}$$
which implies $v = 0$, a contradiction. As conclusion, $|x - z| = 0$ or equivalently, $z = x$.

(2) Pick any $u \in A$. Since $x_\alpha \xrightarrow{A\text{-uo}} x$ and $y_\alpha \xrightarrow{A\text{-uo}} y$, we have
$$\begin{aligned} 0 &\leq |rx_\alpha + sy_\alpha - (rx + sy)| \wedge u \\ &\leq (|r||x_\alpha - x| + |s||y_\alpha - y|) \wedge u \\ &\leq (|r||x_\alpha - x|) \wedge u + (|s||y_\alpha - y|) \wedge u \\ &\leq ((|r| + 1)|x_\alpha - x|) \wedge (|r| + 1)u + ((|s| + 1)|y_\alpha - y|) \wedge (|s| + 1)u \\ &= (|r| + 1)(|x_\alpha - x| \wedge u) + (|s| + 1)(|y_\alpha - y| \wedge u) \xrightarrow{o} 0. \end{aligned}$$
Therefore, $rx_\alpha + sy_\alpha \xrightarrow{A\text{-uo}} rx + sy$, as desired.

(3) Let $u \in A$ be arbitrary. Notice that $||x_\alpha| - |x|| \leq |x_\alpha - x|$. We have
$$0 \leq ||x_\alpha| - |x|| \wedge u \leq |x_\alpha - x| \wedge u \xrightarrow{o} 0,$$
so that $||x_\alpha| - |x|| \wedge u \xrightarrow{o} 0$. Hence, $|x_\alpha| \xrightarrow{A\text{-uo}} |x|$. Furthermore, combining with part (2), we obtain
$$\begin{aligned} x_\alpha \vee y_\alpha &= \frac{1}{2}(x_\alpha + y_\alpha + |x_\alpha - y_\alpha|) \xrightarrow{A\text{-uo}} \frac{1}{2}(x + y + |x - y|) = x \vee y, \\ x_\alpha \wedge y_\alpha &= \frac{1}{2}(x_\alpha + y_\alpha - |x_\alpha - y_\alpha|) \xrightarrow{A\text{-uo}} \frac{1}{2}(x + y - |x - y|) = x \wedge y, \\ x_\alpha^+ &= \frac{1}{2}(x_\alpha + |x_\alpha|) \xrightarrow{A\text{-uo}} \frac{1}{2}(x + |x|) = x^+, \text{ and} \\ x_\alpha^- &= x_\alpha^+ - x_\alpha \xrightarrow{A\text{-uo}} x^+ - x = x^-. \end{aligned}$$

(4) Notice that $x_\alpha \wedge y_\alpha \xrightarrow{A\text{-uo}} x \wedge y$. On the other hand, $x_\alpha \wedge y_\alpha = x_\alpha \xrightarrow{A\text{-uo}} x$. Using Theorem 3.4(1), we have $x = x \wedge y \leq y$.

□

The first result in Theorem 3.4 states that the order density of A in X^+ guarantees the uniqueness of the A-uo limit of A-uo convergence net. We now show that this fact can be strengthened. Specifically, if A and B are order dense set in X^+, the A-uo and B-uo limit of any net (if exists) must be same. The converse is also true.

Theorem 3.5. *Let $A, B \subseteq X^+$ be two nonempty sets. These two statements are equivalent.*

(1) Both A and B are order dense in X^+.

(2) For every net (x_α) in X with $x_\alpha \xrightarrow{A\text{-}uo} x$ and $x_\alpha \xrightarrow{B\text{-}uo} y$ for some $x, y \in X$, we have $x = y$.

In particular, if $A = B$, the order density of A in X^+ is necessary and sufficient to ensure the A-uo limit uniqueness.

Proof. First, we may assume that $A = I^+$ and $B = J^+$ for some ideals I and J. Now, it's known that A and B are order dense in X^+. Since $x_\alpha \xrightarrow{A\text{-uo}} x$, then $x_\alpha \xrightarrow{A\cap B\text{-uo}} x$. On the other hand, since $x_\alpha \xrightarrow{B\text{-uo}} y$, we also have $x_\alpha \xrightarrow{A\cap B\text{-uo}} y$. Notice that $A \cap B$ is order dense in X^+ according to Lemma 2.3. Using Theorem 3.4(1), we can conclude that $x = y$.

For the reverse direction, it's known that for every net (x_α) in X, if $x_\alpha \xrightarrow{A\text{-uo}} x$ and $x_\alpha \xrightarrow{B\text{-uo}} y$, then $x = y$. Suppose A isn't order dense in X^+. It means that there is $z \in X^+$ with $z > 0$ and for any $u \in A$ satisfying $u \leq z$, we must have $u = 0$. Now, pick $a \in A$ arbitrarily. Since $A = I^+$ with I is an ideal, then $a \wedge z \in A$ and $a \wedge z \leq z$, so that $|0 - z| \wedge a = 0$ must hold. It implies, $0 \xrightarrow{A\text{-uo}} z$. However, it's also true that $0 \xrightarrow{B\text{-uo}} 0$. Hence, we must have $z = 0$, which is a contradiction. Therefore, A must be order dense in X^+, and it can be shown analogously that so is B. □

4. EXTENSION OF A TO $\langle A \rangle^+$

In Theorem 3.2, it has been shown that the set A in the definition of A-uo convergence can be expanded to I_A^+. Now, it's natural to ask whether we can extend A to the larger set, such as $\langle A \rangle^+$. This theorem below states that such expansion is possible in the Archimedean Riesz spaces.

Theorem 4.1. *Let X be an Archimedean Riesz space, $A \subseteq X^+$ be a nonempty set, and $x \in X$. For every net (x_α) in X, we have $x_\alpha \xrightarrow{A\text{-}uo} x$ if and only if $x_\alpha \xrightarrow{\langle A \rangle^+\text{-}uo} x$.*

Proof. We first assume that $A = I^+$ for some ideal I in X. Also, we may assume that (x_α) is in X^+ and $x = 0$ based on equivalence 1. The backward implication of this theorem is trivial since $A \subseteq \langle A \rangle^+$. So, it remains to show the forward one.

For the proof, we will use the notion of Dedekind completion of X, denoted by X^δ. Readers who are not familiar with this concept may refer to [8] for details. We also use the fact provided in [9], which states $x_\alpha \xrightarrow{\text{o}} x$ in X is equivalent to $x_\alpha \xrightarrow{\text{o}} x$ in X^δ for any net (x_α) in X and $x \in X$. The assumption of Archimedean space for X is needed since only Archimedean spaces have Dedekind completion.

Now, take any $u \in \langle A\rangle^+$. Consider $(x_\alpha \wedge u)$ as a net on X^δ. Since it is bounded above by u and bounded below by 0, we have

$$\limsup_\alpha x_\alpha \wedge u = \inf_\alpha \sup_{\beta\geq\alpha} x_\beta \wedge u$$

exists in X^δ. Next, we are going to show that $\limsup_\alpha x_\alpha \wedge u = 0$.

Suppose on the contrary that $\limsup_\alpha x_\alpha \wedge u \neq 0$ is true. This means that we can find a lower bound $y \in X^\delta$ for the net $\big(\sup_{\beta\geq\alpha} x_\beta \wedge u\big)$ with $y \nleq 0$. Without loss of generality, we may assume that $y > 0$ (otherwise, pick $y \vee 0$ to replace y). Using the definition of Dedekind completion, there exists $z \in X$ such that $0 < z \leq y$. Notice that for every index α,

$$0 < z \leq y \leq \sup_{\beta\geq\alpha} x_\beta \wedge u \leq \sup_{\beta\geq\alpha} u = u,$$

so, since $u \in \langle A\rangle^+$ and $\langle A\rangle$ is an ideal in X, we get $z \in \langle A\rangle^+$.

Now, using Theorem 7.8 in [6], $z = \sup\{v \in A : v \leq z\}$. Since $z > 0$, we can find $v \in A$ satisfying $0 < v \leq z$. Furthermore, v also satisfy $0 < v < z \leq y \leq u$, so

$$\begin{aligned}
& v \wedge \sup_{\beta\geq\alpha} x_\beta \wedge u \geq v \wedge y \\
\iff & \sup_{\beta\geq\alpha} x_\beta \wedge u \wedge v \geq v \\
\iff & \sup_{\beta\geq\alpha} x_\beta \wedge v \geq v \\
\implies & \limsup_\alpha x_\alpha \wedge v \geq \inf_\alpha v = v > 0.
\end{aligned}$$

On the other hand, $v \in A$ and $x_\alpha \xrightarrow{A\text{-uo}} 0$ on X, so we have $x_\alpha \wedge v \xrightarrow{\text{o}} 0$ on X which implies $x_\alpha \wedge v \xrightarrow{\text{o}} 0$ di X^δ. It contradicts with the fact that $\limsup_\alpha x_\alpha \wedge v > 0$ di X^δ. Hence, it must be true that $\limsup_\alpha x_\alpha \wedge u = 0$ in X^δ.

Last, notice also that $\liminf_\alpha x_\alpha \wedge u \geq \liminf_\alpha 0 = 0$ in X^δ, so it can be conclude that $x_\alpha \wedge u \xrightarrow{\text{o}} 0$ di X^δ which gives $x_\alpha \wedge u \xrightarrow{\text{o}} 0$ in X. As conclusion, $x_\alpha \xrightarrow{\text{uo}} 0$. □

An element $e > 0$ in X is called weak (order) unit if $\langle\{e\}\rangle = X$. If X is Archimedean and has a weak unit e, Samuel Kaplan [3] and Gao and Xanthos [4] gave a fact that to verify the uo convergence of (x_α) to $x \in X$, it is sufficient to check whether $|x_\alpha - x| \wedge e$ is convergent in order to x. In this paper, we have a stronger statement as a corollary of Theorem 2.2 and Theorem 4.1.

Corollary 4.2. *Given an Archimedean Riesz space X, a nonempty set $A \subseteq X^+$, $x \in X$, and a net (x_α) in X.*

(1) *If A is order dense in X^+, then $x_\alpha \xrightarrow{A\text{-}uo} x$ if and only if $x_\alpha \xrightarrow{uo} x$.*
(2) *If X has a weak unit $e > 0$, then $x_\alpha \xrightarrow{uo} x$ if and only if $|x_\alpha - x| \wedge e \xrightarrow{o} 0$.*

In general, A-uo convergence is not equivalent to B-uo convergence for two different sets $A, B \in X^+$. It raises a question about when A-uo and B-uo convergence are actually equivalent. Theorem 4.1 gives a sufficient condition for it in an Archimedean space, that is when $B = \langle A \rangle^+$. In fact, using this theorem, we can deduce a sufficient and necessary condition for the equivalence.

Theorem 4.3. *Let X be an Archimedean Riesz space and $A, B \subseteq X^+$ be two nonempty sets. The following two statements are equivalent.*
(1) *$\langle A \rangle = \langle B \rangle$.*
(2) *For every net x_α in X and $x \in X$, we have $x_\alpha \xrightarrow{A\text{-}uo} x$ if and only if $x_\alpha \xrightarrow{B\text{-}uo} x$.*

Proof. The implication (1) $\implies$ (2) is clear according to Theorem 4.1. It remains to show the reverse implication. To do that, we may assume at first that $A = I^+$ and $B = J^+$ for some ideals $I, J \subseteq X$ based on Theorem 3.2.

Now, take any $b \in B$ and let $D = \{a \in A : a \leq b\}$. For each $a_1, a_2 \in D$, $a_1 \vee a_2 \in A$ and $a_1 \vee a_2 \leq b$ hold, so that $a_1 \vee a_2 \in D$. Consequently, D is a directed set with the same order relation as X. Next, define a net $(x_\alpha)_{\alpha \in D}$ with $x_\alpha = \alpha$ for each $\alpha \in D$. It is clear that $x_\alpha \uparrow$.

Pick any $a \in A$. We obtain $(b - x_\alpha) \wedge a \downarrow$ and 0 is the lower bound of the net $((b - x_\alpha) \wedge a)_{\alpha \in D}$. Next, take any lower bound of the net, say v. It is obtained that $u = v \vee 0$ is also its lower bound or in other words, $u \leq (b - x_\alpha) \wedge a$ for every $\alpha \in D$. This fact gives $u \leq a$, so that $u + \alpha \in A$ for all $\alpha \in D$ since $A = I^+$ for some ideal I. We also have $u + \alpha = u + x_\alpha \leq b$. Therefore, $u + a \in D$, which implies $u \leq (b - x_{u+\alpha}) \wedge a$ for all $\alpha \in D$. Using mathematical induction, it can be shown that for every $n \in \mathbb{N}$ and $\alpha \in D$, we have $u \leq (b - x_{nu+\alpha}) \wedge a$.

From the result above, we obtain $nu \leq nu + \alpha = nu + x_\alpha \leq b$ for all $n \in \mathbb{N}$. It implies that u must be zero since X is Archimedean. This results in $\inf_{\alpha \in D}(b - x_\alpha) \wedge a = 0$, so that $|x_\alpha - b| \wedge a = (b - x_\alpha) \wedge a \xrightarrow{o} 0$. So, since a is arbitrary, we obtain $x_\alpha \xrightarrow{A\text{-uo}} b$.

Finally, using the assumption (2), we get $x_\alpha \xrightarrow{B\text{-uo}} b$, so that $|b - x_\alpha| = |b - x_\alpha| \wedge b \xrightarrow{o} 0$ or equivalently, $x_\alpha \xrightarrow{o} b$. Consequently $x_\alpha \uparrow b$, so that

$$\sup\{a \in A : a \leq b\} = \sup D = \sup\{x_\alpha : \alpha \in D\} = b,$$

Based on Theorem 7.8 in [6], we obtain $b \in \langle A \rangle$, which implies $B \subseteq \langle A \rangle$. Analogously, we can show $A \subseteq \langle B \rangle$. Thus, it is proven that $\langle A \rangle = \langle B \rangle$. $\square$

As a remark, we note that Theorem 4.3 may not be hold in non-Archimedean Riesz spaces. For instance, on the Riesz space $\mathbb{R}^2$ with lexicographic order relation, let $A = \{(0, y) : y \in \mathbb{R}\}$. It can be shown that A is a band in $\mathbb{R}^2$. Now, take any net (x_α) in $\mathbb{R}^2$ and $x \in \mathbb{R}^2$ with $x_\alpha \xrightarrow{A^+\text{-uo}} x$. We have $|x_\alpha - x| \wedge (0,2) \xrightarrow{\text{o}} (0,0)$, so there is a net (z_β) with $z_\beta \downarrow (0,0)$ and for each index β, there is index α_β such that for every index α with $\alpha \geq \alpha_\beta$, $|x_\alpha - x| \wedge (0,2) \leq z_\beta$ is hold.

Since $\mathbb{R}^2$ is totally ordered, there must be an index β' satisfying $z_{\beta'} \leq (0,1)$. Consequently, for any index α with $\alpha \geq \alpha_{\beta'}$, we have

$$|x_\alpha - x| \wedge (0,2) \leq z_{\beta'} \leq (0,1)$$

which gives $|x_\alpha - x| \leq (0,2)$. So, for each index β, there is an index α_0 which is an upper bound of $\{\alpha_\beta, \alpha_{\beta'}\}$ such that for each index α with $\alpha \geq \alpha_0$, it is true that

$$|x_\alpha - x| = |x_\alpha - x| \wedge (0,2) \leq z_\beta,$$

which means $x_\alpha \xrightarrow{\text{o}} x$.

Since an order convergent net is also uo convergent, we obtain $x_\alpha \xrightarrow{\text{uo}} x$. This shows that $x_\alpha \xrightarrow{A^+\text{-uo}} x$ is equivalent to $x_\alpha \xrightarrow{\text{uo}} x$ for each (x_α) net in $\mathbb{R}^2$ and $x \in \mathbb{R}^2$. However, it is easy to see that $\langle A^+ \rangle \neq \big\langle (\mathbb{R}^2)^+ \big\rangle$. Thus, Theorem 4.3 does not hold on the Riesz spaces $\mathbb{R}^2$ with lexicographic order relation.

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